identical transformation $A \rightarrow A$. Then $\overline{U}$ may be chosen of type Δ, and the number $(\Gamma \cdot \Delta)$ obtained is precisely (24).

Let us recall in concluding that the same formulas hold for transformations of compact metric HLC spaces. They are spaces endowed with a strong type of local connectedness in the sense of homology, analogous to that possessed by the so-called absolute neighborhood retracts.†

PRINCETON UNIVERSITY

CIRCLES IN WHICH $|F(x)|$ HAS A SINGULARITY OR ASSUMES PREASSIGNED VALUES

BY J. W. CELL

Let k be a given positive integer and let a_0 and $a_k \neq 0$ be two given constants. Let $F_k(x)$ be any member whatever of the class C_k of functions which are regular in the neighborhood of the origin and which there have the expansion

$$F_k(x) = a_0 + a_k x^k + a_{k+1} x^{k+1} + \cdots ,$$

where a_0 and a_k are the two given constants.

THEOREM 1. *Let $\eta(a_0, a_1) = 0$ if $|a_0| = 1$. In case $|a_0| < 1$, let $\eta(a_0, a_1) = \{1 - |a_0|^2\}/|a_1|$, and if $|a_0| > 1$, let $\eta(a_0, a_1) = \{2|a_0| \log |a_0|\}/|a_1|$. Then in or on the circle $|x| = \eta(a_0, a_1)$, either $F_1(x)$ has a singularity or $|F_1(x)|$ assumes the value one. Moreover, no smaller radius will do for the whole class of functions C_1.*

COROLLARY. $\eta(a_0, 1) = |a_1| \; \eta(a_0, a_1)$.

PROOF. If $|a_0| = 1$, the theorem is granted, so we shall henceforth suppose that this is not the case. If $a_0 = re^{i\alpha}$, $(r \geqq 0)$, we define $E(x) = e^{-i\alpha} F_1(x)$. Then $|E(x)| = |F_1(x)|$ and hence we may, with no loss of generality in the proof, suppose that a_0 is real and non-negative.

CASE 1. $0 \leqq a_0 < 1$. There exists a positive number η such that for $|x| \leqq \eta$, $F_1(x)$ is regular and $|F_1(x)| < 1$. Now form

$$G(x) = \frac{F_1(x) - a_0}{-a_0 F_1(x) + 1}. \tag{1}$$

† See Duke Mathematical Journal, vol. 2 (1936), pp. 435–442.

This transformation maps $|F_1(x)|<1$ upon $|G(x)|<1$ and $|F_1(x)|=1$ upon $|G(x)|=1$. Hence when $|x|\leqq\eta$, $G(x)$ is regular and $|G(x)|<1$. We expand $G(x)$ in a power series which converges for $|x|\leqq\eta$ and obtain

$$(2) \qquad G(x) = \frac{a_1 x}{1-a_0^2} + \cdots .$$

We apply Cauchy's inequality for the first derivative and obtain $\eta\leqq(1-a_0^2)/|a_1|$.

To show that $\eta(a_0, a_1)$ cannot be less than this amount and hence that the equality sign must persist if this is to be true for every member of the class C_1, subject to the condition $0\leqq a_0<1$, it is sufficient to exhibit a particular member of C_1 which is regular for $|x|<(1-a_0^2)/|a_1|$ and which in this circle is less than one in absolute value. To this end we define a particular function $G(x)$ by the first term of the series (2) and thence a particular function $F_1(x)$ by the inverse of (1). This resulting function $F_1(x)$ is easily shown to have the requisite properties.

Let S be the region consisting of the interior and boundary of the circle $|x|=\eta(a_0, a_1)$. To complete the proof of the theorem for this case we must show that if $F_1(x)$ is regular throughout S, then $|F_1(x)|$ must assume the value one at least once in S. To that end we suppose that $F_1(x)$ is regular throughout S and that $|F_1(x)|$ is not equal to one at any point of S. Then, since $|F_1(0)|<1$, we would have $|F_1(x)|<1$ at every point of S. It would follow that $|G(x)|<1$ at every point of S. But by Cauchy's inequality,

$$|G'(0)| \leqq \frac{\max |G(x)|}{\eta(a_0, a_1)},$$

or by using the series (2),

$$\frac{1}{\eta(a_0, a_1)} \leqq \frac{\max |G(x)|}{\eta(a_0, a_1)}.$$

Hence $\max |G(x)|\geqq 1$, which is a contradiction. Thus either $F_1(x)$ is not regular at every point of S, or else $|F_1(x)|\geqq 1$ at some point of S. Now there exist members of the class C_1 which are regular for all values of x in S and hence the contradiction shows, since $|F_1(0)|<1$, that $|F_1(x)|=1$ at some point of S.

CASE 2. $1 < a_0$. If $1 < a_0$, there exists a positive number η such that for $|x| \leqq \eta$, $F_1(x)$ is regular and $|F_1(x)| > 1$. We write

$$K(x) = \frac{\log F_1(x) - 1}{\log F_1(x) + 1}. \tag{3}$$

Let the region in the F_1 plane consisting of the exterior and boundary of the unit circle with center at the origin be the region R. If we use the principal determination for $\log F_1(x)$, this transformation maps R on a circular arc triangle with vertices on the unit circle in the K plane and in such a way that the boundary of R (the circumference of the unit circle and the point at ∞) maps upon part of the circumference of the unit-circle in the K plane and the remainder of R upon the interior of this circle. The map for a general determination of the logarithm is a similar circular arc triangle with the same properties as to the image of the boundary of R and of its interior. Moreover, these circular arc triangles fill up, without overlapping, the interior of the unit circle in the K plane.

If we agree to use the principal determination for $\log F_1(x)$, then for $|x| \leqq \eta$, $K(x)$ is regular and $|K(x)| < 1$. Then $K(x)$ has the power series expansion $K(x) = \alpha_0 + \alpha_1 x + \cdots$, where

$$\alpha_0 = \frac{\log a_0 - 1}{\log a_0 + 1}, \qquad \alpha_1 = \frac{2a_1}{a_0(\log a_0 + 1)^2}.$$

We may apply the results of our former case to $K(x)$. We form $(1-\alpha_0^2)/|\alpha_1|$ and obtain $(2a_0 \log a_0)/|a_1|$, which is $\eta(a_0, a_1)$ for this second case.

We replace a_0 by $|a_0|$ in our results and thus obtain the theorem as stated.

THEOREM 2. *Let $\beta \neq 0$ be a positive real number and let $\phi(a_0, a_1, \beta) = \eta(a_0/\beta, a_1/\beta)$. Then in or on the circle $|x| = \phi(a_0, a_1, \beta)$ either $F_1(x)$ has a singularity or else $|F_1(x)|$ assumes the value β.*

We define $L(x) = F_1(x)/\beta$ and apply Theorem 1 to $L(x)$ to prove this theorem.

THEOREM 3. *Let $\eta(a_0, a_k, k) = \{\eta(a_0, a_k)\}^{1/k}$. Then in or on the circle $|x| = \eta(a_0, a_k, k)$, either $F_k(x)$ has a singularity or else $|F_k(x)|$ assumes the value one.*

The proof of this theorem follows the same method as the proof of Theorem 1 with the exception that we now use Cauchy's inequality for the kth derivative instead of for the first derivative.

THEOREM 4. *Let* $\theta(a_0, a_1)=0$ *if* $a_0=0$ *or if* $|a_0|=1$. *Otherwise let* $\theta(a_0, a_1)=|\{2a_0 \log |a_0|\}/a_1|$. *Then in or on the circle* $|x|=\theta(a_0, a_1)$, *either* $F_1(x)$ *has a singularity or else* $|F_1(x)|$ *assumes the value zero or the value one.*

We may, in the proof, suppose that a_0 is positive and that $a_0\neq 1$. If $0<a_0<1$, we write

$$G(x) = \frac{\log F_1(x) + 1}{\log F_1(x) - 1},$$

and if $1<a_0$,

$$H(x) = \frac{\log F_1(x) - 1}{\log F_1(x) + 1}.$$

In both cases the resulting function, for x interior to or on the boundary of an appropriate circle, is less than one in absolute value. We may then apply the result of the first case of Theorem 1 to complete the proof.

It is clear that theorems similar to 2 and 3 may be established as generalizations of Theorem 4. We remark that the existence of the various radii in these theorems is implied by the Picard-Landau theorems.*

NORTH CAROLINA STATE COLLEGE

---

* See E. Landau, *Darstellung und Begründung einiger neuer Ergebnisse der Funktionentheorie*, 1929.