SOME RECENT CONTRIBUTIONS TO ALGEBRAIC GEOMETRY*

BY VIRGIL SNYDER

1. *Introduction.* During the last decade mathematical literature has been enriched by over a thousand contributions to algebraic geometry, including about eight hundred to the narrower field of rational transformations. It would be an ambitious task to report on this immense field in one short talk. Today I wish to speak of only three problems, each one somewhat well defined, or even narrow. Of these, one furnishes a striking example of mathematical elegance in providing one solution to what was regarded as several distinct problems. The other two are capable of unlimited extension, each new enlarged field furnishing phases not existing in the earlier ones.

2. *Series of Composition of Veneroni Transformations.* The Cremona transformations determined by a system of bilinear equations between the systems of coordinates of the two associated spaces were among the first to be considered in S_2 and S_3. For the general case the configuration of fundamental and principal elements can be at once expressed in terms of the vanishing of certain determinants of a matrix. (Segre.†) This has recently been generalized to spaces of higher dimensions by various authors, in particular to S_n by Godeaux.‡ Numerous particular cases have been considered. If $x_ix_i' = x_kx_k'$, the extreme case of inversion results. Transformations made up of this inversion and of collineations have no fundamental curves of the first kind; they have a series of composition somewhat similar to that of the general case for S_2.

If in every bilinear equation the coefficients a_{ik} and a_{ki} are equal, the transformation is involutorial and can be expressed in terms of polarity as to a series of quadrics or as to null systems. In case all the polarities are quadric and the quadric primals are independent, there are only a finite number of invariant

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† C. Segre, Rendiconti dei Lincei, (2), vol. 9 (1900), pp. 253–260.

‡ L. Godeaux, Lombardo Istituto Rendiconti, (2), vol. 43 (1910), pp. 116–119.

points. These have been discussed by M. del Re.* The case of $n=4$ has been investigated by Miss Alderton,† and various further properties have been given by Wong‡ and Roth.§

The general Cremona transformation for S_r, $r>2$, does not have a series of composition, that is, it cannot be expressed as the product of transformations of lower order.

A Cremona transformation in S_3 of order $n=4k+r$ having as base curve of maximal multiplicity a sextic curve C of genus $p=3$ and of multiplicity $k+1$, is the product of a (3, 3) and of another transformation of order n. (Piazzolla-Beloch.‖)

Similarly, a $T_{2n-1}: C_3^{(n-1)}$, the only other base-system consisting of bisecants of C_3, has a definite series of composition. (Montesano.¶) This case has been developed further by Tinto.**

3. *Case* $n=3$. The most important particular case from this point of view for $n=3$ is that in which the base C_6 is composed of four skew lines r_i and their two transversals u, v. The inverse is defined by cubic surfaces through four skew lines r_i' and their two transversals u', v'. A point on r_i has for image a line meeting three lines r_k', so that r_i itself is transformed into the quadric through three lines r_k'. A point on u (or v) has the whole line u' (or v') for image, and conversely. These transversals are simple fundamental lines of the second kind.

The image of an arbitrary line is a space cubic meeting each r_i' twice, but not meeting u' or v'. A line r meeting u and v is transformed into a line meeting u' and v'. The (1, 1) correspondence between this line and its image is a collineation, such

* Maria del Re, Naples Rendiconti, (3), vol. 28 (1922), pp. 203–211, and (3), vol. 35 (1929), pp. 208–217.

† Nina Alderton, University of California Publications in Mathematics, vol. 1 (1923), pp. 345–358.

‡ B. C. Wong, Annals of Mathematics, (2), vol. 27 (1926), pp. 330–336, and vol. 28 (1927), pp. 251–262; American Journal of Mathematics, vol. 49 (1927), pp. 383–388; this Bulletin, vol. 35 (1929), pp. 829–832.

§ L. Roth, Proceedings of the Cambridge Philosophical Society, vol. 29 (1933), pp. 178–183.

‖ M. Piazzolla-Beloch, Annali di Matematica, (3), vol. 16 (1909), pp. 27–98.

¶ D. Montesano, Naples Rendiconti, (3), vol. 27 (1921), pp. 116–127 and 164–175.

** F. J. Tinto, Proceedings of the Edinburgh Mathematical Society, vol. 34 (1916), pp. 133–145.

that the points r, u and r, v are transformed into r', u' and r', v'. Through a point P in (x) passes a single line meeting u, v; through its image P' passes a single line meeting u', v'. The collineation is now uniquely fixed. Let u be taken as $x_1=0$, $x_2=0$ and v as $x_3=0$, $x_4=0$. For r_1, r_2 the lines $x_1=0$, $x_3=0$ and $x_2=0$, $x_4=0$ may be taken. Then for r_3 take $ax_1+bx_2=0$, $cx_3+dx_4=0$ and r_4' is $a'x_1+b'x_2=0$, $c'x_3+d'x_4=0$.

Since the quadric surface through any three lines r_i contains the two transversals, a quadric and any plane of the pencil through the residual line will be a cubic surface of the homaloidal system. If $H_{13}=0$ is the equation of the quadric through all the lines except $x_1=0$, $x_3=0$, etc., we may write

$$x_1 = x_1H_{13}, \quad x_2' = x_2H_{24}, \quad x_3' = x_3H_{13}, \quad x_4' = x_4H_{24}.$$

Similar pencils are of the forms

$$(ax_1 + bx_2)H_{a,c} + (cx_3 + dx_4)H_{a,c} = 0,$$
$$(a'x_1 + b'x_2)H_{a',c'} + \text{etc.} = 0.$$

By writing the equations of H_{13}, H_{24} we can now obtain the forms of the defining bilinear equations

$$x_1'x_3 - x_1x_3' = 0, \qquad x_2'x_4 - x_2x_4' = 0,$$
$$(cd')(ax_1 + bx_2)(b'x_3' - a'x_4')$$
$$- (ab')(d'x_1' - c'x_2')(cx_3 + dx_4) = 0.$$

4. *Products of* (3, 3) *Cremona Transformations.* Now consider the product of $T_1{:}r_iu, v; r_i'u', v'$ and $T_2{:}r_i'u', v'; r_i'', u'', v''$ in which the same lines u', v' appear in T_1^{-1} and in T_2. The lines r_i', r_i' may all be distinct or in part the same. Similarly for the continued product $T_1, T_2, \cdots, T_k$. For every value of k the transformation is determined by two lines u, v of the same multiplicity and of a number of lines, of various multiplicities, meeting both of them, and of no other base elements.

Conversely, consider a transformation T defined by the two properties:

(a) To a line r meeting two skew lines u, v shall correspond a line r' meeting two skew lines u', v'.

(b) The projectivity established between r and r' shall never be degenerate, and the point r, u shall correspond to the point r', u'; similarly for r, v and r', v'.

The transformations determined by these conditions have u, v to the same multiplicity ν; a number of lines meeting both are base elements of the web, and there can be no other base elements. The transformation is of order $2\nu+1$. We may then write

$$|S_2| \sim |\phi_{2\nu}\,\phi_{2\nu+1}| : u^{\nu} v^{\nu} r_1^{\rho_1} \cdots r_p^{\rho_p},$$
$$|S_1| \sim |C_{2\nu}\,C_{2\nu+1}| : r_1^{2\rho_1} \cdots r_p^{2\rho_p}.$$

A plane through u (or v) is transformed into a ruled surface $R_{2\nu+1}: u'^{\nu}\, v'^{\nu+1} r^{2\rho_1} \cdots r_p^{2\rho_p}$.

Each line r_i is transformed into a ruled surface; each point of $r_i \sim$ a rational curve o' of order ρ_i', not meeting u', v', which generates $R_2\rho_i: u'^{\rho_i'}\, v'^{\rho_i'}$.

An arbitrary plane ω meets u, v each in a point and contains the line l joining them. Let ω' be any plane (not containing u', v') through the image of this line. Given any point T in ω, through it passes a unique line c meeting u, v. The image line c' meets u', v' and pierces ω' in T'. Between ω, ω' exists a plane Cremona transformation. Let T describe a line p in ω. The lines u, v, p determine a quadric surface, the image of which is a surface of order $4\nu+2$. The section by ω' contains l' and the image of p, hence

$$c_1 \sim c'_{4\nu+1}: U'^{2\nu+1} V'^{2\nu+1} \sum_{i=1}^{p'} R_i^{\rho_i'}.$$

Since in a plane Cremona transformation the number of F points in the two planes is the same, it follows that $p = p'$.

From the two fundamental relations of a plane Cremona transformation, we have further $\sum \rho_i = 4$, $\sum \rho_i^2 = 2\nu(\nu+1)$. Let $\rho_1 \geqq \rho_2 \geqq \cdots \geqq \rho_p$. Then

$$\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4(\rho_4 + \cdots + \rho_p) \geqq 2\nu(\nu+1),$$
$$\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4(4\nu - \rho_1 - \rho_2 - \rho_3) \geqq 2\nu(\nu+1),$$
$$\nu(\rho_1 - \rho_4) + \nu(\rho_2 - \rho_4) + \nu(\rho_3 - \rho_4) + 4\nu\rho_4 \geqq 2\nu(\nu+1),$$
$$\rho_1 + \rho_2 + \rho_3 + \rho_4 \geqq 2(\nu+1).$$

From this last equation it follows that if the web of surfaces, homaloids of the planes of space, is transformed by means of the (3, 3) transformation defined by r_1, r_2, r_3, r_4, the order of the surfaces of the web will be lowered. By continuing this process it follows that every transformation of this kind can be expressed as the product of (3, 3) transformations defined by four skew base lines. (Montesano.*) The complete table of characteristics can now be constructed, and all the details of incidence determined exactly as for a plane Cremona transformation.

5. *Case* $n=4$. In S_4, consider five planes π_i, no two lying in an S_3. The system $|\phi_3^4|$ of quartic primals passing through them form a system ∞^4 of homaloids. The base M_2^{10} consists of the five planes π_i and of a ruled surface R_5 formed by the ∞^1 lines which meet all five planes. Lines meeting any four of the planes π_i are all P lines. They generate a Segre 10 nodal cubic primal V_3^3, $\psi_i:4\pi_k$, not π_i. This cubic primal and any S_3 through π_i together form a quartic primal of the homaloidal system. The equations of the associated Cremona transformation therefore become $x_i' = \psi_i S_{3,i}:\pi_i$. Each ψ_i contains ∞^2 lines, images of the points of π_i. The inverse system is of the same form as the given one. Any two of the base planes, as π_1, π_2, meet in a point π_{12}. The plane through π_{12}, π_{23}, π_{31} meets π_1, π_2, π_3 each in a line lying on ψ_4 and ψ_3. This plane is transformed into the point P'_{45} common to π_4', π_5'; it is a double point on ψ_1', ψ_2', ψ_3'.

The primals ψ_4, ψ_5 intersect in π_1, π_2, π_3 and the plane π_{45} determined by π_{12}, π_{23}, π_{31}, and R_5. A section by any S_3 consists of three skew lines, a line meeting all of them, and of a quintic curve which is therefore elliptic. *The base R_5 is an elliptic ruled surface.* Every generator of R_5 is a base line and a P line having a line in (x') for image in such a way that any point on either line has the whole other line for image, $R_5 \sim R_5'$.

Two primals ϕ of the homaloidal system intersect in a surface of order 16, consisting of $5\pi_i$, R_5, and a variable surface f of order 6. A section of f by an S_3 is a sextic curve C_1 of $p=3$, meeting R_5 in 5 points. The surface f meets each π_i in an elliptic cubic curve, and R_5 in 5 generators. Three primals ϕ not belonging to the same pencil meet in a normal C_4 which has three points in each base π_i. Since a line in (x) is uniquely fixed by two points on it, it follows that a unique C_4 can be passed through two points and have five given planes of S_4 for trisecants. (Veneroni,† White,‡ Todd.§) Consider the ∞^6 lines of

* D. Montesano, Naples Atti, (2), vol. 18, (1930), 44 pp.

† E. Veneroni, Lombardo Istituto Rendiconti, vol. 34 (1901), pp. 640–644.

‡ F. P. White, Proceedings of the Cambridge Philosophical Society, vol. 21 (1922), pp. 216–227.

§ A. J. Todd, ibid., vol. 26 (1930), pp. 323–333.

S_4; ∞^6 $|C_4|$ have 5 given planes for trisecants. Of these, ∞^3 pass through a given point, hence a finite number have a sixth plane for trisecant, images of lines meeting f (image of the sixth plane) in three points. This number is one (White*); hence a normal C_4 can be passed through a given point, having six given planes for trisecants. It was seen that R_5 is cut by S_5 in a C_5, $p=1$; hence an S_3 through a generator g of R_5 will contain an elliptic C_4, and an S_3 through two generators meets R_5 in an elliptic cubic, hence a plane curve. There are ∞^2 pairs of generators on R_5, but if π is a plane containing a cubic curve γ_3 on R_5, through π can be passed $\infty^1|S_3|$, each containing a pair of generators of R_5; hence R_5 *contains* ∞^1 *plane cubic curves* γ_3. The base planes π_i are among the planes π containing curves γ_3. Any plane π meeting R_5 in γ_3 is transformed into a plane π', since through every point of γ_3 passes a generator g of R_5; hence R_5' is a component of the image f' of this plane.

Let π, π' be a pair of associated planes in $T_{4,4}$. Each contains a cubic curve γ_3, γ_3' and these curves are projectively equivalent. A line l in π meets γ_3 in three points, hence its image C_4' consists of three generators of R_5' and of a line l' in π'.

The (1, 1) *correspondence between the points of two associated planes* π, π' *is a collineation.*

Now consider the homaloidal system defined by $\pi_1 \cdots \pi_5 R_5$ in (x), and its inverse $\pi_1' \cdots \pi_5' R_5'$ in (x') and a second one in (x') having R_5' as before, but a system of transversal planes $\bar{\pi}_i$, which may be distinct from π_i' or partly identical, and its inverse $\pi_1'' \cdots \pi_5'' R_5''$ in (x''). By means of the product of the associated Cremona transformations, a plane π meeting R_5 in γ_3 is transformed into a plane π'' meeting R_5'' in a γ_3'', and the (1, 1) correspondence between these planes is a collineation. Similarly for any number of components. The resulting product will always be of the form

$$T \qquad S_3 \sim \phi_{3\nu+1} : R_5^{\nu} \pi_1^{\rho_1} \cdots \pi_p^{\rho_p},$$

and the inverse will be of the same form.

Now the question arises whether every Cremona transformation of this form in S_4 can be expressed as the product of $T_{4,4}$ of the type just considered. An arbitrary S_3 meets R_5 in C_5, $p=1$; the trisecants of C_5 form a ruled surface σ_5 of order 5,

* F. P. White, loc. cit.

having C_5 for complete double curve. The image of each of these generating trisecants is a line, and these image lines describe a ruled surface σ_5' having a double curve C_5' in S_3' (not the image of S_3 in $T_{4,4}$). A (1, 1) correspondence between the points of S_3, S_3' can be established as follows: from any point P in S_3 can be drawn one and only one trisecant of R_5. (White.*) If P is not on σ_5, its associated trisecant t is not in S_3. The image t' of t meets S' in P'. Let P describe a line p in S_3. It will meet 5 lines of σ_5. Construct the ruled surface of lines t for each position of P on p, get its transform in T, then cut the transform by S_3'. The section will consist of the image of p, and of five generators of σ_5'.

Since p is a simple directrix line on the ruled surface of trisecants, its order is one greater than the number of generators in any S_3 through p. Since any S_3 through p meets R_5 in C_5 and contains a surface σ_5 of trisecants, the surface is of order 6. Its image in T is therefore of order $6(3\nu+1)$ and in S_3 the image of p is a curve of order $18\nu+1$, containing C_5' to multiplicity 6ν and lines $\omega_1, \cdots, \omega_p$ to multiplicities $6\rho_i$:

$$p \sim C_{18\nu+1}: C_5'^{6}\omega_1^{\rho_1'} \cdots \omega_{p'}^{\rho'_{p'}}.$$

For $\nu=1$ this is a particular case of the transformation belonging to a tetrahedral complex. The C_{10} of $p=11$ of the linear complex is replaced by C_5, $p=1$, and 5 of its trisecants. (Sharpe and Snyder.†)

By repeating the argument already used, and using the relations analogous to those used by Montesano,‡ the condition $\rho_1+\cdots+\rho_5 \geqq 3(\nu+1)$ can be established; hence by using the $T_{4,4}$ determined by those π_i of highest multiplicity, the order of the primals of the homaloidal net can be reduced. The transformation T can be expressed in terms of $T_{4,4}, \cdots$.

An S_3 meets ϕ_i, ϕ_k of the system in quartic surfaces through C_5 and 5 of its trisecants. The residual intersection is C_6, $p=3$, meeting C_5 in 5 points and the 5 trisecants each in three. Hence the surface f thus meets each of the five plane π_i in cubic curves, and R_5 in a composite quintic curve consisting of 5 generators.

* F. P. White, loc. cit.

† F. R. Sharpe and V. Snyder, Transactions of this Society, vol. 21 (1920), pp. 52–78.

‡ D. Montesano, Naples Rendiconti, loc. cit.

Two surfaces f meet in 36 points. The point π_{ij} is double for each primal, hence quadruple on their surface of intersection. Of these, three are accounted for by the fact that π_{ij} lies on π_i, π_j, R_5; hence f passes through π_{ij} simply.

The cubic curve in which f_6 meets π_i passes through the four points π_{ij} which lie in this plane, so that two such cubics have five intersections at points not coinciding with π_{ij}. Two f's, then, have 10 π_{ij}, 5 other points in each of the planes π_i, and one variable intersection.

A plane ω meets R_5 in 5 points, and meets each ψ_i in a cubic curve c_3 belonging to the pencil $4\pi_{ik}5\Gamma$. The image f_6' of ω contains 5 lines g' of R_5' and meets π_i' (image of ψ_i) in a cubic curve c_3', in (1, 1) correspondence with c_3. The lines g' pierce π_i' in 5 points Γ', all on c_3'. But these points also lie on γ_3', section of R_5' by π_i'. Through the five lines $g_1', \cdots, g_5'$, in which f_6' meets R_5', pass ∞^1 surfaces f_6'.

The five points G' in which any f_6' meets γ_3' in π_i' apart from the points π_{ik} are base points of a pencil of cubic curves. Let ω meet R_5 in 5 points G. Through each point G passes a generator g_i of R_5. These lines Gg_i are met by ∞^1 planes ω. Two such planes have for images surfaces f_6' having 10 points π_{ik} and 5 lines g' in common.

6. *Case* $n=5$. Let $\sigma_1, \cdots, \sigma_6$ be six three-spaces in [5]. They determine a homaloidal system of order 5. Any five of the σ_i, as $\sigma_1, \cdots, \sigma_5$, determine a quartic primal ψ_6, and similarly for five others. These are images of the base system σ_i' in another [5]. Consider $\sigma_1, \cdots, \sigma_4$ common to ψ_5, ψ_6. Any two of them intersect in a line, thus defining six lines l_{ik}, three skew lines in each of the spaces $\sigma_1, \cdots, \sigma_4$. The lines are all double lines on the primals of the system; the [3] determined by σ_{12}, σ_{34} lies entirely on ψ_5, ψ_6 and similarly for the other two [3] determined respectively by σ_{13}, σ_{24}; σ_{14}, σ_{23}. The section of ψ_5, ψ_6 consists of the four spaces $\sigma_1, \cdots, \sigma_4$, the three transversals just described, and a residual ruled variety of order nine, common to all the homaloids of the system.

The base of the ∞^5 *homaloidal system is a three-way variety of order* 9, *locus of the* ∞^2 *lines that meet all six three-spaces of the bases* σ_i.

7. *Case n General.* This system is always included among those

determined by a system of bilinear equations; hence all the configurations are expressed by the vanishing of determinants of a matrix. We may write

$$\sum a_{ik}^{(r)} x_i x_k' = 0, \qquad (r = 1, \cdots, n).$$

If $\sum_{i=1}^{n+1} a_{i,k}^{(r)} x_i = A_k^{(r)}$, then $\sum A_k^{(r)} x_k' = 0$, and the birational transformation is expressed by $x_p' = \|A_k^{(r)}\|$, $(k \neq p)$, and similarly $x_p = \|A_k'^{(r)}\|$, $(k \neq p)$. The primals $\|A_k^{(r)}\|$ are of order n, and have in common a manifold M of dimensionality $n-2$ and of order $n(n+1)/2$, consisting of the $n+1$ spaces $[n-2]$ and of a ruled variety of order $(n+1)(n-2)/2$.

The formula given by Veneroni* and repeated in the Encyklopädie (vol. III, Chapter 7, p. 967) does not give the base, but the entire intersection. For further general properties, see M. Mikam.†

Any primal of order n through M belongs to the system. The transversals of the primals in pairs include $(n-2)(n-3)/2$ spaces analogous to l_{12}, l_{24}, etc.

An $[n-r]$ of $[n]$ defined by r n-primals of $[n]$ is transformed into a manifold of dimensionality $n-r$ of order ${}_nC_{n-r}$, having with M a manifold N of dimensionality $n-r-1$ and of order $(r(n+1)/(n-r)_nC_{n-r-1}$ including the $n+1$ intersections with the base. The lines which meet the composite M in n points generate a ruled primal of order n^2-1, the Jacobian of the Cremona transformation. No line meets M in more than n points. (Segre,‡ Godeaux.§)

8. *Related Questions.* A number of questions closely related to those here considered have been solved recently. Thus, the algebraic surfaces on a cubic primal of S_4 (Archbald||); octavic normal surfaces with a double line in [5] (Babbage¶); isolated singular points in the theory of algebraic surfaces (Babbage**); some chains of theorems derived by successive projection

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* Loc. cit.

† M. Mikam, Prag Ceske Akademie Rozpravy, vol. 39 (1929), No. 21.

‡ C. Segre, loc. cit.

§ L. Godeaux, loc. cit.

|| J. W. Archbald, Proceedings of the Cambridge Philosophical Society, vol. 27 (1931), pp. 405–406, and vol. 29 (1933), pp. 484–486.

¶ D. W. Babbage, ibid., vol. 29 (1933), pp. 95–102 and 405–406.

** D. W. Babbage, ibid., vol. 29 (1933), pp. 212–230.

(Lob*). Segre's theorem concerning curves on ruled surfaces has been considered and extended by Welchman.† A useful scheme for mapping spaces S_k of an S_m upon the points of a third of order $(m-k)(k+1)$ is given by Room,‡ and Babbage gives a helpful discussion of the transformations of certain surfaces having special singularities. Welchman§ has generalized certain incidence formulas to apply to incidence scrolls, ruled surfaces generated by certain directrix spaces. It would be well to extend this investigation to apply to varieties generated by planes, [3], by similar methods. (Welchman.‖)

A discussion of the forms and possible transformations of isolated singular points of algebraic surfaces, extending the work of Godeaux,¶ was made by Babbage.** Roth†† gives further examples of surfaces in hyperspace containing triple curves and obtains some further results concerning composite surfaces contained within a given system in higher spaces.‡‡

Six lines of S_3 can have various relations to each other discussed by Todd,§§ generalizing Ursell's‖‖ study of seven lines on a quartic surface, the results of Richmond,¶¶ and the earlier paper by Wakeford.***

A detailed analytic account of the ∞^{n-2} lines generating the V_{n-1} in $[n]$ is given by Eiesland.††† These manifolds are the

* H. Lob, Proceedings of the Cambridge Philosophical Society, vol. 29 (1933), pp. 45–51.

† W. G. Welchman, ibid., vol. 29 (1933), pp. 382–388.

‡ T. G. Room, ibid., vol. 29 (1933), pp. 331–336.

§ W. G. Welchman, ibid., vol. 29 (1933), pp. 235–254.

‖ W. G. Welchman, Proceedings of the Cambridge Philosophical Society, vol. 29 (1933), pp. 103–115.

¶ L. Godeaux, Bulletin de l'Académie Royale de Belgique, Classe des Sciences, (5), vol. 15 (1929), pp. 317–318.

** Loc. cit.

†† L. Roth, Proceedings of the London Mathematical Society, (2), vol. 30 (1929), pp. 118–126; Proceedings of the Cambridge Philosophical Society, vol. 28 (1932), pp. 300–310, and vol. 29 (1933), pp. 178–183.

‡‡ L. Roth, ibid., vol. 28 (1932), pp. 45–52, and vol. 29 (1933), pp. 88–94.

§§ J. A. Todd, ibid., vol. 29 (1933), pp. 52–68.

‖‖ H. D. Ursell, ibid., vol. 25 (1929), pp. 31–38.

¶¶ H. W. Richmond, Journal of the London Mathematical Society, vol. 7 (1932), pp. 113–117.

*** E. K. Wakeford, Proceedings of the London Mathematical Society, (2), vol. 21 (1933), pp. 89–113.

††† J. Eiesland, Palermo Rendiconti, vol. 54 (1930), pp. 335–365.

single $[n-2]$ of the base primes. He gives much more information concerning questions discussed by Wong.* Particular attention is given to $n=5$. Various particular cases are included. Another derivation of the equation of V_{n-1}^{n-1} is given by Rupp.†

9. *Transformations Defined by Projective Systems.* An important type of Cremona transformations, applicable to spaces of any number of dimensions and including a large number of cases in each, can be illustrated by the simplest case as follows.

Given four points A, B, C, D in the plane, and a line d in the plane, not passing through any of them. Make the conics of the pencil projective with the points of d. A point y fixes a conic and a point Z on d. The line yZ meets the conic belonging to Z in a point y'. The correspondence y, y' is an involutorial Cremona transformation of the type $c_1 \sim c_7 : 4^3 3^2$ in which the three double points are all on d. They are defined as those points Z which lie on the conics associated with them. When one of these conics is composite, the component through Z divides out, lowering the order of the transformation. In the plane, all of these are particular cases of the Geiser involution. Interesting variations occur when the four points approach each other in various ways. The projectivity may be replaced by a $(1, k)$ correspondence, k conics being associated with the same point Z.

In space, the simplest case is that of a pencil of quadrics, the base curve not meeting the line d. Among the possible types are those discussed by Montesano,‡ furnishing an illustration of a congruence of lines being transformed into itself. The pencil of quadrics may be any of the possible types, and the correspondence may be $(1, 1)$ or $(1, k)$. The interesting features are the appearance and behavior of parasitic lines.

The quadrics may be replaced by a pencil of surfaces of order n having d to multiplicity $n-2$. When the residual curve is nonreducible, these cases have all been considered by Carroll.§ The application to pencils having a composite residual base curve has thus far been limited to pencils of cubics having d as a simple base line.

* B. C. Wong, this Bulletin, vol. 34 (1928), pp. 829–832.

† C. A. Rupp, this Bulletin, vol. 35 (1929), pp. 319–320.

‡ D. Montesano, Giornale di Matematiche, vol. 31 (1893), pp. 36–50.

§ E. T. Carroll, American Journal of Mathematics, vol. 54 (1932), pp. 707–717.

The configuration of the homaloids along the various base lines includes an unexpectedly large number of variations. (Rusk.*) It would be well worth while to make the corresponding study of pencils of surfaces of larger order, and to extend it to pencils of primals in hyperspace.

The straight line d may be replaced by a rational curve. Here the possible number of types is limited, but new arrangements of fundamental elements appear. The case of an irreducible residual base curve has been completed, and that of composite curves is limited to the features characteristic of the composite case. (Black.†) Various other types have been considered by Davis,‡ in connection with Cremona transformations contained in a special or non-special linear line complex.

Isolated cases had been discussed earlier by Pieri and by Montesano. The latter proposed to classify involutions according to the product of the order of the complex to which each belongs and the number of pairs of conjugate points on each line. The involution belonging to the general cubic primal of S_4 belongs to a special linear complex, and each line has two pairs; it is probably irrational, but no case has as yet been proved definitely irrational. Illustrations defined by pairs of planes in a curiously simple form were given by Snyder.§ All of these cases are rational. (Dye.‖) Since every line of a surface of a pencil that passes through its associated point is a fundamental line of the second kind, a particularly instructive case is that in which every surface of the pencil is ruled. Through every point of the rational curve d pass a number of generators, hence the surface generated by them is a component of every homaloid. The base curves of the pencil and isolated generators form the entire system of fundamental curves. Along the directrix curves the surfaces have contact of maximum order, furnishing a prob-

* Evelyn T. Carroll ≡ (Mrs.) E. C. Rusk, American Journal of Mathematics, vol. 56 (1934), pp. 96–108.

† A. H. Black, Transactions of this Society, vol. 34 (1932), pp. 795–810; this Bulletin, vol. 40 (1934), pp. 417–420.

‡ H. A. Davis, American Journal of Mathematics, vol. 52 (1930), pp. 58–71, and vol. 53 (1931), p. 72; H. A. Davis and Tyr. Davis, Tôhoku Mathematical Journal, vol. 33 (1931), pp. 53–58.

§ V. Snyder, this Bulletin, vol. 36 (1930), pp. 89–93; Atti, Bologna Congress, vol. 4 (1930) [1928], pp. 13–21.

‖ L. A. Dye, Transactions of this Society, vol. 32 (1930), pp. 251–261.

lem in postulation and equivalence not accounted for by known formulas. Apart from a few isolated cases, all of these types have for directrix system a line and a rational curve of order m meeting it in $m-1$ points. (Snyder.*)

A non-linear null system defined by a cubic surface gives rise to a Cremona involution by associating each surface of a pencil of cubics with the points of a line, each surface passing through its associated point but not containing the line. (Snyder and Schoonmaker.†) Applications of the method to obtain certain space generalizations of the Geiser and Bertini plane involutions have been made by Dye and Sharpe.‡

10. *Double Fives*. B. Segre§ considers the problem: Is it possible to find a pyramid (simplex) which is simultaneously inscribed in and circumscribed to a quadric primal Q in S_4? Let the faces α_i touch Q in B_i, and four faces meet in A_i. If such a simplex is possible, then both pyramids $A_1 \cdots A_5$ and $B_1 \cdots B_5$ are both inscribed and circumscribed, and are polar conjugates as to Q. It is shown that pyramids do exist. A face (three space) α_5 cuts Q in a point cone, vertex at B_5, and containing the lines B_5A_i, $(i=1, 2, 3, 4)$. On the cone these four generators are equianharmonic. Two pairs of conjugate pyramids are projectively equivalent. One pair is transformed into the other by 120 collineations and by 120 dualities. Together these form a group under which Q remains invariant.

Five associated planes of S_4 (as defined by C. Segre‖) always admit a pair of conjugate pyramids, and consequently define the quadric Q. The five planes are the intersections of the five pairs of opposite faces. By polarity as to Q these are changed into five associated lines, related to the given planes. These lines are the joins of associated vertices of the two conjugate pyramids. If Q be projected stereographically from a point L upon it into an S_3 not passing through L, a fundamental conic γ results. The twenty lines that are projections of A_iB_k, $(i\neq k)$, are all unisecants of γ.

* V. Snyder, Transactions of this Society, vol. 35 (1933), pp. 341–347.

† V. Snyder and H. E. Schoonmaker, American Journal of Mathematics, vol. 54 (1932), pp. 299–304.

‡ L. A. Dye and F. R. Sharpe, Transactions of this Society, vol. 36 (1934), pp. 292–305.

§ B. Segre, Lincei Memorie, (6), vol. 2 (1928), pp. 204–209.

‖ C. Segre, Torino Memorie, (2), vol. 39 (1903), pp. 1–48.

The projections of the points A_i, B_i form a double five of points related to the conic. If γ is projected from any one of them, the resulting quadric cone contains four other points of the double five, and on the cone the generators containing them are always equianharmonic. An irreducible conic has ∞^{10} related double fives of points.

11. *Sets of Five Lines in* S_3. If now the well known mapping of the lines of S_3 on the points of an irreducible quadric primal of S_5 be employed, corresponding to double fives of points relative to the various hyperplane sections of Q, there exist configurations of lines in S_3 which have the following properties.

Let a_i, b_i be two sets of five lines each in S_3 such that (1) they all belong to the same non-special linear complex L, (2) that each line a_i is incident to four lines b_k, but not to b_i, and each line b_i is incident with all lines a_k except a_i. Such a configuration of lines is called a double five. The four points in which a_i meet b_k are equianharmonic.

Given a line b_5 in a non-special linear complex L, and four lines $a_1, \cdots, a_4$ of L meeting it, the reguli defined by these lines a_i, three at a time, contain one and only one line of L, apart from b_5. The necessary and sufficient condition that the four lines thus obtained have only one common transversal is that the set of lines a_i be equianharmonic.

A number of interesting properties follow from this simple construction of a double five of lines in S_3. This result was obtained in terms of lines of S_3 by B. Segre in an earlier paper.* The same result was obtained by Weitzenböck.† (See also Schaake.‡) The general properties of double fours and of double fives were given by Maurer.§

12. *Sets of Oriented Circles in the Plane.* Another interpretation can be given in terms of oriented circles in the plane, mapping each on a point of a general quadric Q primal of S_4, in such manner that those oriented circles which touch each other in the same point are mapped on the points of a line of Q. In a plane are ∞^{10} double fives of oriented circles, such that each circle is

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* B. Segre, Rendiconti dei Lincei, (6), vol. 2 (1925), pp. 539–542.

† R. Weitzenböck, Amsterdam Proceedings, vol. 31 (1928), pp. 133–137.

‡ G. Schaake, Amsterdam Proceedings, vol. 31 (1928), pp. 715–717.

§ A. Maurer, Bonn disseration, 1929.

touched by 4 circles, belonging to the quintuple to which the given circle does not belong. Upon the latter circle the four points of contact are equianharmonic. This last property was found by a different method by Barbilian,* and further discussed by Tzitzeica.† The relations between the two interpretations were shown by Weiss.‡ Part of these relations were obtained by Study,§ who also obtains a double five, but these are not equivalent to those here considered, as the complex containing the ten lines is so particularized that the S_4 represented by it in S_5 cuts Q in a quadric Q' having the property that all ten lines touch it.

Gambier‖ applies the ideas of orientation to circles and spheres in contact, including a metric generalization of the Morley-Peterson property to S_n; he¶ makes an application to orthogonal systems by methods of pure geometry, the case $n=4$ producing the results obtained by Barbilian. E. Ciani** points out that the whole idea of double fives is due to him.††

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* D. Barbilian, Bucharest Akademie Bulletin, Mathematics and Physics, vol. 1 (1929), pp. 12–16.

† G. Tzitzeica, Bucharest Akademie Bulletin, Mathematics and Physics, vol. 1 (1929), pp. 16–21.

‡ E. A. Weiss, Amsterdam Proceedings, vol. 35 (1932), pp. 969–978.

§ E. Study, Berlin Sitzungsberichte, 1926, pp. 360–380.

‖ B. Gambier, Comptes Rendus, vol. 190 (1930), pp. 157–159.

¶ B. Gambier, Bulletin des Sciences Mathématiques, vol. 55 (1931), pp. 75–96; Journal de Mathématiques, (9), vol. 11 (1931), pp. 377–387.

** E. Ciani, Annali di Matematica, (3), vol. 8 (1902), pp. 1–21.

†† E. Ciani, Rendiconti dei Lincei, (6), vol. 17 (1933), pp. 215–217.