STONE ON HILBERT SPACE

Linear Transformations in Hilbert Space and their Applications to Analysis.
  By M. H. Stone. New York, American Mathematical Society, 1932. viii
  +622 pp.

The attitude of mathematicians towards the applications varies. Some feel
a need of rationalizing their personal interests by pointing out their importance
for the applications, and the applications know them not. Others assert with
pride that their results cannot possibly be applied, and fate whimsically har-
nesses their purest dreams to the chariot of industry. Whether we want our
results to be applied or not, we probably all agree that the applications have
a way of posing stimulating questions which spur the progress of our science.
Thus we probably owe more of our advance in analysis to the prying curiosity
of the physicist than to any other agent.

We recall how the problem of wave motion and heat conduction led to the
development of the function concept and the introduction of orthogonal series,
basic elements of present day analysis. Dirichlet's problem in potential theory
had a profound influence on the calculus of variations and led to Fredholm's
theory of integral equations. We know what happened to this theory in the
masterly hands of Hilbert, how it became a theory of orthogonal transforma-
tions and reduction of quadratic forms, how Hilbert revived the interest in
orthogonal functions, and created the atmosphere which stimulated, for ex-
ample, F. Riesz in his basic discoveries of different types of convergence in
function spaces. One is perhaps justified in regarding F. Riesz's *Systèmes
Linéaires d'une Infinité d'Inconnues* as the climax of the development of this
period, though its novel points of view really mark it off as the forerunner of
much of the modern theory.

It is only fair to admit that much of this development seemed of little in-
terest to the average physicist who, of course, knew that Dirichlet's problem
could be solved long before we actually solved it. But there was heavy weather
ahead for the physicists; the foundations of physics were taken for a ride, and a
bewildered physicist had to reorient himself in a strange world, perhaps not
created by a mathematician, but at least one where the mathematician played
the role of the little tin god on wheels. Space and time became differential
geometry and tensor analysis. Energy and matter disappeared in a cloud of
operators, abstract algebra, probabilities, and boundary value problems.

The analytical problems of quantum and wave mechanics can be thought
of in terms of linear transformations in a space of infinitely many dimensions
with certain specified properties, the abstract Hilbert space. To Hilbert and
his pupils we owe a theory of completely continuous and of bounded trans-
formations of this space. An excellent account of this theory is to be found in
the Encyclopädie article of Hellinger and Toeplitz. The modern theory of
unbounded transformations is connected chiefly with the names of T. Carle-
man, J. von Neumann, F. Riesz, M. H. Stone, and A. Wintner. Carleman's
work on integral equations started in 1916; a powerful analytical technique
enabled him to push this theory far beyond the bounds reached by the Hilbert

school. His older work is collected in his *Équations Intégrales Singulières* [Upsala, 1923]; his later results have been presented in outline in a Poincaré Institute lecture and in congress addresses. If Carleman be regarded as the heir of the Fredholm tradition, von Neumann's leaning towards algebraic methods and abstract formulations makes him a worthy successor of Hilbert. His investigations in this field, started in 1927, culminated in his book *Mathematische Grundlagen der Quantenmechanik* (Springer, 1932). Wintner also leans towards the Hilbert tradition as formulated by Toeplitz; his investigations are collected in his book *Spektraltheorie der Unendlichen Matrizen* (Hirzel, 1929). F. Riesz, finally, has shown how the simple and elegant methods which he employed in the bounded case can be used also for unbounded transformations [Acta Szeged, vol. 5 (1930)].

The first results of Stone's investigations appeared in three notes in the Proceedings of the National Academy of Sciences in 1929–30, of which the third one contained the spectral resolution of unitary groups of transformations in Hilbert space. The present book is an outgrowth of these preliminary communications. The author has set himself the task of giving a thorough and systematic exposition of the main features of the theory, based on his own investigations, but also containing the principal results of Carleman and von Neumann. Let it be said at once that he has accomplished this task in an excellent manner. It is only to be regretted that considerations of space forced him to omit all group theoretical questions.

It is difficult for an outsider to ascribe credit where credit is due. A sizeable portion of the material in Stone's book is of course introductory or common property. Other portions are avowedly based upon outside sources, but the material is so well assimilated, digested, and developed that complete unity is preserved. Finally the author's own investigations appear to be the main source for large portions of the book, especially in chapters five to seven and in the middle of chapter ten. The author has evidently tried to be strictly fair in his references to the work of other writers; in some places one feels, however, that the theorems have been improved beyond recognition. That the author is indebted in various ways to the writings of von Neumann is obvious, and he makes generous acknowledgment of the fact. It is also obvious that he has penetrated the difficult writings of Carleman, and has shouldered the important task of bringing Carleman's theory in line with the later development. This is particularly desirable since Carleman's work has not always received adequate recognition by the Hilbert school.

In what follows, we shall outline the contents of Stone's treatise.

The first chapter gives the definition of the abstract Hilbert space and its main properties, as well as various realizations of the space. This space $\mathfrak{H}$ is characterized by five properties: (i) it is a linear vector space, (ii) to any pair of elements f and g corresponds a complex-valued functional (f, g), with certain properties, which defines the metric in $\mathfrak{H}$, the norm of f being $\|f\| = (f, f)^{1/2}$, (iii) for every n there exist n linearly independent elements, (iv) $\mathfrak{H}$ is separable, and (v) complete.

Transformations are introduced in Chapter 2. A careful study is made of the notions of domain, range, extension, inverse, closure, continuity, linearity, and adjointness in connection with transformations. T_1 is adjoint to T_2, in symbols $T_1 \wedge T_2$, if $(T_1 f, g) = (f, T_2 g)$ in the domains of the transformations.

To every T corresponds a unique adjoint T^* such that any T_1 with $T_1 \wedge T$ admits of T^* as an extension. The author calls a transformation H symmetric [Hermitian in usual terminology] if $H \wedge H$ and the smallest linear manifold containing the domain of H is $\mathfrak{H}$ itself. A transformation H is maximal if it admits of no proper symmetric extension, essentially self-adjoint if $H^* = (H^*)^*$, and self-adjoint if $H = H^*$. The rest of the chapter deals with bounded transformations $[\|Tf\| < C\|f\|]$, projections [bounded, maximal symmetric transformations E such that $E \cdot E = E$], isometric transformations $[(Uf, Ug) = (f, g)]$, and unitary transformations [isometric with range and domain $\mathfrak{H}$]. An isometric non-unitary transformation has a deficiency-index (m, n), m and n being the dimension numbers of the manifolds orthogonal to the domain and range of Uf, respectively.

Chapter 3 illustrates these concepts by examples from the theory of infinite matrices, integral operators including Fourier and Hankel transforms, and differential operators.

In Chapter 4 the main problem is the study of the inverse of the transformation $T_l \equiv T - lI$, where l is a complex number. This leads to the spectrum [point, continuous, and residual spectra in general] and to the resolvent R_l of T. Particular reference is made to the symmetric case.

The detailed study of self-adjoint transformations is taken up in Chapter 5, based upon the earlier publications of the author. The gist of the method goes back to Stieltjes and Carleman, but notation and terminology agree essentially with those of von Neumann. To a self-adjoint H corresponds a family of projections $E(\lambda)$, known as a resolution of the identity such that

$$(R_l f, g) = \int_{-\infty}^{\infty} (\lambda - l)^{-1} d(E(\lambda)f, g).$$

The domain of H consists of those and only of those elements of $\mathfrak{H}$ for which $\int_{-\infty}^{\infty} \lambda^2 d \| E(\lambda)f \|^2 < \infty$, and for such f's and arbitrary g's

$$(Hf, g) = \int_{-\infty}^{\infty} \lambda d(E(\lambda)f, g).$$

The points of discontinuity of $E(\lambda)$ furnish the point spectrum, while points of continuity not interior to intervals of constancy give the continuous spectrum. There is no residual spectrum.

Chapter 6 deals with operational calculus. The basic problem is to give a sense to the notion of a function of a transformation. Let H be self-adjoint, $E(\lambda)$ its resolution of the identity, and $F(\lambda)$ a complex-valued function of a real variable; then we define $F(H)$ by

$$(F(H)f, g) = \int_{-\infty}^{\infty} F(\lambda) d(E(\lambda)f, g),$$

where the integral is of the Radon-Stieltjes type, and the domain of the transformation $F(H)$ has to be determined.

On the basis of this operational calculus the author solves in Chapter 7 the difficult problem of unitary equivalence of two self-adjoint transformations H_1 and H_2, that is, the question of the existence of a unitary transformation U such that $H_1 = UH_2U^{-1}$. For the bounded case this problem was solved by

Hellinger and Hahn. The solution leads to a complete characterization of the various possible types of self-adjoint transformations.

In Chapter 8, the basic idea is that of permutability of two transformations which leads to various complications in the unbounded case. It contains further a study of unitary and normal $[T \cdot T^* = T^* \cdot T]$ transformations, the theory of which is closely allied to the self-adjoint case.

The general symmetric transformations are taken up in Chapter 9. With a symmetric H the author, following von Neumann, associates an isometric transformation V such that $H \equiv i(I+V)(I-V)^{-1}$, and assigns to H the same deficiency-index as V has. This gives a basis for the classification. It is followed by a theory of formally real transformations, and an abstract formulation of the approximation theorems of Carleman with additions and elaborations.

Chapter 10 is devoted to applications. It occupies 218 pages, more than one third of the whole book, and still it is not long enough. The author starts with the theory of unbounded integral operators, that is, essentially the theory of integral equations with kernels of the Carleman type, which is a special case and also a forerunner of the general theory. This is followed by the theory of differential operators in which the author gives a very thorough and systematic discussion of differential systems of the first and second order. These operators provide fairly simple examples of symmetric transformations with deficiency-index (1, 1) or (2, 2). The author has here carried investigations initiated by Weyl and Carleman to an elegant finish. The last topic treated is the theory of Jacobi matrices which includes the moment problem for infinite intervals and the theory of continued fractions. Here we encounter the results of Carleman, Hamburger, R. Nevanlinna, and M. Riesz in new clothing. There are many other applications one should have liked to have included here. The important applications to the ergodic hypothesis [the long wished-for child which had so many fathers!] and to non-linear differential equations in general, obviously could not be included since the basic papers appeared while the book was in press.

To sum up, Professor Stone has done us a real service by providing a clear, systematic, and readable account of this important theory which is still in statu nascendi. The book is not always easy reading, but the author is fair to the reader: nothing essential is withheld, the terminology is clearly defined and strictly adhered to, there are enough facts provided for the imagination to feed on, but no loose statements.* We hope that the author will give us a second volume with further applications and aspects of this fascinating theory before long.

EINAR HILLE

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* According to the author there is at least one mistake in the book: on p. 229, line 4 from the bottom, the reference should read "Theorem 4.18" instead of "Theorem 2.18." The reviewer has not found any errors.