FURTHER MEAN-VALUE THEOREMS*

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The present note is a sequel to my recent article† in which certain mean-value theorems due to Weierstrass and Fekete were generalized. The generalizations resulted from replacing a positive real weight-function by one assuming only values in the angular region $0 \leqq \arg w \leqq \gamma < \pi$. Here the generalizations will be extended in such a manner as to yield analogous theorems in which the weight-function takes on arbitrary real values (Corollary 2, Theorem 3) or more generally any values in the double angle $0 \leqq \arg (\pm w) \leqq \gamma < \pi$ (Corollary 1, Theorem 3). Incidentally, these extensions yield (as Corollary 3, Theorem 3) the generalization of the Gauss-Lucas theorem which formed the principal result of another paper.‡

In what follows we shall denote by $f(Z)$ the point set $w = f(z)$ obtained on letting point z vary over the point set Z; by $\Delta \arg (Z-p)$ the magnitude of the smallest angle, with vertex at the point p, enclosing the point set Z; by $K(Z)$ the smallest convex region containing set Z, and, finally, by $S(Z, \theta)$ the star-shaped region composed of all points from which the set Z subtends an angle of not less than θ. The regions $K(Z)$ and $S(Z, \theta)$ can also be defined as the loci of all points p which satisfy respectively the inequalities $\Delta \arg (Z-p) \geqq \pi$, $\Delta \arg (Z-p) \geqq \theta$. Obviously, $S(Z, \theta) \equiv S(K(Z), \theta)$ and hence $S(Z, \theta)$ always contains $K(Z)$. Finally, in what follows, the two rectifiable curves

$$C: \quad z = z(s), \quad a \leqq s \leqq b; \qquad \Gamma: \quad \lambda = \lambda(t), \quad \alpha \leqq t \leqq \beta,$$

will serve as the curves of integration, and, unless further qualified, all functions introduced hereafter will be supposed to be continuous on these curves except perhaps for a finite number of finite jumps.

* Presented to the Society, April 15, 1933.

† M. Marden, this Bulletin, vol. 38 (1932), pp. 434–441.

‡ M. Marden, *On the zeros of certain rational functions*, Transactions of this Society, vol. 32 (1930), pp. 658–668.

THEOREM 1. *Let there be given the real numbers m_i and the functions $f_i(z)$ and $g(z)$ with Δ arg $g(C) \leqq \gamma < \pi$. Then each point σ as defined by the equation*

$$\int_a^b g(z) \prod_{i=1}^{h} [f_i(z) - \sigma]^{m_i} ds = 0 \tag{1}$$

lies in the region $S[K(f_1(C), f_2(C), \cdots, f_h(C)), (\pi-\gamma)/m]$, where $m = \sum_1^h |m_i|$.

THEOREM 2. *Let, in particular, each $f_i(z)$ be a rational function with exactly n_i finite zeros and p_i finite poles, none of the latter lying in the region $S(C, (\pi-\gamma)/n)$, where $n = \sum_1^h |m_i| (p_i + q_i)$ and $q_i = \max (n_i, p_i)$. Then for each value σ as defined by* (1) *there exists at least one integer i, $1 \leqq i \leqq h$, and at least one point z in $S(C, (\pi-\gamma)/n)$, such that $f_i(z) = \sigma$.*

Suppose Theorem 1 were not true; that is, suppose that, for some σ and for all i,

$$\Delta \arg [f_i(C) - \sigma] < \frac{\pi - \gamma}{m} \cdot$$

Then

$$\Delta \arg [f_i(C) - \sigma]^{m_i} < \frac{\pi - \gamma}{m} | m_i | ,$$

and hence

$$\Delta \arg g(C) \prod_{i=1}^{h} [f_i(C) - \sigma]^{m_i} < \pi .$$

Accordingly, the left hand-side of (1) is a sum of vectors each drawn from $w=0$ to points on the same side of some line through $w=0$. As such a sum cannot vanish, the assumption that Theorem 1 is false contradicts equation (1). Hence Theorem 1 must be true.

Similarly, let us suppose Theorem 2 to be false; that is, writing

$$f_i(z) - \sigma = A_i \frac{(z - a_{i1})(z - a_{i2}) \cdots (z - a_{iq_i})}{(z - b_{i1})(z - b_{i2}) \cdots (z - b_{ip_i})},$$

let us suppose that for all j and k

$$\Delta \arg (C - a_{jk}) < \frac{\pi - \gamma}{n} \cdot$$

Since we know by hypothesis, for all j and k, that

$$\Delta \arg (C - b_{jk}) < \frac{\pi - \gamma}{n},$$

it would follow that

$$\Delta \arg [f_i(C) - \sigma] < \frac{\pi - \gamma}{n}(p_i + q_i),$$

and hence

$$\Delta \arg g(C) \prod_{i=1}^{h} [f_i(C) - \sigma]^{m_i} < \pi.$$

Again equation (1) would be contradicted and hence Theorem 2 must be true.

On setting each $m_i = h = 1$, we obtain from Theorem 1 a previous generalization* of Weierstrass' mean-value theorem, and on setting also $\gamma = 0$, we derive his original theorem.†

The choice $m_i - 1 = p_i = h - 1 = 0$ for all i reduces Theorem 2 to the previous generalization of Fekete's theorems,‡ particularly of his following two theorems.

(1) *If $f(z)$ is a polynomial of degree n and $f(\alpha) \neq f(\beta)$, $\alpha \neq \beta$, it assumes every value between $f(\alpha)$ and $f(\beta)$, that is, on the line-segment joining $f(\alpha)$ and $f(\beta)$, at least once in S* (seg $\alpha\beta$, π/n).

(2) *If $P(z)$ is a polynomial of degree n and $P(\alpha) = P(\beta)$, $\alpha \neq \beta$, then $P'(z) = 0$ at least once in S*(seg $\alpha\beta$, $\pi/(n-1)$).

The first of Fekete's theorems is analogous to the Bolzano theorem for continuous functions of a real variable. The second Fekete theorem is analogous to Grace's theorem§, that under the same assumptions, $P'(z) = 0$ at least once in the circle with its center at $(\alpha+\beta)/2$ and with a radius of $\frac{1}{2}|\beta - \alpha|$ ctn (π/n). It is interesting to note that, since the circle of Grace's theorem passes through the centers of the two circles bounding

* M. Marden, this Bulletin, loc. cit., p. 435.

† Osgood, *Lehrbuch der Funktionentheorie*, 1923, vol. I, p. 212.

‡ See this Bulletin, loc. cit., p. 438 and p. 440. Also M. Fekete, Acta Szeged, vol. 1 (1923), pp. 98–100, and vol. 4 (1929), pp. 234–243; Mathematische Zeitschrift, vol. 22 (1925) pp. 1–7; Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 32 (1923), pp. 299–306, and vol. 34 (1926), pp. 220–233. J. v. Sz. Nagy, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 32 (1923), pp. 307–309.

§ P. J. Heawood, Quarterly Journal of Mathematics, vol. 38 (1907), pp. 84–107.

$S(\text{seg } \alpha\beta, \pi/(n-1))$, a better approximation to a zero of $P'(z)$ is obtained through use of both Grace's and Fekete's theorems than through either separately.

Theorem 1 may be stated in the following more general form.

THEOREM 3. *Let there be given the functions* $f_i(z, \lambda)$ *and* $g(z, \lambda)$ *with* $\Delta \arg g(C, \Gamma) \leqq \gamma < \pi$. *Then each point* σ *defined by the equation*

$$(2) \qquad \int_\alpha^\beta \int_a^b g(z, \lambda) \prod_{i=1}^{h} [f_i(z, \lambda) - \sigma]^{m_i} ds dt = 0,$$

lies in

$$S\left[K(f_1(C, \Gamma), f_2(C, \Gamma), \cdots, f_h(C, \Gamma)), \frac{\pi - \gamma}{m} \right],$$

where $m = \sum_1^h |m_i|$.

This theorem may be proved precisely as was Theorem 1.

If in Theorem 3 we specialize $\lambda(t) \equiv 0$ for $\alpha = 0 \leqq t \leqq 1 = \beta$ and $\lambda(t) \equiv 1$ for $1 < t \leqq 2 = \beta$ and $f(z, 0) = -f(z, 1) = f(z)$, and if we let $g_1(z) = g(z, 0)$ and $g_2(z) = g(z, 1)$, we derive the following result.

COROLLARY 1. *Let there be given the functions* $f(z)$, $g_1(z)$, *and* $g_2(z)$ *with* $0 \leqq \arg g_i(z) \leqq \gamma < \pi$ *for* $i = 1, 2$; *then the point* σ, *as defined by the equation*

$$\int_a^b [g_1(z) - g_2(z)] f(z) ds = \sigma \int_a^b [g_1(z) + g_2(z)] ds,$$

lies in

$$S(\pm f(C), \pi - \gamma).$$

This result leads us to a mean-value theorem in which the weight-function $g(z)$ is real, but not necessarily positive. We may indeed define two functions $g_1(z)$ and $g_2(z)$ so that

$$\begin{aligned} g_1(z) &\equiv g(z), \quad g_2(z) \equiv 0, &&\text{for } g(z) \geqq 0, \\ g_1(z) &\equiv 0, \quad g_2(z) \equiv -g(z), &&\text{for } g(z) \leqq 0. \end{aligned}$$

These functions $g_1(z)$ and $g_2(z)$ fulfill the requirements of Corollary 1 with $\gamma = 0$ and

$$g_1(z) - g_2(z) = g(z), \qquad g_1(z) + g_2(z) = |g(z)|.$$

The resulting theorem may be stated as follows.

COROLLARY 2. *Let there be given the complex function $f(z)$ and the real function $g(z)$. Then the point σ, as defined by the equation*

$$\int_a^b g(z)f(z)ds = \sigma \int_a^b |g(z)|\,ds,$$

lies in $K(\pm f(C))$.

Finally in Theorem 3 let us specialize as follows:

$$\alpha = 0,\ \beta = 1,\ a = 0,\ b = r \text{ (an integer)};$$
$$z(s) \equiv z_j \text{ for } j - 1 \leqq s < j;$$
$$\lambda(t) \equiv 0; \qquad g(z_j, 0) = \alpha_j;$$
$$m_k = 1, \quad f_k(z_j, 0) = a_{jk} \text{ for } \quad 1 \leqq k \leqq n;$$
$$m_k = -1,\ f_k(z_j, 0) = b_{jk} \text{ for } \ n + 1 \leqq k \leqq n + m = h;$$

and thus obtain the following corollary.*

COROLLARY 3. *Let α_i be any complex numbers such that for all i, $0 \leqq \arg \alpha_i \leqq \gamma < \pi$, and let a_{jk} and b_{jk} for all j and k be points of a given convex region K. Then all the zeros of the function*

$$\Phi(z) = \sum_1^r \alpha_i \phi_i(z),$$

where

$$\phi_i(z) = \frac{(z - a_{i1})(z - a_{i2}) \cdots (z - a_{in})}{(z - b_{i1})(z - b_{i2}) \cdots (z - b_{im})},$$

lie in the region

$$S\left(K, \frac{\pi - \gamma}{m + n}\right).$$

The particular case of this corollary $\gamma = \alpha_i - 1 = n = m - 1 = 0$ yields the theorem that the zeros of the partial fraction sum $\sum_1^r (z - z_i)^{-1}$ lie in the smallest convex region enclosing the points z_i. As this partial fraction is the logarithmic derivative of the polynomial $f(z) = A(z - z_1)(z - z_2) \cdots (z - z_r)$, this special case is identical with the Gauss-Lucas theorem for the zeros of the derivative of a polynomial.

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* See Marden, Transactions of this Society, loc. cit.