INDEPENDENT POSTULATES FOR AN "INFORMAL PRINCIPIA SYSTEM WITH EQUALITY"*

BY E. V. HUNTINGTON

1. *Introduction.* The Postulates P1–P8 in the preceding paper define an abstract mathematical system $(K, C, +, ')$ which has there been called an "informal Principia system."

An interesting concrete illustration of this abstract mathematical theory may be developed as follows.

2. *A System of "Verdicts" and the Idea of Equality.* Consider a group of people, A, B, C, $\cdots$, each of whom is supposed to be either "guilty" or "not guilty" on some charge before a court. A judgment of the court such as "A is guilty" or "B is not guilty" may be called a "simple verdict," while any combination of two or more such simple verdicts by means of the connectives "or" and "and" may be called a "compound verdict." For example, the statement "A-is-guilty-and-B-is-not-guilty or C-is-guilty" is a compound verdict. (The word "or" is to be understood in the sense of "at least one.")

Clearly, if a is a verdict and b is a verdict, then the statement "a and b" is a verdict, and also the statement "a or b" is a verdict.

To every verdict, a, there corresponds a "contradictory" verdict, which we shall denote by $-a$. If a is a verdict, then the "contradictory of a" is to be understood in the usual linguistic sense. For example:

if $a =$ "A is guilty," then $-a =$ "A is not guilty";
if $a =$ "A is not guilty," then $-a =$ "A is guilty";
if $a =$ "A is guilty and B is not guilty," then
$-a =$ "A is not guilty or B is guilty";
if $a =$ "A is guilty or B is not guilty," then
$-a =$ "A is not guilty and B is guilty";
if $a =$ "A-is-guilty-and-B-is-not-guilty or C is-guilty," then
$-a =$ "A-is-not-guilty-or-B-is-guilty and C-is-not-guilty"; etc.

Clearly, if a is any verdict, than $-a$ is also a verdict.

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Further, every verdict is supposed to be either "correct" or "incorrect" under the law.

It must be noted, however, that if a is a verdict, then the statement "a is correct" is not itself a verdict; and similarly, if a is a verdict, then the statement "a is incorrect" is not itself a verdict. In other words, the attributes "guilty" and "not guilty" apply only to the people, A, B, C, $\cdots$, while the attributes "correct" and "incorrect" apply only to the verdicts of the court.

Further, we observe that among the "correct" verdicts there are some which are "necessarily correct," irrespective of the law in the case. For example, if $a=$"A is guilty or A is not guilty," then a is a necessarily correct verdict. Such verdicts we may call "truistic verdicts," or simply "truisms."

Similarly, among the "incorrect" verdicts there are some which are "necessarily incorrect." For example, the verdict "A is guilty and A is not guilty" is a necessarily incorrect verdict. Such verdicts we may call "absurd verdicts" or simply "absurdities."

Finally, verdicts may be classified according to the effects which they produce on the personal fate of the individual defendents A, B, C, $\cdots$. For example, the verdict "A is guilty" will send Mr. A to jail, while the verdict "B is not guilty" will free Mr. B from the charge against him. The verdict "A is guilty or B is guilty," though it does not directly send anybody to jail, does affect the legal status of both A and B; for if this verdict is followed later by the verdict "B is not guilty," the result will be to send A to jail. On the other hand, a truistic verdict, like "A is guilty or A is not guilty," or an absurd verdict, like "A is guilty and A is not guilty," will have no effect on anybody's fate.

If two correct verdicts, a and b, are "equal" with respect to legal effect—that is, if a and b can be interchanged in the history of the decisions of the court without affecting, directly or ultimately, the legal fate of any person in the group considered —then we write "$a=b$." Similarly, if two incorrect verdicts, a and b, are equal with respect to legal effect, then also we write "$a=b$."

For example, suppose a means "A is guilty" and b means "A-is-guilty-and-B-is-guilty or A-is-guilty-and-B-is-not-guilty," then $a=b$; for the legal effect of the two verdicts is precisely the same, namely, to send A to jail and to do nothing to B or to any other defendant. Again, any truistic verdict will be "equal" to any other truistic verdict, and any absurd verdict will be "equal" to any other absurd verdict; but no correct verdict will be "equal" to any incorrect verdict. It need hardly be said that the notation "$a=b$" does not mean that a and b are "identical in all respects," which is an indefensible concept; here, as elsewhere in mathematics, $a=b$ means merely that a and b are identical in some respect, which must be specified.

The idea of equality does not occur in the primitive propositions of the Principia, and hence the equality sign is not used in our Postulates P1–P8. In the next section we propose to extend the abstract theory of systems $(K, C, +, ')$ so as to include systems, $(K, C, +, ')$ making use of the idea of equality. Such systems for lack of a better notation, we shall indicate by $(K, C, +, ', =)$, although the role of the "$=$" sign is clearly not quite analogous to the role of the other variables.* That is, "$=$" is not a wholly undefined relation; $a=b$ indicates merely that a and b are equal in some undefined respect, it being understood that if a and b are "equal" in this respect, either one may be replaced by the other wherever it occurs within the system $(K, C, +, ', =)$.

3. *Systems* $(K, C, +, ', =)$. *Postulates* P1–P11. Let us consider an abstract system $(K, C, +, ', =)$, where

$K=$ an undefined class of elements, $a, b, c, \cdots$;

$C=$ an undefined subclass within K assumed non-empty;

$a+b=$ the result of an undefined binary operation on a and b;

$a'=$ the result of an undefined unary operation on a; and

$a=b$ means a and b are equal in some undefined respect (see above);

and let us impose the following postulates, P1–P11:

POSTULATE P1. *If a is in K and b is in K, then $a+b$ is in K.*

POSTULATE P2. *If a is in K, then a' is in K.*

POSTULATE P3. *If a is in C, then a is in K.*

POSTULATE P4. *If $a+b$ is in C, then $b+a$ is in C.*

POSTULATE P5. *If a is in C, then $a+b$ is in C.*

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* Compare K. Yoneyama, in the Tôhoku Mathematical Journal, vol. 22 (1923), p. 101, and vol. 24 (1925), pp. 287–293.

DEFINITION. The notation (a is in C') means (a is in K and a is not in C).

POSTULATE P6. *If a is in K and a' is in C, then a is in C'.*

POSTULATE P7. *If a is in K and a' is in C', then a is in C.*

POSTULATE P8. *If $a+b$ is in C and a' is in C, then b is in C.*

In the following postulates, a, b, $a+b$, a', b', etc., are understood to be elements of K.

POSTULATE P9. $a+b=b+a$.

POSTULATE P10. $(a+b)+c=a+(b+c)$.

POSTULATE P11. $(a'+b')'+(a'+b)'=a$.

[If we introduce the usual definition of ab, namely,

DEFINITION. $ab=(a'+b')'$,

then Postulate P11 may be replaced by the following:

POSTULATE P11a. $ab+ab'=a$.]

Now any system (K, C, $+$, $'$) which satisfies Postulates P1–P8 has just been called an "informal Principia system." Hence any system (K, C, $+$, $'$, $=$) which satisfies Postulates P1–P11 may be called an "*informal Principia system with equality.*"

On the other hand, any system (K, $+$, $'$, $=$) which satisfies the five postulates P1, P2, P9, P10, P11 is known to be a Boolean algebra.* Hence any system (K, C, $+$, $'$, $=$) which satisfies Postulates P1–P11 may be called a "*Boolean algebra with subclass C.*"

This result, that an "informal Principia system with equality" is the same thing as a "Boolean algebra with subclass C," has an immediate bearing on the much-discussed problem of the proper relation between the "algebra of propositions" and the "algebra of classes."

4. *On the Independence of Postulates* P1–P11. In regard to the independence of Postulates P1–P11, it is obvious that Postulate P4 becomes redundant as soon as P9 is added. With the exception of Postulate P4, all the postulates are independent.

To prove this, we notice first that all the Examples P1–P8 above (with the added specification that "$=$" shall mean "equal with respect to numerical value") are found to satisfy Postulates P9–P11, except that Example P4 does not satisfy Postulate P9. Hence Postulates P1–P3, P5–P8 remain independent even after

* See my *fourth set* of postulates for Boolean algebra, Transactions of this Society, vol. 35 (1933), p. 280 and p. 557; or Mind, vol. 42 (1933), pp. 203–207.

P9–P11 have been added. The following supplementary examples prove the independence of P9–P11.

EXAMPLE P9.
$K=1, 2, 3, 4, 5, 6$;
$C=1, 2, 3$;
$a+b$ and a' as in the table;
"$=$" means equal in numerical value. Here P9 fails, since $4+5=4$ and $5+4=5$. All the other postulates (including P4) are satisfied.

$+$	1	2	3	4	5	6	$'$
1	1	1	1	1	1	1	6
2	1	2	2	1	1	2	5
3	1	3	3	1	1	3	4
4	1	1	1	4	4	4	3
5	1	1	1	5	5	5	2
6	1	2	3	4	5	6	1

EXAMPLE P10.
$K=1, 2, 3, 4, 5, 6, 7, 8$;
$C=1, 2, 3, 4$;
$a+b$ and a' as in the table;
"$=$" means equal in numerical value.
Here P10 fails, since $(2+5)+7\neq 2+(5+7)$. All the other postulates are satisfied.

$+$	1	2	3	4	5	6	7	8	$'$
1	1	1	1	1	1	1	1	1	8
2	1	2	3	4	3	4	1	2	7
3	1	3	3	1	3	1	1	3	6
4	1	4	1	4	1	4	2	4	5
5	1	3	3	1	5	7	5	5	4
6	1	4	1	4	7	6	7	6	3
7	1	1	1	2	5	7	7	7	2
8	1	2	3	4	5	6	7	8	1

EXAMPLE P11.
$K=1, 2, 3, 4$;
$C=1, 2$;
$a+b$ and a' as in the table;
"$=$" means equal in numerical value.
Here P11 fails when $a=1$, $b=2$. All the other postulates are satisfied.

$+$	1	2	3	4	$'$
1	1	1	1	1	4
2	1	2	2	2	3
3	1	2	3	3	2
4	1	2	3	4	1

We have thus established the fact that no one of the eleven postulates P1–P11 can be deduced from the other ten (except that P4 is a consequence of P9).

5. *Examples of Systems* (K, C, $+$, $'$, $=$) *which Satisfy all the Postulates.* The Examples 0.1, 0.2, and 0.3, given on page 136, will be found to satisfy all the postulates P1–P11, when "$=$" is understood to mean "equal with respect to numerical value." More interesting examples are the following.

EXAMPLE 0.4.

K = the class of "verdicts," $a, b, c, \cdots$, as explained above.

C = the subclass of verdicts which are "correct";

$a+b$ = the statement "a or b";

a' = the statement contradictory to a;

$a=b$ means that a and b are "equal with respect to legal effect."

Here all the postulates are found to be satisfied. The subclass C' comprises the "incorrect" verdicts, and ab = the statement "a and b." The "truistic" verdicts are all equal to the "universe element," U, of the Boolean algebra.

EXAMPLE 0.5.

K = the class of "verdicts," $a, b, c, \cdots$, as above:

C = the subclass of verdicts which are "incorrect";

$a+b$ = the statement "a and b";

a' = the statement contradictory to a;

$a=b$ means that a and b are "equal with respect to legal effect."

Here again all the postulates are found to be satisfied. The subclass C' comprises the "correct" verdicts, and ab = the statement "a or b." The "truistic" verdicts are all equal to the "zero element," Z, of the Boolean algebra.

These last two examples illustrate the "principle of duality" between $+$ and $\times$. In Example 0.4, $+$ = "or," $\times$ = "and," and C = "correct." In Example 0.5, $+$ = "and," $\times$ = "or," and C = "incorrect."

The existence of any one of these five examples establishes the "consistency" of Postulates P1–P11.

Any one of these five systems is an example of an "informal Principia system with equality" (or a "Boolean algebra with subclass C").*

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* *Note on the "formal" theory of the Principia.* Any system (K, C, $+$, $'$, $=$) which satisfies all the postulates P1–P11 except P7, and violates P7, may be called a *"formal Principia system with equality."*

For, it can easily be shown that from the Postulates P1–P11 without P7 we can deduce all the "formal" propositions P20–P28, but not the "informal" proposition P29. And conversely, from the "formal" propositions P20–P28, taken in conjunction with the propositions about equality, P9–P11, we can deduce all the Postulates P1–P6, P8–P11, but not P7. (Here, for the moment,