A POLAR REPRESENTATION OF SINGULAR MATRICES

BY JOHN WILLIAMSON

Let $A=(a_{ij})$, $(i=1, 2, \cdots, m; j=1, 2, \cdots, n)$, be a matrix of m rows and n columns, whose elements a_{ij} are complex numbers. It has been shown† that, if $m=n$ and A is non-singular, $A=P_1U=UP_2$, where U is a unitary matrix, while P_1 and P_2 are positive definite hermitian matrices. Moreover in such a polar representation of A, as it has been called, the matrices P_1, P_2, and U are uniquely determined. We shall show that, if $m=n$ and the rank of A is $r<n$, $A=P_1U=UP_2$, where P_1 and P_2 are uniquely determined positive hermitian matrices of rank r and U is unitary but no longer unique. Any such representation of course is impossible if $m\neq n$, as by definition both hermitian and unitary matrices are square, but it will be shown that somewhat analogous results exist in this case as well.

As is customary we shall denote the conjugate transposed of A by $A^*=(a_{ij}^*)$, where $a_{ij}^*=\bar{a}_{ji}$, the complex conjugate of a_{ji}. We shall use this notation, even if A is a vector, that is, a matrix of one row, so that in this case AA^* will simply denote the norm of the vector A. For the sake of brevity we shall use the notations E_j for the unit matrix of order j and $0_{i,j}$ for the zero matrix of i rows and j columns.

The matrix $N_1=AA^*$ is a square matrix of order m and the matrix $N_2=A^*A$ is a square matrix of order n, and since $N_1=N_1^*$ and $N_2=N_2^*$, both of these matrices are hermitian. Moreover, if the rank of A is r, the rank of N_1 is r and so is the rank of N_2. For, if K is the rth compound‡ of A, at least one element k_{ij} of K is different from zero. The element in the ith place of the leading diagonal of the product matrix KK^* is $\sum_t k_{it}\bar{k}_{it}$, which is a positive real number, since k_{ij} is not zero. Accordingly there is at least one r-rowed determinant of N_1

† L. Autonne, Bulletin de la Société Mathématique, vol. 30 (1902), pp. 121–134. A. Wintner and F. D. Murnaghan, *On a polar representation of non-singular matrices*, Proceedings of the National Academy of Sciences, vol. 17 (1931), pp. 676–678.

‡ Turnbull and Aitken, *The Theory of Canonical Matrices*, p. 27.

which is not zero, so that the rank of N_1 is at least r. Since the rank of N_1 cannot exceed the rank r of A, the rank of N_1 is exactly r. Similarly the rank of N_2 is r.

Since N_1 is a hermitian matrix of rank r, there exists an m-rowed unitary matrix X such that†

$$(1) \qquad XN_1X^* = D = \begin{pmatrix} D_{11} & 0_{r,m-r} \\ 0_{m-r,r} & 0_{m-r,m-r} \end{pmatrix},$$

where D_{11} is a diagonal matrix of order r. If $B = XA$, $B^* = A^*X^*$, so that

$$(2) \qquad BB^* = XAA^*X^* = XN_1X^* = D.$$

If we denote the row vectors of B by b_i, $(i = 1, 2, \cdots, m)$, the column vectors of B^* are b_i^*, $(i = 1, 2, \cdots, m)$, and the element in the ith row and jth column of BB^* is $b_ib_j^*$. It therefore follows from (1) and (2) that $b_ib_i^* = 0$, $(i = r+1, \cdots, m)$. Hence B is a matrix whose last $m-r$ rows are zero so that

$$(3) \qquad B = \begin{pmatrix} B_1 \\ 0_{m-r,n} \end{pmatrix},$$

where B_1 is a matrix of r rows and n columns. By a similar argument applied to N_2 instead of to N_1, it can be shown that there exists an n-rowed unitary matrix Y such that

$$(4) \qquad AY = (B_2 \quad 0_{m,n-r}),$$

where B_2 is a matrix of m rows and r columns. From (3) and (4) we deduce that

$$(5) \qquad XAY = C = \begin{pmatrix} C_{11} & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix},$$

where C_{11} is an r-rowed square matrix, which is non-singular since the rank of A is r. Since $XN_1X^* = XAYY^*A^*X^* = CC^*$, it follows from (1) and (5) that

$$(6) \qquad C_{11}C_{11}^* = D_{11}.$$

Denoting by c_i, $(i = 1, 2, \cdots, r)$, the row vectors of C_{11}, and by d_i the element in the ith place of the diagonal matrix D_{11}, we

† Turnbull and Aitken, op. cit., p. 85.

deduce the equalities $c_i c_j^* = 0$, $(i \neq j)$, $c_i c_i^* = d_i$, $(i, j = 1, 2, \cdots, r)$. Hence d_i is a positive real number and the vector $c_i / d_i^{1/2}$ is a normalized vector. Consequently the matrix

$$V_{11} = \begin{bmatrix} d_1^{-1/2} & 0 & \cdots & 0 \\ 0 & d_2^{-1/2} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & d_r^{-1/2} \end{bmatrix} C_{11} = (D_{11})^{-1/2} C_{11}$$

is unitary and

$$C_{11} = Q_{11} V_{11}, \tag{7}$$

where V_{11} is a unitary matrix of order r and $Q_{11} = (D_{11})^{1/2}$ is a positive definite hermitian matrix of order r. Using the value of C given by (5), we have, from (7),

$$\begin{aligned} C &= \begin{pmatrix} Q_{11} V_{11} & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix} \\ &= \begin{pmatrix} Q_{11} & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix} \begin{pmatrix} V_{11} & 0_{r,n-r} \\ 0_{n-r,r} & E_{n-r} \end{pmatrix}. \end{aligned} \tag{8}$$

Now if $m \leqq n$, this last result may be written in the form

$$C = (Q \quad 0_{m,n-m}) V, \tag{9}$$

where

$$Q = \begin{pmatrix} Q_{11} & 0_{r,m-r} \\ 0_{m-r} & 0_{m-r,m-r} \end{pmatrix}, \qquad V = \begin{pmatrix} V_{11} & 0_{r,n-r} \\ 0_{n-r,r} & E_{n-r} \end{pmatrix},$$

Q being a positive hermitian matrix of order m and rank r, while V is a unitary matrix of order n. Moreover, if

$$C = (Q_1 \quad 0_{m,n-m}) V_1 \tag{10}$$

is another such representation of C, where Q_1 is a positive hermitian matrix of order m and rank r and V_1 is a unitary matrix of order n, we see that $CC^* = Q_1 Q_1^* = Q_1^2 = D$, so that $Q_1 = D^{1/2} = Q$. Accordingly we may write (10) in the form

$$C = \begin{pmatrix} Q_{11} & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix} \begin{pmatrix} W_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix},$$

where W_{11} is an r-rowed square matrix, V_{22} an $(n-r)$-rowed square matrix, V_{12} an r by $n-r$ and V_{21} an $n-r$ by n matrix. Hence

$$C = \begin{pmatrix} Q_{11}W_{11} & Q_{11}V_{12} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix},$$

and comparison with (8) shows that $W_{11}=V_{11}$ and $V_{12}=0_{r,n-r}$. Since $V_1V_1^*=E_n$, it follows easily that $V_{21}=0_{n-r,r}$ and that $V_{22}V_{22}^*=E_{n-r}$. Hence the matrix V_1 in (10) is of the form VW, where

$$W = \begin{pmatrix} E_r & 0_{r,n-r} \\ 0_{n-r,r} & V_{22} \end{pmatrix}$$

and V_{22} is an arbitrary unitary matrix of order $n-r$. We have therefore proved the following lemma.

LEMMA. *The matrix C can be represented in the form $C=(Q\quad 0_{n-m})\,V_1$, where Q is a positive hermitian matrix of order m and rank r and V_1 is a unitary matrix of order n. The matrix Q is unique while the matrix V_1 is one of a set $[V_1]=[VW]$, where V is a fixed unitary matrix and W ranges over a group G of unitary matrices of order n simply isomorphic with the group of all unitary matrices of order $n-r$.*

Since $A=X^*CY^*$, by (9),

$$A = X^*(Q\qquad 0_{m,n-m})VY^* = X^*(Q\qquad 0_{m,n-m})X_1X_1^*VY^*,$$

where

$$X_1 = \begin{pmatrix} X & 0_{m,n-m} \\ 0_{n-m,m} & E_{n-m} \end{pmatrix}.$$

Hence

$$A = (X^*QX\qquad 0_{m,n-m})X_1^*VY^* = (P\qquad 0_{m,n-m})U,$$

where $P=X^*QX$ is a positive hermitian matrix of order m and rank r, while $U=X_1^*VY^*$ is a unitary matrix of order n. If $A=(P_1\ \ 0_{m,n-m})U_1$ is another such representation of A, it follows easily from the previous lemma that $XP_1X^*=Q$, so that $P_1=P$, and that $U_1=UZ$, where $Z=YWY^*$. Accordingly we have proved the following theorem.

THEOREM. *If A is a matrix of m rows and n columns of rank r and $m \leqq n$, A can be represented in the form*

$$A = (P_1 \quad 0_{m,n-m})U_1, \tag{11}$$

where P_1 is a positive hermitian matrix of order m and rank r and U_1 is a unitary matrix of order n. The matrix P_1 is unique while the matrix U_1 is one of a set $[U_1] = [UZ_1]$, where U is a fixed unitary matrix and Z_1 one of a group G_1 of matrices, simply isomorphic with the group of all unitary matrices of order $n-r$.

COROLLARY. *Under the above hypotheses the totality of unitary matrices Z_1 for which $AZ_1 = A$ forms a group simply isomorphic with the group of all unitary matrices of order $n-r$. This group is the group G_1.*

For if $AZ_1 = A$, $(P_1 \quad 0_{m,n-m})UZ_1 = (P_1 \quad 0_{m,n-m})U$, and Z^1 must lie in G_1. Similarly if Z_1 lies in G_1, $AZ_1 = A$.

If $m \geqq n$, A^* is a matrix in which the number of its rows is at most equal to the number of its columns. Accordingly, under this hypothesis our theorem is true if A is replaced by A^*. Hence, if $m \geqq n$, A can be represented in the form

$$A = U_2 \begin{pmatrix} P_2 \\ 0_{m-n,n} \end{pmatrix}, \tag{12}$$

where P_2 is a uniquely determined positive hermitian matrix of order n and rank r and U_2 is one of a set $[U_2] = [Z_2U]$, where Z_2 ranges over a group G_2 simply isomorphic with the group of all unitary matrices of order $m-r$.

When $m = n$, that is, when the matrix A is square, some further results follow. In this case equations (11) and (12) become

$$A = P_1U_1, \tag{13}$$

and

$$A = U_2P_2, \tag{14}$$

respectively, where P_1 and P_2 are positive hermitian matrices of rank r and U_1 and U_2 are unitary matrices. *The two groups G_1 and G_2 are simply isomorphic; the two sets $[U_1]$ and $[U_2]$ coincide; if U_1 belongs to the set $[U_1]$, $P_2 = U_1^*P_1U_1$.* The first statement is obviously true; if U_1 lies in $[U_1]$, $A = U_1U_1^*P_1U_1$ and, since $U_1^*P_1U_1$ is a positive hermitian matrix of rank r, $U_1^*P_1U_1 = P_2$ and U_1 lies in $[U_2]$. Similarly any member U_2 of $[U_2]$ lies in $[U_1]$. Further *the matrix P_2 is invariant under unitary transformation by any matrix of the group G_1, and P_1 under transformation by any matrix of the group G_2.* For if Z_1 lies in G_1, $AZ_1 = A$ so that $A = U_2Z_1Z_1^*P_2Z_1$, and accordingly, $Z_1^*P_2Z_1 = P_2$.

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ON A THEOREM OF FÉRAUD

BY D. C. LEWIS, JR.*

The Birkhoff-Pfaffian equations of dynamics are written in variational form as follows:

$$\delta \int \left[\sum_{i=1}^{2m} X_i \left(\frac{dx_i}{dt} \right) + Q \right] dt = 0,$$

where Q and the X's are functions of $x_1, \cdots, x_{2m}$ and, in general, depend also periodically upon t, and where the skew-symmetric determinant $|a_{ij}|$, $(a_{ij} = \partial X_i/\partial x_j - \partial X_j/\partial x_i)$, does not vanish in the regions considered. We restrict attention to the neighborhood of a generalized equilibrium point, that is, a point where all the $\partial Q/\partial x_i - \partial X_i/\partial t$ vanish identically in t. We take this point at the origin, $x_i = 0$, $(i = 1, 2, \cdots, 2m)$.

The problem of reducing the Pfaffian system to a Hamiltonian system can be reduced to that of finding a non-singular transformation, $x_i = x_i(y_1, \cdots, y_{2m})$, leaving the origin invariant (and depending in general periodically upon t) which reduces the linear differential form $\sum_{i=1}^{2m} X_i dx_i$ to the form $\sum_{i=1}^{m} y_{2i} dy_{2i-1} + dw$, where dw is an exact differential in $y_1, \cdots, y_{2m}$, the coefficients of which are independent of t. This same problem also will play an important role in a future paper of mine on "conservative" transformations in $2m$-dimensional spaces.

The problem has been considered by Féraud,† who obtained a

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† *Extension au cas d'un nombre quelconque de degrês de libertê d'une propriêtê relative aux systèmes Pfaffiens*, Comptes Rendus, vol. 190 (1930), pp. 358–360.