ON ARITHMETIC INVARIANTS OF BINARY CUBIC AND BINARY TRILINEAR FORMS*

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1. *Introduction.* It is the purpose of this note to show that binary cubic and binary trilinear forms can be completely and very simply characterized by *arithmetic rank invariants* for non-singular linear transformations in the complex field. We define the factorization rank† of a matrix $A=(a_{ijk})$ of order n and its associated trilinear form to be the minimum value of ϵ such that A can be "factored" into the form

$$(1) \qquad A = \left(\sum_{\alpha=1}^{\epsilon} a_{\alpha i} b_{\alpha j} c_{\alpha k}\right), \qquad (i, j, k = 1, 2, \cdots, n).$$

Hitchcock‡ obtained minimum values of ϵ for certain polyadics. The number ϵ is invariant under non-singular linear transformations on the variables in the trilinear form

$$\sum a_{ijk} x_i y_j z_k$$

associated with A. In a paper which appeared in this Bulletin,§ I classified binary trilinear forms by means of ranks and a property of invariant factors, which are invariant under non-singular linear transformations in the complex field. *The ranks of that paper and the rank defined above form a complete invariant system for these forms. For binary cubic forms the factorization rank alone forms a complete invariant system.*

2. *Factorization Ranks.* The canonical binary trilinear forms are

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† This rank was used in developing a general theory of non-singular p-way matrices in the paper *Non-singular multilinear forms and certain p-way matrix factorizations*, Transactions of this Society, vol. 39 (1936), pp. 422–455. If factorization rank is similarly defined for 2-way matrices, it is found that the factorization rank of a 2-way matrix is n if and only if its ordinary rank is n.

‡ F.L. Hitchcock, *The expression of a tensor or polyadic as a sum of products*, Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 6 (1927), pp. 164–189.

§ *On canonical binary trilinear forms*, vol. 38 (1932), pp. 385–387.

$$
\begin{aligned}
R &= x_1y_1z_1 + x_2y_2z_2,\\
L &= x_1y_1z_1 + x_2y_2z_1 + x_2y_1z_2,\\
H &= x_1y_1z_1 + x_1y_2z_2,\\
K &= x_1y_1z_1.
\end{aligned}
$$

The forms H, K can be distinguished from each other and from L, R by the ranks of my Bulletin paper referred to above. Since the i-, j-, and k-ranks* of L equal 2, the factorization rank ϵ of L is $\geqq 2$. If ϵ were equal to 2 for L, the form L would be equivalent to R, which is impossible. For then the matrix A of L is of the form (1), where $A=(a_{\alpha i})$, $B=(b_{\alpha j})$, $C=(c_{\alpha k})$ are non-singular, whence transformations on L with A^{-1}, B^{-1}, C^{-1} give R. Since the matrix of L can be written in the form (1), where

$$
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix},
$$

it follows that $\epsilon=3$.

The factorization rank ϵ for H is $\geqq 2$, since the j- and k-ranks of H equal 2. That $\epsilon=2$ is evident since the matrix of H can be written in the form (1), where

$$
A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad B = C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

The matrix of K can be written in the form (1), where

$$
A = B = C = (1,\ 0),
$$

whence $\epsilon=1$.

The canonical binary cubic forms for the complex field are well known.† They are

$$
P = x^3 + y^3, \qquad Q = x^2y, \qquad S = x^3.
$$

Since the matrix of P can be taken to be the same as the matrix of R, its factorization rank is 2. The matrix of $3Q$ can be taken to be the matrix A whose only non-vanishing elements are

* The i-rank of $A=(a_{ijk})$, $(i,\ j,\ k=1,2)$, is 2 if the minors (a_{1jk}) and (a_{2jk}) are linearly independent.

† L. E. Dickson, *Modern Algebraic Theories*, pp. 7–9.

$a_{112}=a_{121}=a_{212}=1$. The trilinear form associated with this matrix is $x_1y_1z_2+x_1y_2z_1+x_2y_1z_1$, which is equivalent to L under the transformations $x_1=x_2'$, $x_2=x_1'$. Since the matrices of S and K can be taken to be the same, the factorization rank of S is 1. We have proved the following result.

THEOREM. *The factorization ranks of the forms in the sets* (K, S), (R, P, H), *and* (L, Q) *are* 1, 2, *and* 3, *respectively.*

The equivalence of cubics to P, Q, S can be recognized very simply without the use of factorization rank from the theory of my previous Bulletin paper.

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A NOTE ON THE DEGREE OF POLYNOMIAL APPROXIMATION*

BY J. H. CURTISS

Let C be a rectifiable Jordan curve of the finite z plane. We shall say that a function $f(z)$ belongs to the class Lip (C, j, α) if $f(z)$ is regular in the limited region bounded by C (which we shall call the interior of C), if $f(z)$ is continuous in the corresponding closed region, and if the jth derivative of $f(z)$ is also continuous in this closed region and satisfies a Lipschitz condition with exponent α on C:

$$\left| f^{(j)}(z_1) - f^{(j)}(z_2) \right| \leqq M \left| z_1 - z_2 \right|^{\alpha},$$

z_1, z_2 on C. The number α will be positive and not greater than unity. The number j will be a positive integer or zero; we define $f^{(0)}(z)$ to be identically $f(z)$. The object of this note is to establish the following existence theorem.

THEOREM. *Let the point set S consist of a finite number of closed limited Jordan regions of the z plane bounded by the mutually exterior analytic curves* $C_1, C_2, \cdots, C_\lambda$. *Let the functions* $f_1(z), f_2(z), \cdots, f_\lambda(z)$ *belong respectively to the classes*

$$\text{Lip } (C_1, j, \alpha), \qquad \text{Lip } (C_2, j, \alpha), \qquad \cdots, \qquad \text{Lip } (C_\lambda, j, \alpha),$$

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