THE APPARENT CONTOUR OF THE GENERAL V_3^n IN S_4*

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It is the purpose of this paper to study the surface obtained by intersecting with a 3-space the hypercone of tangents drawn to a hypersurface V in S_4 from a point O. Or we may say that we project from O upon S_3 the surface which is the intersection of V and the first polar hypersurface of O. It may be recalled that when V is the locus of lines that meet four planes, and therefore a fifth, and O is a general point of V, the Kummer surface is the result. In this paper V will be assumed non-singular and of order n. But the surface obtained has all the ordinary singularities, including a cuspidal curve. The two cases when the center of projection is not a point of V, and when it is on V, will be considered in that order.

If V is a non-singular hypersurface in S_4 of order n, and O any point not on V, the hypercone of tangents to V from O meets a 3-space in a surface F' of order $n(n-1)$. For a line in S_3 determines with O a plane that meets V in a curve to which $n(n-1)$ tangents can be drawn from O. As remarked, F' is the projection from O of the surface F, which is the intersection of V and the first polar hypersurface of O. The second polar hypersurface of O meets F in a curve C of order $n(n-1)(n-2)$ whose projection is the cuspidal curve of F'. For, if P is a general point of C, the tangent plane to F at P passes through O. A 3-space containing OP, but not the tangent plane, meets F in a curve to which OP is a simple tangent. Hence its projection is a plane curve on F', having a cusp at P', the projection of P. The cuspidal tangent is the intersection with S_3 of the osculating plane to the space curve at P. As the 3-space section through OP varies there are ∞^1 distinct osculating planes. They lie in the 3-space which is tangent to V at P. For this 3-space intersects V in a surface having a node at P, and intersects the first polar of O in a surface whose tangent plane at P is the tangent plane to F. The second of these surfaces is the polar of O with respect to the first. Furthermore, OP meets V in three points at P. From all of which

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it follows that the tangent plane to F at P is tangent along OP to the quadric cone of lines that meet V three times at P, and that the 3-space in question intersects F in a curve having a cusp at P, whose cuspidal tangent is OP. The tangent plane to F is the osculating plane to this curve and meets it, and therefore F, in four points at P. Therefore this 3-space, tangent to V at P, meets S_3 in a plane π, which contains the cuspidal tangents to all sections of F' by planes through P'. The plane π meets F' in a curve having a triple point, whose only real tangent is the tangent to the cuspidal curve, that is, the intersection with S_3 of the tangent plane to F at P. A general 3-space of the pencil that contains the tangent plane to F at P meets V in a surface to which that plane is a stationary tangent plane, and meets F in a curve which has a double point at P. The projections of these curves give the pencil of plane sections of F' whose axis is the tangent to the cuspidal curve. Such a section has a tacnode at P' instead of a cusp. The order of the cuspidal curve is called c. Thus we have $c=n(n-1)(n-2)$.

To find the double curve and the triple points, let Q' with coordinates $(x_1':x_2':x_3':x_4':x_5')$ be a point of V, and $Q(x_1:x_2:x_3:x_4:x_5)$ a general point of S_4. Then the coordinates of a point on $Q'Q$ are $x_i'+\lambda x_i$, $(i=1, 2, \cdots, 5)$. We suppose that $Q'Q$ meets V in two points at Q', and we require that it be tangent to V elsewhere. This means the vanishing of the discriminant of the residual equation in λ of order $n-2$. This discriminant is of order $n(n-1)-6$ in the coordinates of Q, and combined with the tangent 3-space at Q' represents the cone of lines tangent to V at Q' and once elsewhere. The same discriminant is of order $(n-2)(n-3)$ in the coordinates of Q'. Hence if Q, or O, is a fixed point, not on V, the points of contact of the double tangents to V from O lie on a hypersurface of order $(n-2)(n-3)$. They lie also on the surface F, and hence fill a curve B of order $n(n-1)(n-2)(n-3)$, which lies on a cone of half that order with vertex at O. This cone meets S_3 in the double curve of F', whose order is therefore $\frac{1}{2}n(n-1)(n-2)(n-3)$. The order of the double curve is called b.

To find the triple points we require that the above equation of order $n-2$ in λ have two distinct pairs of equal roots. We thus get a restricted system, whose order is $4n(n-4)(n-5)+\frac{1}{2}(n-2)(n-3)(n-4)(n-5)$ or $\frac{1}{2}(n-4)(n-5)(n^2+3n+6)$ in the coordinates of Q, and is $\frac{1}{2}(n-2)(n-3)(n-4)(n-5)$ in the coordinates of Q'.* This means that, from a point Q' on V, $\frac{1}{2}(n-4)(n-5)(n^2+3n+6)$ lines can be drawn tangent to V at Q' and twice elsewhere. If Q, or O, is a point not on V it means that the number of points of contact of lines drawn from O tangent three times to V is $\frac{1}{2}n(n-1)(n-2)(n-3)(n-4)(n-5)$; and hence the number of triple tangents to the hypersurface V from O, or the number of triple points on the double curve of F' is $\frac{1}{6}n(n-1)(n-2)(n-3)(n-4)(n-5)$. The number of these points, which are triple on the double curve and triple on the surface, is denoted by t.

To find the intersections of the double and cuspidal curves we note first that their originals B and C in S_4 do not in general have an apparent double point as seen from O. For that would demand that a line through O satisfy four conditions. But the curves B and C do meet on F where the second polar hypersurface of O meets B. These intersections of the curves B and C are of two sorts. First there are the $n(n-1)(n-2)(n-3)$ points common to V and the first three polar hypersurfaces of O. At such a point, B and C are tangent. Hence the number of remaining intersections is $n(n-1)(n-2)^2(n-3)-2n(n-1)(n-2)(n-3)$ $=n(n-1)(n-2)(n-3)(n-4)$. If P is a point of the first sort, OP meets V four times at P and the first polar of O, and therefore F, three times. Also OP is a simple tangent to C, and therefore the cuspidal curve of F' has itself a cusp at P'. The line OP is tangent to B at P; but since B meets twice each generator of the projecting cone, P' is not a cusp of the double curve, but a point where it changes from crunodal to acnodal. Now a general 3-space containing OP, but not the tangent plane to F, meets F in a curve to which OP is an inflectional tangent. Therefore its projection is a plane curve for which P' is not a cusp, but a triple point with one real tangent. These tangents form a plane pencil as before. Now the tangent plane to F at P meets F in six points there. A general 3-space containing this plane meets F in a curve having a double point P, one of whose tangents is OP; and the tangent 3-space to V meets F in a curve having a tacnode whose tangent is OP. Again if P is an intersection of B and C of the second sort, OP will meet V three times at P and be tangent to V at one other point. The point P' will be a point of F' where the cuspidal curve is intersected by a sheet

* Salmon: *Higher Algebra*, §278.

belonging to the double curve, and will be a cusp of the double curve. The number of intersections of the cuspidal curve and double curve which are cusps on the former is called β, and the number that are cusps on the latter is called γ. From what precedes, therefore, we see that $\beta = n(n-1)(n-2)(n-3)$ and $\gamma = n(n-1)(n-2)(n-3)(n-4)$.

We now consider the tangent cone to F' from a general point O' of S_3. The first polar hypersurface of O' meets F in a curve A, the intersection of V and the first polars of O and O'. At a point P of A the tangent 3-space to V passes through O and O', and meets F in a curve with a double point at P, whose projection lies in a plane through O' tangent to F' at P'. Thus the order a of the tangent cone to F' is $n(n-1)^2$. The class of that cone, or class of F', is the number of 3-spaces through a plane $OO'O''$ tangent to V. It is denoted by n' and is $n(n-1)^3$. The number Δ of planes through a line that meet twice the curve A, whose projection is the curve of contact, is given by

$$2\Delta = n(n-1)^2[n(n-1)^2 - 3n + 4] = n(n-1)^2(n-2)(n^2-2).$$

Hence to find κ and δ, the number of cuspidal and double edges of the tangent cone, we have

$$2(\kappa + \delta) = n(n-1)^2(n-2)(n^2-2)$$

and

$$n(n-1)^2[n(n-1)^2 - 1] - 2\delta - 3\kappa = n(n-1)^3.$$

Hence $\kappa = 2n(n-1)^2(n-2)$ and $\delta = \frac{1}{2}n(n-1)^2(n-2)(n^2-6)$. The intersections of the curve of contact with the double curve and cuspidal curve correspond to the intersections of A with B and C, and are $n(n-1)^2(n-2)(n-3)$ and $n(n-1)^2(n-2)$ in number. These numbers which give the class of the nodal developable and that of the cuspidal developable are called ρ and σ, respectively. For the curve C, we have as above

$$2\Delta = n(n-1)(n-2)[n(n-1)(n-2) - 3(n-3) + 2].$$

Dividing by 2 and subtracting the β cusps, we obtain h, the number of apparent double points of the cuspidal curve. Thus $h = \frac{1}{2}n(n-1)(n-2)[n^3 - 3n^2 - 3n + 11]$. To find k, the number of apparent double points of the double curve, we find two expressions for the rank, or class, of B, one of which involves k. The order of the hypersurface which meets B in the points where the tangent meets a given plane is $n-1+n-2+(n-2)(n-3)-1$, or $(n-1)(n-2)$. Now the three points of contact on one of the triple tangents to V from O are double points of B. For such a tangent is a triple generator of the cone that projects B from O, and therefore a general three-space containing it meets B in $n(n-1)(n-2)(n-3)-6$ other points. Also if P_1 is a point on γ, OP_1 is tangent to B at P_1 and meets B again at P_2, where B is not tangent to the generator, but has a cusp. Hence the rank of B is $n(n-1)^2(n-2)^2(n-3)-6t-3\gamma$. Again if we take an arbitrary plane through O, the rank of B is given by $\beta+\gamma$, the number of tangents to B that pass through O, plus the number of tangents to B that meet the arbitrary plane elsewhere. Since a general generator of the cone meets B twice, the latter number is twice the rank of the cone, that is, twice the rank of the double curve of F'. Hence we have the equation

$$\begin{aligned} n(n-1)^2(n-2)^2(n-3) - 6t - 3\gamma \\ = \beta + \gamma + 2[N(N-1) - 2k - 3\gamma - 6t], \end{aligned}$$

where $N=\frac{1}{2}n(n-1)(n-2)(n-3)$, the order of the double curve. Hence

$$k = n(n-1)(n-2)(n-3)[n^4 - 6n^3 + 7n^2 + 14n - 28]/8.$$

We may summarize what precedes as follows. When V is of order n and the center of projection is not on V, the order of F', the surface obtained in S_3, is $n(n-1)$. The orders, c and b, of its cuspidal and double curves are $n(n-1)(n-2)$ and $\frac{1}{2}n(n-1)(n-2)(n-3)$. The numbers, β and γ, of intersections of the two curves which are cusps on the former and latter, respectively, are

$$n(n-1)(n-2)(n-3) \quad \text{and} \quad n(n-1)(n-2)(n-3)(n-4).$$

The number, t, of points triple on the double curve and triple on F' is $n(n-1)(n-2)(n-3)(n-4)(n-5)/6$. The order, a, of a general tangent cone is $n(n-1)^2$. Its class, n', or the class of F', is $n(n-1)^3$. The numbers, σ and ρ, of intersections of the curve of contact with the cuspidal and double curves are $n(n-1)^2(n-2)$ and $n(n-1)^2(n-2)(n-3)$. The numbers, κ and δ, of cuspidal and double edges of the tangent cone are $2n(n-1)^2(n-2)$ and $\frac{1}{2}n(n-1)^2(n-2)(n^2-6)$. The numbers, h and k, of apparent double points of the cuspidal and double curves were given in the last paragraph. There are apparently no close points on the cuspidal curve, nor points at which the cuspidal and double curves meet but which are singular on neither. It will be found that the above results satisfy identically the established formulas for surfaces in S_3; see, for example, Salmon's *Solid Geometry*, 5th edition, the six equations A and B, §§610 and 612. In those equations n, which denotes the order of the surface, must be replaced by $n(n-1)$.

The Hessian of V meets the surface F in a curve H of order $5n(n-1)(n-2)$ whose projection is the parabolic curve of F'. At a point of H the quadric cone of lines that meet V three times there becomes two planes, and the tangent 3-space to V meets F in a curve having a cusp. The Hessian meets the curve A in $5n(n-1)^2(n-2)$ points. This number is the class of the spinode torse of F'. The Hessian meets C in $5n(n-1)(n-2)^2$ points. At such a point one of the pair of planes just mentioned passes through O and is the tangent plane to F. The tangent three-space to V meets C three times there. Hence at the projected point, where the parabolic curve of F' meets the cuspidal curve, the osculating plane to the latter is the tangent plane to the surface. Applying Plücker's equations to the tangent cone, we find the class of the bitangential developable to be

$$\tfrac{1}{2}n(n-1)^2(n-2)(n^4-2n^3+2n^2-15).$$

We infer that the bitangential curve is the projection of a curve on F in S_4 of order $n(n-1)(n-2)(n^4-2n^3+2n^2-15)$. This number is checked by use of the formulas for surfaces in S_3. It appears also that the locus of points of contact of 3-spaces doubly tangent to V is the intersection of V with a hypersurface of order $(n-2)(n^4-2n^3+2n^2-15)$; this might be proved otherwise.

It remains to indicate briefly the modifications to be made when the center of projection O is a point of V. Now the polar hypersurfaces of O have in common the tangent 3-space, the quadric cone of lines that meet V three times at O, and the six lines that meet it four times. The order of F is the same; but the order of F' is $n(n-1)-2$. The curve C, the intersection of V with the first two polars of O, has a sextuple point at O, the tangents being the six lines that meet V four times there. This curve is met by a general 3-space in six points at O, but by the tangent 3-space in 24. Thus c, the order of the cuspidal curve of F', is $n(n-1)(n-2)-6$. A general 3-space through O meets F in a curve having a double point there. The cone projecting this curve from O has $\frac{1}{2}n(n-1)(n-2)(n-3)-3[n(n-1)-6]$ double edges. This is the number b, the order of the double curve of F'. It is the previous value diminished by two for each line that can be drawn (in the 3-space section) tangent to V at O and once elsewhere. Now

$$2b = n(n-1)[(n-2)(n-3)-2] - 2(n-4)(n+3).$$

This suggests that the points of contact of the double tangents to V from O lie on a hypersurface of order $(n-2)(n-3)-2$, and fill a curve B of order $n(n-1)[(n-2)(n-3)-2]$ on F that has a multiple point of order $2(n-4)(n+3)$ at O. This is seen by the method used for the similar problem above. If Q, or O, is on V, the coefficient of the highest power of λ in the equation of order $n-2$ vanishes and the discriminant is divisible by the square of the preceding coefficient, which, when the primed letters are variable, represents the three-space tangent to V at O. With regard to the multiple point of the curve at O, we note that the quadric cone of lines meeting V three times at O and the residual cone of order $n(n-1)-6$ are tangent along the six lines that meet V four times at O, and hence have in common $2(n-4)(n+3)$ other generators. Such a generator meets V three times at O, and is tangent to V elsewhere. It is evidently a generator of the cone of double tangents drawn to V from O. The quadric cone of lines that meet V three times at O gives on the surface F' a trope, that is, a conic whose plane is tangent to F' along the conic. The residual cone at O, in the tangent 3-space, has $n(n-1)(n-2)-24$ cuspidal edges, and $\frac{1}{2}(n-4)(n-5)(n^2+3n+6)$ double edges, that is, lines tangent to V at O and twice elsewhere. Hence the cuspidal curve on F' meets the plane of the trope three times where each of the six lines meets it; and the double curve is tangent to the plane of the trope where each of the $2(n-4)(n+3)$ generators meets it. For the difference between the number of double tangents that can be drawn to V from O in a general 3-space and in the tangent 3-space is $4(n-4)(n+3)$. The number, β, of points common to V and the first three polars of O is $n(n-1)(n-2)(n-3)-24$. The second polar of O meets B $3\cdot 2(n-4)(n+3)$ times at O, and hence

$$\begin{aligned}\gamma &= n(n-1)(n-2)[(n-2)(n-3)-2]-6(n-4)(n+3)\\ &\qquad -2[n(n-1)(n-2)(n-3)-24]\\ &= n(n-1)(n-2)(n-3)(n-4)-2(n-4)(n^2+4n+15).\end{aligned}$$

The order and class of the general tangent cone to F' and its other characteristics remain as before. The number σ is the same; but ρ is $n(n-1)^2[(n-2)(n-3)-2]$, that is, reduced by $2n(n-1)^2$. We can infer also that the number of triple tangents to V from O is the number in the previous case diminished by two for each of the $\frac{1}{2}(n-4)(n-5)(n^2+3n+6)$ lines that can be drawn tangent to the hypersurface V at O and twice elsewhere. Thus the number of triple points is found to be $(n-4)(n-5)(n-6)(n+1)(n^2-n+6)/6$; a result which can be checked by the relation $b(n-2)=\rho+2\beta+3\gamma+3t$, on replacing n by $n(n-1)-2$. To find h, the number of apparent double points of the cuspidal curve, we note that the rank of C is reduced by 18 by the sextuple point at O. For, taking the origin at O, we easily see that the hypersurface which meets C in the points where the tangent meets a given plane has a triple point at O. Thus the rank of C is $3n(n-1)(n-2)^2-18$. But the rank of C is the number β of tangents to C from O, plus the six tangents at O taken twice, plus the rank of the cone projecting C, that is, the rank of the cuspidal curve. Hence we have $3n(n-1)(n-2)^2-18=\beta+12+M(M-1)-2h-3\beta$, where $M=n(n-1)(n-2)-6$. Hence we shall have also the relation $h=\frac{1}{2}n(n-1)(n-2)[n^3-3n^2-3n-1]+60$. It would be possible, but too tedious, to make the corresponding modification for B, and so find k independently.

Finally it must not be forgotten that some of the statements in the last paragraph require obvious modification when $n=3$. Thus when O is a point of a general cubic hypersurface we get a quartic surface in S_3 with no double or cuspidal curve, but having a trope on which are six nodes, the points in which its plane is met by the six lines through O that lie on V and on F. For a plane through one of them meets V in a conic to which two tangents can be drawn from O. The order and class of the tangent cone are as above $n(n-1)^2$ and $n(n-1)^3$, that is, 12 and 24. Similarly for its other characteristics. The trope is the 16th trope of the Kummer surface obtained when the cubic hypersurface is of the sort mentioned in the introduction to this paper.

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