THE DIOPHANTINE EQUATION $x_1x_2a_1+x_2x_3a_2+\cdots+x_nx_{n+1}a_n=\delta$

BY E. J. FINAN

The object of this paper is to prove the following theorem.

THEOREM. *If δ is the greatest common divisor of $a_1, a_2, \cdots, a_n$, then the Dophantine equation*

$$x_1x_2a_1 + x_2x_3a_2 + \cdots + x_nx_{n+1}a_n = \delta \tag{1}$$

always has a solution.

We shall first prove the following lemma.

LEMMA. *If the greatest common divisor of α and β is γ, there exist integers x and y such that $x\alpha+y\beta=\gamma$ and such that x is prime to any previously given integer d.*

If $\beta=0$ the lemma is evident. Hence assume $\beta\neq 0$. Let x and y be any pair of integers for which $x\alpha+y\beta=\gamma$. Let $\alpha=\alpha_1\gamma$ and $\beta=\beta_1\gamma$. Then

$$x\alpha_1 + y\beta_1 = 1. \tag{2}$$

This gives $y=(1-x\alpha_1)/\beta_1$. Hence x must satisfy the congruence $\alpha_1 x\equiv 1$, mod β_1. Since α_1 and β_1 are relative prime, all the solutions are given by $x=x_1+k\beta_1$, where x_1 is a particular solution and k is an integer. Evidently all such values of x satisfy (2). Dirichlet's theorem on the infinitude of primes in an arithmetical progression assures us of the existence of a k that will satisfy the conditions of the lemma.

We shall now prove the theorem by induction. Since we may divide (1) by δ it is quite general to assume that the a's are relatively prime and δ is unity.

When n is even the terms of (1) may be grouped in the following manner:

$$\begin{aligned} x_2(x_1a_1 + x_3a_2) + \cdots + x_{r+1}(x_ra_r + x_{r+2}a_{r+1}) \\ + x_{r+3}(x_{r+2}a_{r+2} + x_{r+4}a_{r+3}) + \cdots \\ + x_n(x_{n-1}a_{n-1} + x_{n+1}a_n) = 1, \end{aligned} \tag{3}$$

where r is odd. We shall now determine the x's that will satisfy (3). Let δ_i be the greatest common divisor of $a_1, a_2, \cdots, a_i$, $(i=1, 2, \cdots, n)$. We shall find x_{r+4} so that (a) $x=x_{r+2}a_{r+2}+x_{r+4}a_{r+3}$ is prime to δ_{r+1} except for factors in δ_{r+3}, and (b) x_{r+4} is prime to δ_{r+3}, if (c) x_{r+2} is prime to δ_{r+1}.

Let $\delta_{n+3}=q_1^{\alpha_1}q_2^{\alpha_2}\cdots q_m^{\alpha_m}$, $a_{r+3}=\delta_{r+3}fa'_{r+3}$, and $a_{r+2}=\delta_{r+3}fa'_{r+2}$, where a'_{r+2} and a'_{r+3} are relatively prime. Let $\delta_{r+1}=\delta_{r+3}d_1$, and $p=p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k}$ be the greatest common divisor of d_1 and a'_{r+2}. Then set $a'_{r+2}=p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k}a''_{r+2}$, $d_1=pd_2$. But since d_2 may contain some of the p's, write $d_2=p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}d_3$, where d_3 contains none of the p's as a factor and hence is prime to a'_{r+2}. Finally let $d_3=q_1^{t_1}q_2^{t_2}\cdots q_m^{t_m}d_4$, so that d_4 is prime to δ_{r+3}.

We assert that d_4 is the x_{r+4} we want. Evidently (b) is satisfied by the definition of d_4. We shall show that (a) is satisfied. We now have $a_{r+2}=\delta_{r+3}fp_1^{\beta_1}\cdots p_k^{\beta_k}a''_{r+2}$, $x_{r+4}=d_4$,

$$\delta_{r+1}=\delta_{r+3}p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}q_1^{t_1}q_2^{t_2}\cdots q_m^{t_m}d_4,$$

$(s_i=e_i+\beta_i;\ i=1, 2, \cdots, k)$. Consider

$$x=x_{r+2}\delta_{r+3}fp_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k}a''_{r+2}+d_4\delta_{r+3}fa'_{r+3}$$

and

$$\delta_{r+1}=\delta_{r+3}p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}q_1^{t_1}q_2^{t_2}\cdots q_m^{t_m}d_4.$$

Evidently δ_{r+3} is a common factor.

If $p_i\neq q_j$, $(i=1, 2, \cdots, k; j=1, 2, \cdots, m)$, p_i cannot divide x because p_i is prime to d_4 and to f because δ_{r+3} is the greatest common divisor of $a_1, a_2, \cdots, a_{r+3}$, and to a'_{r+3} because p_i divides a'_{r+2} while a'_{r+2} and a'_{r+3} are relatively prime. Also d_4 is prime to x because it is prime to x_{r+2} due to the fact that d_4 is contained in δ_{r+1} and by (c) δ_{r+1} is prime to x_{r+2}. Finally, d_4 is prime to f since δ_{r+3} is the greatest common divisor of $a_1, a_2, \cdots, a_{r+3}$, and d_4 is prime to a''_{r+2} by definition of p. Hence (a) is satisfied, for the only factors common to δ_{r+1} and x are the factors q_i which are all found in δ_{r+3}.

Now by the lemma we can find x_1 and x_3 so that $x_1a_1+x_3a_2=\delta_2$ and x_3 is prime to δ_2. Then by the method given above we can find x_5 so that $x_3a_3+x_5a_4$ is prime to δ_2 except for factors in δ_4.

We continue in this way until we have determined all the x's in parentheses in (3). The quantities in these parentheses are relatively prime, for if g were a common factor it would be in δ_2. Then according to the derivation of the x's given above g can appear in the second parenthesis as a factor only if δ_4 contains g, and in the third parenthesis only if δ_6 contains g, and so on. But since $\delta_n = 1$, g must be a unit. If the quantities in parenthesis are relatively prime it is well known that we can find $x_2, x_4, \cdots, x_n$ so that (3) is satisfied.

If n is odd the procedure is just the same with the exception that the single term in the last parenthesis is automatically prime to those in the preceding parentheses which have been determined as above, and as before we have only to determine $x_2, x_4, \cdots, x_n$.

The above theorem not only proves the existence of the x's but also enables us to calculate them.

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ON CONTINUED FRACTIONS WHICH REPRESENT MEROMORPHIC FUNCTIONS*

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1. *Introduction.* In this paper I shall give sufficient conditions in order that a continued fraction of the form

$$\frac{b_1}{1} + \frac{b_2 z}{1} + \frac{b_3 z}{1} + \cdots \tag{1}$$

with arbitrary real or complex coefficients not zero shall represent a meromorphic function of z. Van Vleck† has shown that a sufficient condition is the following:

$$\lim_{n=\infty} b_n = 0. \tag{2}$$

Stieltjes‡ proved that (2) is necessary as well as sufficient when the b_n are real and positive. Van Vleck§ proved that when the b_n are real and $b_{2n}b_{2n+1} > 0$, and the roots of the denominators, D_{2n+1}, of the $(2n+1)$th convergents of (1) have distinct limits, not zero, for $n = \infty$, then the condition (2) is necessary and sufficient in order that (1) shall represent a meromorphic function of

* Presented to the Society, April 15, 1933.

† E. B. Van Vleck, Transactions of this Society, vol. 2 (1901), pp. 476–483.

‡ Stieltjes, Oeuvres, vol. 2, pp. 560–566.

§ E. B. Van Vleck: Transactions of this Society, vol. 4 (1903), pp. 309–310.