

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

123. Professor G. A. Bliss and Dr. M. R. Hestenes: *A note on the problem of Bolza in the calculus of variations.*

In this paper it is shown by a simple reapplication of the first necessary condition on a minimizing arc for the problem of Bolza that for every such arc there exists a function $F = \lambda_0 f + \lambda_\alpha \phi_\alpha$ for which $F - y'_i F_{y'_i}$ and F_x satisfy a Du Bois Reymond equation similar to those satisfied by the functions $F_{y'_i}$ and $F_{y''_i}$. These equations are dependent for a minimizing arc which is normal and non-singular. For other arcs no proof has as yet been given that this is so. Consequences of these results are given. It is shown that for a normal minimizing arc the continuity of $F - y'_i F_{y'_i}$ is a consequence of the continuity of $F_{y'_i}$ and the necessary condition of Weierstrass. (Received March 22, 1933.)

124. Dr. M. R. Hestenes: *Sufficient conditions for the problem of Lagrange in the calculus of variations.*

In this paper a set of sufficient conditions for the problem of Lagrange with fixed end points is given in which it is assumed that the arc E_{12} under consideration is normal on the interval $x_1 x_2$ but no assumption is made regarding normality on the sub-intervals of E_{12} . This result is attained by replacing the usual condition that E_{12} contains no point conjugate to the point 1 by a new condition which states that a certain quadratic form is positive on the interval $x_1 x_2$ and positive definite at one value of x on $x_1 x_2$. This new condition is a consequence of the theory of broken extremals applied to the problem of the second variation. It is closely related to a condition given by Bliss for variable end point problems in the plane and also the fundamental quadratic form of Morse. The relations between the sufficient conditions here given and known sufficient conditions are discussed. In particular an example is given in which the minimizing arc E_{12} satisfies the conditions of this paper but which is not normal on every sub-interval $x_1 x_3$ nor on every subinterval $x_3 x_2$ and hence does not satisfy the sufficient conditions given heretofore. (Received April 8, 1933.)

125. Professor Oswald Veblen: *Geometry of two-component spinors.*

This paper shows how a theory of two-component spinors can be derived from the geometry of the light-cones in the tangent spaces of the space-time

manifold of general relativity. It is to be published in the Proceedings of the National Academy of Sciences for April, 1933. (Received April 15, 1933.)

126. Professor R. E. Langer: *An inverse problem in differential equations.*

In an investigation (by Professor L. B. Slichter of the Massachusetts Institute of Technology) of the electrical resistivity of the earth's crust at inaccessible depths, an electrode is made to supply a current to the earth at its surface. The electrical potentials resulting at the surface in the surrounding region are observable, and constitute the sole obtainable data. From them the conductivity of the earth as a function of the depth is to be computed. The problem leads to the following formulation. In the equation $\{d(\sigma(x)dy/dx)\}/dx - \lambda^2\sigma(x)y=0$, λ is a parameter, $0 < \lambda < \infty$, and $\sigma(x)$ a positive, differentiable, but unknown function. If $y_1(x, \lambda)$ is the solution of the equation which vanishes at $x = \infty$, then $k(\lambda) \equiv [y_1(x, \lambda)/\{d(y_1(x, \lambda))/dx\}]_{x=0}$ is a known function of λ . From it $\sigma(x)$ is to be computed. In order to do this, the equation is transformed into $d(\log \sigma(x))/dx = \{v'(x, \lambda)+1\}/\{v(x, \lambda)\} - \lambda^2v(x, \lambda)$. If one sets $v(x, \lambda) = -1(\sum_{j=0}^{\infty} a_j(x)/\lambda^j)/\lambda$, and substitutes this in the equation, the condition that the result be independent of λ , together with $v(0, \lambda) \equiv k(\lambda)$, yields a system of recurrence relations from which the Taylor's series for $\sigma(x)$ may be computed. (Received March 22, 1933.)

127. Professor I. S. Sokolnikoff and Dr. E. S. Sokolnikoff: *On a resolution of linear differential systems.*

A problem of resolving a linear differential system into a product of two or more systems of lower order is considered in this paper. The fact that such a resolution is not always possible, even when the given differential operator is factorable, is obvious. This paper furnishes the necessary and sufficient conditions for resolving the system into two equivalent systems of lower order when the given differential operator is decomposable into two or more factors. The results are specialized for the case of a differential system of second order, and some examples of resolution of differential systems are given. (Received March 21, 1933.)

128. Dr. H. P. Thielman: *On the invariance of a generalized Gramian under the group of linear functional transformations of the third kind.*

In this paper a Riemannian function space is considered (A. D. Michal, American Journal of Mathematics, vol. 50, p. 516) in which the inner product of two vectors $y_1(s)$, $y_2(s)$ is defined in the following way: $(y_1y_2)_g = \int_a^b \int_a^b g(\alpha, \beta) \cdot y_1(\alpha)y_2(\beta)d\alpha d\beta + \int_a^b g(\alpha)y_1(\alpha)y_2(\alpha)d\alpha$, $g(\alpha, \beta) = g(\beta, \alpha)$, $g(\alpha) \neq 0$ in (a, b) , and $(yy)_g > 0$ for all real, continuous functions which do not vanish identically in (a, b) . In such a space the volume of a parallelepiped made by the vectors $y_1(s)$, $y_2(s)$, \dots , $y_n(s)$ having a common origin is defined by means of the determinant $|(y_iy_j)_g|$ whose general element is $(y_iy_j)_g$, $i, j = 1, 2, \dots, n$. The following theorem is then proved: *A necessary and sufficient condition that the generalized Gramian $|(y_iy_j)_g|$ ($i, j = 1, 2, \dots, n$) be invariant under the group of*

linear functional transformations: $y(s) = K(s)\bar{y}(s) + \int_a^b K(s, \alpha)\bar{y}(\alpha)d\alpha$, $K(s) \neq 0$ in (a, b) , is that the group be the group of motion, i.e. that it leave $(y_i y_i)_0$ invariant. If $g(\alpha, \beta) \equiv 0$, $g(\alpha) \equiv 1$, a well known result is obtained (I. A. Barnett, *Annals of Mathematics*, vol. 25, p. 220, T. S. Peterson, *American Journal of Mathematics*, vol. 51, p. 427). (Received March 22, 1933.)

129. Professor Theodore Bennett: *Note on (1, 2) algebraic correspondences.*

If in a (1, 2) algebraic correspondence between the points of two planes a point P corresponds to a single point P' and a point P' corresponds to two points P_1, P_2 , then the locus of points P' for which the corresponding points P_1, P_2 coincide is called the *branch locus*. It has been stated repeatedly in the literature that the necessary and sufficient condition that the image points P_1, P_2 of a variable point P' describe separate loci is that the locus of P' be tangent to the branch locus at every common point except the fundamental points. In this note an example is constructed which proves that this condition is not sufficient. Similar examples can be constructed in projective spaces of any number of dimensions. (Received March 22, 1933.)

130. Professor Arnold Emch: *On an involutorial Cremona transformation in S_r .*

In the study of the symmetric involutorial (n, n) -correspondence and its application to problems of closure an interesting problem concerning an involutorial Cremona transformation in S_r presented itself. Let r hypercones of order two be given in S_r . Now r generic tangent hyperplanes, one for each hypercone, intersect in a point $P(\lambda_1, \lambda_2, \dots, \lambda_r)$, in which the λ 's are the parameters associated with these cones and tangent hyperplanes. From P , r other hyperplanes with the parameters $\lambda'_1, \lambda'_2, \dots, \lambda'_r$ may be drawn. Thus the set (λ') is involutorially connected with the set (λ) . The object of the paper is to investigate this involution, with particular reference to the cases $r=2$, and $r=3$. (Received March 24, 1933.)

131. Professor Arnold Emch: *Symmetric monoids and cones of higher order in S_3 .*

If $\phi_1 = \Sigma x_i$, $\phi_2 = \Sigma x_i x_k$, $\phi_3 = \Sigma x_i x_j x_k$, $\phi_4 = x_1 x_2 x_3 x_4$, then $\Sigma k_i \phi_1^{m_1} \cdot \phi_2^{2m_2} \cdot \phi_3^{3m_3} \cdot \phi_4^{4m_4} = 0$, $m_1 + 2m_2 + 3m_3 + 4m_4 = n$, represents a symmetric n -ic in S_3 . The number of linearly available constants depends upon the solution of the indicated partition problem in number theory. The author's paper is a continuation of previous studies of geometric forms which are invariant in symmetric substitution groups. The immediate objects are the symmetric monoids and the symmetric cones. As an example may be mentioned a pencil of symmetric sextic cones which has the property, that *every proper quadric of a symmetric pencil cuts every sextic cone of the symmetric pencil of cones in a space curve which breaks up into two space cubics*. (Received March 24, 1933.)

132. Mr. J. R. Mayor: *A generalization of the Steiner and Veronese surfaces.*

The linear system of all hyperquadrics in S_r enables one to map the points of S_r upon a locus $G_r^{2^r}$ in S_R , $R=r(r+3)/2$, such that the hyperplanes of S_R correspond to the hyperquadrics of S_r . Any locus of dimension r in a space S_k , $k < R$, on which the points of S_r are mapped by quadratic forms may be obtained as a projection of the variety $G_r^{2^r}$ in S_R . For $r=2$, $G_r^{2^r}$ is the Veronese surface in S_5 . The projection of $G_r^{2^r}$ upon an S_{r+1} from a general $S_{(r^2+r-4)/2}$ which does not meet it may be considered as a generalization of the Steiner surface. By making a special choice of the space of projection, a hypersurface is obtained in S_{r+1} upon which the points of S_r are mapped by the squares of linear forms; the properties of this hypersurface are more closely analogous to those of the Steiner surface. (Received March 22, 1933.)

133. Professor V. G. Grove: *Contributions to the theory of transformations of nets in a space S_n .*

In this paper, the transformations C and E in a euclidean space of $n \geq 3$ dimensions are studied. The relation between a congruence and the curves of a C net is studied. In particular if the lines of a congruence are perpendicular to the tangents of one (only) of the families of curves of the net N in which the developables of the congruence intersect a surface, the lines of the congruence enjoy the same property with respect to any E transform of the net. A transformation E , which is also radial, is a transformation by reciprocal radii and conversely. The theorems developed are independent of the dimension of the space in which the configurations are immersed. (Received March 9, 1933.)

134. Dr. C. B. Morrey, Jr. (National Research Fellow): *The topology of (path) surfaces.*

A hemicactoid shall be a continuous curve in 3-space consisting of a plane continuous curve without complementary domains plus a finite or denumerable set of cactoids each having exactly one point in common with the plane set and no other points in common with the other cactoids; the simple closed surfaces involved may each be taken analytic except for two vertices. It is shown that: (1) given any hemicactoid, H , and a continuous vector function, $X(U)$, defined thereon, there exists a continuous (vector) function, $U(u)$, carrying a Jordan region, \bar{r} , into H , such that the set of points of \bar{r} corresponding to each point of H is a continuum; (2) to each such function corresponds a surface, $x=x(u)=X[U(u)]$, all these surfaces being identical (Fréchet)—thus we may represent surfaces on hemicactoids; (3) every surface can be represented on some hemicactoid, no continuum being carried into a point. Finally (a) the relations between different representations of the same surface are discussed; (b) a criterion, in terms of its given representation on a Jordan region, that a surface be non-degenerate (author's abstract, January 1933) is developed; and (c) every saddle surface bounded by a Jordan curve is shown to be non-degenerate. (Received March 24, 1933.)

135. Dr. C. B. Morrey, Jr. (National Research Fellow): *An analytic characterization of surfaces of finite Lebesgue area (II).*

In abstract No. 64 of the January issue of this Bulletin, the author characterized non-degenerate surfaces of finite Lebesgue area. In the present paper,

the results are extended to arbitrary surfaces of finite area. First of all, it is shown that if S is represented on a hemicactoid (see the preceding abstract), $L(S)$ is the sum of the areas of the portions of S corresponding to the non-degenerate simple links of H (i.e., the Jordan regions of the plane set or the simple closed surfaces of the cactoids); the part of S corresponding to a portion of H containing no point of a non-degenerate simple link contributes nothing to $L(S)$ even though this portion may fill a cube. It is then shown that if S is of finite Lebesgue area, there exists a hemicactoid, H , each of whose simple closed surfaces is analytic except for two vertices, on which S can be represented continuously, the representation possessing the additional property that the part of S corresponding to any non-degenerate simple link of H is represented generalized conformally thereon, the area of that portion being given by the usual formula. (Received March 24, 1933.)

136. Dr. Gordon Pall: *Applications of the automorphs of sums of three squares to regularity of ternary quadratic forms.*

The forms of the rational automorphs of denominator p of $x^2+y^2+z^2$ are used to prove the regularity (hitherto in doubt) of certain positive ternary quadratic forms. (Received March 20, 1933.)

137. Dr. Gordon Pall: *On the relations between sums of three and four squares.*

There is established a precise correspondence between the representation of a prime p as a sum of four squares, the solutions of the congruence $x^2+y^2+z^2 \equiv 0 \pmod{p}$, and the rational automorphs of denominator p of a sum of three squares. (Received March 20, 1933.)

138. Dr. Gordon Pall: *Towards a unification of the arithmetic of quadratic forms.*

It is generally agreed that the use of Kronecker binary quadratic forms leads to a theory of greater simplicity than that of the Gauss forms with their distinction between properly and improperly primitive forms. In this paper the various g.c.d.'s, invariants, and concomitants of any quadratic form in s variables ($s \geq 2$) are so defined as to secure much greater simplicity in the general arithmetical theory of quadratic forms and to obviate the necessity of considering separately proper and improper classes or concomitant classes. (Received March 20, 1933.)

139. Professor A. A. Albert: *On universal sets of positive ternary quadratic forms.*

It is well known that no positive ternary quadratic form represents all positive integers. Hence a theory of universal positive ternaries cannot exist. In the present paper the author introduces the notion of universal sets of forms (such that every positive integer is represented by at least one form of the set) and chains of forms (universal sets with the property that the omission of a single form renders the set not universal). The chains of forms are to replace universal forms in the theory. In particular, chains of *sets of forms* are consid-

ered, each set consisting of *all* forms of a given determinant, and all such chains are completely determined. Every such chain is shown to consist of exactly two sets and moreover the author determines in this case precisely what integers are not represented by sets of *sets of forms* which are not universal. (Received March 22, 1933.)

140. Professor A. A. Albert: *A note on the Dickson theorem on universal ternaries.*

L. E. Dickson has proved (in his *Studies in the Theory of Numbers*, pp. 17–21) his theorem that every universal ternary quadratic form is a zero form. His proof is long, complicated, and very technical, and contains a considerable division into special cases. In the present note the author gives a short proof of Dickson's theorem. In fact a rational proof is given and it is shown that a non-singular ternary quadratic form with integer coefficients is a zero form *if and only if* it represents all integers for rational values of its variables. (Received March 22, 1933.)

141. Professor Dunham Jackson: *The convergence of series of orthogonal trigonometric sums.*

Some further theorems are obtained with regard to the convergence of the series discussed in a paper presented at the Los Angeles meeting of the Society, (Abstract No. 38–7–191). (Received March 24, 1933.)

142. Professor C. N. Moore: *On criteria for Fourier constants of L integrable functions.*

In a note published in the Proceedings of the National Academy of Sciences in May, 1932, the author obtained certain criteria which would assist in determining whether or not a set of constants $a_n (n=0, 1, 2, \dots; a_n \rightarrow 0)$ were the Fourier constants of an L integrable function. In the present paper a more general criterion is obtained for the case where the constants do correspond to an integrable function. The added condition on the set of constants a_n is that the series $\sum n^r |\Delta^{r+1} a_n|$ should converge, where $0 < r < 1$ and $\Delta^{r+1} a_n$ represents the $(r+1)$ st difference of the a 's. This last criterion also includes a criterion due to Kolmogoroff (Bulletin de l'Académie Polonaise, 1922–23), which came to the attention of the author subsequent to the writing of the note referred to above. (Received March 22, 1933.)

143. Dr. Wilhelm Maier: *Lamé's functions and Lucas' numbers.*

Assume $r=0, 1, 2, \dots$ a parameter; $S_r(u)=S_r$ an analytic function, $\partial S_r / \partial u = \sigma_r$. The system of three non linear differential equations $S_{r-1} S_r + S_{r+1} = 0$; $S_r = S_{r+3}$ was treated by Euler, Jacobi and Poinso. Let $n=2, 3, 4, \dots$; and $\partial \alpha / \partial u = 0$; for appropriate parameters α_r , solutions of $S_{r-1} S_r + S_{r+1} + \alpha_r S_{r+1} = 0$; $S_r = S_{r+n}$ depend on Lamé's functions, involving their arithmetic properties. The classification of algebraic problems which arise leads to Lucas' numbers, namely to diophantine properties of $5^{1/2}$. (Received April 6, 1933.)

144. Professor Morris Marden: *Further mean-value theorems.*

The two principal theorems of the present note unite the results previously obtained by the author on *A generalization of Weierstrass' and Fekete's mean-value theorems* and *on the zeros of certain rational functions* (this Bulletin, vol. 38 (1932), pp. 434-441; Transactions of this Society, vol. 32 (1930), pp. 658-668). Among the new corollaries of the theorems are the following. *If, on the rectifiable curve $C: z=z(s)$, $g(z)$ is real, the point σ , $\sigma = (\int_a^b g(z)f(z)ds)/(\int_a^b |g(z)|ds)$, lies in the smallest convex region containing the two curves into which $w = \pm f(z)$ map C . If, in particular, $f(z)$ is a polynomial of degree n , it assumes this value σ at least once in the region consisting of all points from which C subtends an angle not less than $\pi/(2n)$.* The proofs of the two theorems follow methods of the above-mentioned articles. (Received March 23, 1933.)

145. Professor H. S. Wall: *On continued fractions which represent meromorphic functions.*

A sufficient condition (found by Van Vleck) that an infinite continued fraction (1) $[b_n z/1]^{\infty}$ shall represent a meromorphic function of z is that $\lim b_n = 0$. The condition is necessary if the b_n are real and positive. On p. 309 of vol. 4 (1903) of the Transactions of this Society, Van Vleck stated that the condition is necessary when the b_n are real and $b_{2n}b_{2n+1} > 0$. He proved that the roots of the denominator D_n of the n th convergent of (1) are real and simple, and that if $\dots < x_{-3} < x_{-2} < x_{-1} < 0 < x_1 < x_2 < x_3 < \dots$ are the roots of D_{2n+1} , and those with increasing n approach distinct limiting positions not 0, then $\lim b_n = 0$ and (1) is meromorphic. He thought, and so stated (without attempting proof) that the hypothesis of this theorem is fulfilled whenever (1) is meromorphic. The author supplements the above theorem by showing that (1) can be meromorphic when $\limsup |b_n|$ is > 0 , or ∞ , and the b_n real, $b_{2n}b_{2n+1} > 0$; and gives sufficient conditions applicable when the b_n are arbitrary real or complex numbers not 0. (Received March 16, 1933.)

146. Dr. P. W. Ketchum: *Expansions of two arbitrary analytic functions in series of rational functions.*

The particular series $\Sigma(\alpha_n + \beta_n x)F_n(x)$, where $F_n(x) = x^n/(1-x^{2n+1})$, converges to analytic functions inside a circle C of radius $R \leq 1$ and outside a circle C' of radius $1/R$, and diverges between these circles and the unit circle. R is the smallest of the reciprocals of the superior limits of the n th roots of $|\alpha_n|$, $|\beta_n|$, and 1. Given any analytic function $f_1(x)$ one of whose singularities nearest to zero is at γ , and another function $f_2(x)$ one of whose farthest singularities is at γ' , the α 's and β 's may be chosen so that the above series converges to $f_1(x)$ inside C and to $f_2(x)$ outside C' , where the corresponding R is the smallest of $|\gamma|$, $1/|\gamma'|$ and 1. In general the coefficients α_n and β_n may be found in terms of certain contour integrals. Other analytic functions $F_n(x)$ satisfying the equation $F_n(1/x) = \pm xF_n(x)$ may be used, but the character of the convergence is in general not so simple. (Received March 22, 1933.)

147. Dr. E. S. Akeley: *An invariantive characterization of Riemannian geometries.*

Let $(\lambda_1 \cdots \lambda_n)$ be the characteristic values of λ for the pencil of forms $(G_{\mu\nu} - \lambda g_{\mu\nu})dx_\mu dx_\nu$ where $g_{\mu\nu}$ is a positive definite quadratic differential form in n dimensions, and $G_{\mu\nu}$ is its associated contracted Riemannian-Christoffel tensor. If $\partial(\lambda_1 \cdots \lambda_n)/\partial(x_1 \cdots x_n) \neq 0$, one can solve for the x 's in terms of the λ 's, and $(\lambda_1 \cdots \lambda_n)$ will form a canonical coordinate system. If $ds^2 = \bar{g}_{\alpha\beta} d\lambda_\alpha d\lambda_\beta$, the $\bar{g}_{\alpha\beta}$ form a set of n^2 invariants which completely characterize the geometry. Many of the known Riemannian geometries are degenerate in the sense that the above Jacobian is zero. This is always true for $n=2$, and for spherically and axially symmetric solutions in higher dimensions. In order to find examples of non-degenerate geometries, an expansion of the type $g_{\mu\nu} = \delta_{\mu\nu} + \epsilon \gamma_{\mu\nu} + \cdots$, $G_{\mu\nu} = \epsilon H_{\mu\nu} + \cdots$ where ϵ is a parameter was used. The $H_{\mu\nu}$ were furthermore restricted to be linear functions of the $\lambda_i(i)$. The following results were obtained: (1) $n=2$, no solutions exist, (2) $n=3$, the solutions are uniquely determined and have been found, (3) $n>3$, a certain class of solution has been found. An expansion of the more general type $g_{\mu\nu} = C_{\mu\nu} + \epsilon \gamma_{\mu\nu} + \cdots$ has been used, where the $C_{\mu\nu}$ is a constant positive definite matrix, and corresponding theorems for this case are being obtained. (Received April 18, 1933.)

148. Professor T. R. Hollcroft: *Characteristics of multiple curves and their residuals.*

In this paper, relations between the respective characteristics of the two component curves of the complete intersection of two algebraic surfaces are derived, when one of the curves is i_1 -fold on one surface and i_2 -fold on the other. Formulas are also found for the characteristics of a complete intersection curve of given multiplicities on both surfaces. (Received March 24, 1933.)

149. Dr. Amos Black: *Types of involutorial space transformations associated with certain rational curves. Composite basis elements.*

Given a pencil of cubic surfaces $|F_3|$ which contains a rational curve r of order $m(m=2, 3, 4, 5)$. Make the surfaces of the pencil and the points of r projective. A point P in space will uniquely determine a surface F_3 and a point O on r . The line PO cuts F_3 in a third point P' . Thus on each line of the complex of lines which meet r is a pair of points P, P' in involution. The residual basis curve γ of $|F_3|$ is of order $9-m$. The case in which γ is irreducible was discussed in the Transactions of this Society, vol. 34 (1932), pp. 795-810. It is the purpose of the present paper to discuss the involutions when γ is composite, and when the projectivity between $|F_3|$ and r is specialized so as to reduce the degree of the involution. (Received March 21, 1933.)

150. Dr. J. M. Clarkson: *Involutorial line transformations defined by Cremona plane involutions.*

Consider in plane α a Cremona involution of one of the four fundamental types, and in plane β a Cremona involution of the same or of some other type.

An arbitrary line (y) of space, having Plücker coordinates y_1, \dots, y_6 , meets α in A whose image by the involution in α is A' , and β in B whose image by the involution in β is B' . The line $(x) \equiv A'B'$ is the transform of (y) , and the Plücker coordinates x_i are functions of y_j of degree equal to the sum of the orders of the two involutions. The various combinations of the four types of Cremona plane involutions are considered and the invariants and singular elements of the line transformations discussed. (Received March 18, 1933.)

151. Dr. S. S. Cairns: *On the triangulation of regular loci.*

In an n -space (y_1, \dots, y_n) we call an r -cell *regular* if it is definable by a homeomorphism $y_i = y_i(u_1, \dots, u_r)$ with a convex polyhedral r -cell throughout some neighborhood of which the y 's are defined, of class C' , and possess some non-vanishing jacobian, $J(y_{i_1}, \dots, y_{i_r})$. An r -manifold, M_r , means a compact connected point set consisting of (1) *interior points*, M'_r , each with an r -cell for a neighborhood on M_r , and possibly (2) a *boundary*, B_{r-1} , any point, P , of which has for a neighborhood on M_r the closure of an r -cell with P on its boundary. A *completely regular r -manifold* is one satisfying the above definition read for regular r -cells, provided B_{r-1} is a finite set of distinct regular unbounded $(r-1)$ -manifolds. (This assumes the definition of "regular k -manifold," see below, for $k < r$.) A *regular r -locus*, L_r , means a finite set of completely regular k -manifolds ($k=0, \dots, r$), called *elements* of L_r , any two of which (M_i, M_j) are so related that if M_j contains an interior point of M_i , then M_i is an element of the boundary of M_j . A *regular r -manifold* means a regular r -locus which is a manifold. The author triangulates L_r , and various modifications thereof, thus extending the theorem of Abstract No. 38-11-257 (to appear in the Annals of Mathematics). (Received March 22, 1933.)

152. Professor H. M. Gehman: *Concerning the definition of limit point.*

In this paper it is shown that the elementary theorems concerning limit point, closed set, and connected set are true, not only for the usual definition of limit point, but for any definition of limit point which has the following three properties: (1) A limit point of a set M is a limit point of any set containing M . (2) A limit point of the sum of two sets having no points in common is a limit point of at least one of the sets. (3) No point is a limit point of a set consisting of a single point. Certain groups of theorems require only one or two of these properties. In particular, a large number of theorems depend upon the first two properties only, and hence it is possible to consider a definition of limit point in a space consisting of a finite number of points. (Received February 14, 1933.)

153. Dr. I. J. Schoenberg: *A problem of Klein concerning the real roots of an algebraic equation.*

In a paper entitled *Geometrisches zur Abzählung der reellen Wurzeln algebraischer Gleichungen*, *Gesammelte Werke*, vol. 2, pp. 198-208, Felix Klein initiated the problem of comparing the efficiency of various theorems (Budan-Fourier, Descartes-Jacobi, Newton-Sylvester) which give upper bounds for the

number of real roots of an algebraic equation within a given interval. Klein has shown geometrically that for equations of the second degree the upper bound given by the theorem of Descartes-Jacobi never exceeds the bound furnished by the theorem of Budan-Fourier (see H. Weber, *Lehrbuch der Algebra*, 2d edition, vol. 1, pp. 354-357). H. Weber asks (loc. cit., p. 357) if this is true for equations of any degree. In the present paper this question is answered in the affirmative by analytical methods. It is also shown that the method of Descartes-Jacobi is always more efficient than a method due to Laguerre (*Oeuvres*, vol. 1, p. 10). Incidentally, this investigation shows that the theorems of Budan-Fourier and Laguerre are simple corollaries of Descartes' rule of signs. This paper will appear in the *Mathematische Zeitschrift*. (Received March 17, 1933.)

154. Dr. M. F. Deuring: *On the zeros of certain zeta functions.*

Riemann conjectured that all roots of the zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, except the trivial ones at $-2, -4, \dots$, have the real part $\frac{1}{2}$. This hypothesis, not yet proved, has been extended to more general functions, as they arise from the theory of algebraic numbers. Such functions are

$$Z_d(s) = \sum_{n,m=-\infty}^{\infty} (n^2 + bnm + cm^2)^{-s},$$

where $x^2 + bxy + cy^2$ is a definite quadratic form with discriminant $-d = b^2 - 4c < 0$. These functions have many properties in common with $\zeta(s)$. For the functions $Z_d(s)$, the following theorem is proved in this paper: There exists a positive number c_0 , such that all roots of $Z_d(s)$ in the region $0 < t < d^{c_0}$ have the real part $\frac{1}{2}$. This is a sort of "approximation" of the Riemann hypothesis for these functions. (Received March 7, 1933.)

155. Mr. Casper Shanok: *Convex polyhedra and criteria for irreducibility.* Preliminary report.

This paper gives an application of Minkowski's theory of convex polyhedra to the construction of irreducibility criteria for polynomials in several variables. The theorems obtained generalize the results of Dumas on polynomials in one variable, and they contain as special cases various previously known theorems, e.g., a theorem by Glenn. The theory also gives a method of determining irreducibility in numerical cases through an analysis of the network obtained by the projection of the sides of the polyhedra on suitable planes. (Received March 23, 1933.)

156. Professor Edward Kasner: *Normal congruences of parabolas.*

The author discusses the conditions under which a congruence of ∞^2 parabolas, orthogonal to a given plane, constitute a normal congruence, that is, admit ∞^1 orthogonal surfaces. In addition, all *approximately* normal congruences (of various orders, in the sense defined by the author in his discussion of the converse of the theorem of Thomson and Tait, *Transactions of this Society*, vol. 11 (1910), p. 126, and the Princeton Colloquium Lectures) are determined explicitly, and appropriate geometric constructions are obtained. It turns out that when the approximation is of the third order, the congruence

is exactly normal. Each parabola cuts the given plane in two points and thus a definite transformation T is induced in the plane. A relation of T to Darboux transformations is obtained, and thereby connection is made with the theory of natural families of curves. Finally an analogue to Ribaucour's theorem (on normal congruences of circles) is derived in connection with the problem of finding all normal congruences in the system of ∞^5 vertical parabolas (see Princeton Colloquium Lectures, p. 64). (Received March 23, 1933.)

157. Mr. H. S. Grant: *Concerning powers of certain classes of ideals in a cyclotomic realm which give the principal class.*

It is a familiar theorem that if a prime $q \equiv 1, \text{ mod } n$, then, in the cyclotomic number-realm determined by the n th roots of unity, the principal ideal $[q]$ is expressible as the product of $\phi(n)$ prime ideals of the first degree. Several results are known concerning the products of powers of these ideals which give the principal class. Results of this kind are obtained, for n a prime, by use of the Lagrange resolvent function and the associated Jacobi Ψ -function, and for n composite by use of their generalizations as given by Kummer and Stickelberger. The main purpose of this paper is to show, for n the power of an odd prime, that no matter how q^H be represented as the product of two conjugate imaginary ideal factors that are relatively prime, these factors are principal ideals, H being the "first-factor" of the class number of the realm. The method used is an extension of that employed by H. H. Mitchell (Transactions of this Society, 1918) for the case of an imaginary subfield of the cyclotomic field determined by the p th roots of unity, p an odd prime. It involves the use of the group-determinants of Frobenius (Berlin Berichte, 1896-1903). (Received March 23, 1933.)

158. Mr. W. S. Claytor: *Topological immersion of Peanian continua in a spherical surface.*

The author solves completely the problem of finding necessary and sufficient conditions under which a Peanian continuum will be topologically contained in a spherical surface. Kuratowski has handled the special case of this problem in which the Peanian continua contain at most a *finite* number of simple closed curves (Fundamenta Mathematicae, vol. 15, p. 271). In addition the present author obtains new, and very closely related, characterizations of both the simple closed surface and the closed 2-cell. (Received April 14, 1933.)

159. Dr. I. J. Schoenberg: *Convex domains and linear combinations of continuous functions.*

The object of this note is to give a simple geometric proof and a generalization of a theorem of L. L. Dines (Transactions of this Society, vol. 30 (1928), pp. 425-438) and some results of J. Favard (Bulletin de la Société Mathématique de France, vol. 59 (1931), pp. 229-255). One of the theorems obtained is as follows. Let $\phi_0(x) = 1, \phi_1(x), \phi_2(x), \dots$ be a sequence of real continuous functions for $a \leq x \leq b$ ($-\infty < a < b < +\infty$), and $c_0 > 0, c_1, c_2, \dots$ a given sequence of real constants. Every linear combination $\Phi(x) = a_0\phi_0(x) + a_1\phi_1(x) + \dots + a_n\phi_n(x)$, whose coefficients satisfy the relation $a_0c_0 + a_1c_1 + \dots$

$+a_n c_n = 0$, will vanish at some point of $[a, b]$, if and only if the system $\int_a^b \phi_k(x) d\psi(x) = c_k (k=0, 1, 2, \dots)$ admits a monotonic solution $\psi(x)$. This paper will appear in this Bulletin, in an early issue. (Received March 17, 1933.)

160. Dr. G. T. Whyburn: *Decompositions of continua by means of local separating points.*

If X is a subcontinuum of a continuum M , $L(X)$ will denote the set of local separating points of X . For each $p \in M$, there exist maximal subcontinua $B(p)$, $C(p)$, $D(p)$, and (for suitably restricted M) $E(p)$ containing p and such that $L[B(p)]$ and $L(M) \cdot C(p)$ are countable, $L[E(p)]$ and $L(M) \cdot D(p)$ are punctiform. No point belongs to two different sets B , C , D , or E . Thus we have decompositions of M into disjoint continua. The decompositions into sets $C(p)$ and $D(p)$ are upper semi-continuous; the hyperspaces C and D are regular curves; every subcontinuum of C contains uncountably many points of both $L(C)$ and $L(M)$; for no subcontinuum X of D is $X \cdot L(D)$ punctiform; thus no cyclic element of D has a continuum of condensation. $B(p)$ is the sum of all totally imperfect connected subsets containing p . If M is hereditarily locally connected, $E(p)$ exists and is the sum of all punctiform connected subsets containing p ; in this case also the decompositions into sets $B(p)$ and $E(p)$ are upper semi-continuous; the hyperspaces B and E are regular curves, for each subcontinuum X of B , $L(X)$ is uncountable; B contains no totally imperfect connected set, for no subcontinuum X of E is $L(X)$ punctiform; hence E contains no punctiform connected set. (Received March 6, 1933.)

161. Professor Leonard Carlitz: *On a certain function connected with polynomials in a Galois field.*

The zeros of the function discussed in this paper are the set of polynomials in a single indeterminate over a fixed Galois field. Some of the properties of this function are developed, and applications are made to (1) the evaluation of sums of the type $\sum F^{-k}$ (summed over polynomials F); and (2) to two classes of congruences to a polynomial modulus. (Received March 23, 1933.)

162. Mr. R. H. Cameron: *New necessary and sufficient conditions that a transformation be almost periodic.*

In this paper a theorem is proved which enables one to tell whether a given transformation is almost periodic without first determining whether or not it possesses a single-valued inverse. (Received March 3, 1933.)

163. Dr. Hassler Whitney (National Research Fellow): *Differentiable functions defined on closed sets.*

Let A be a closed set in n -space E , let $f_{k_1 \dots k_n}(x_1, \dots, x_n)$ be defined on A for $k_1 + \dots + k_n \leq m$, and suppose that $f_{0 \dots 0}(x_1 + h_1, \dots, x_n + h_n) = \sum f_{k_1 \dots k_n}(x_1, \dots, x_n) h_1^{k_1} \dots h_n^{k_n} / (k_1! \dots k_n!) + S(x_1, \dots, x_n; h_1, \dots, h_n)$ in A , where for every bounded subset B of A and $\epsilon > 0$ there is a $\delta > 0$ such that $|S(x_1, \dots, x_n; h_1, \dots, h_n)| < (\sum h_i^2)^{m/2} \epsilon$ for (x_1, \dots, x_n) in B if $(\sum h_i^2)^{1/2} < \delta$. Then if A satisfies certain conditions, for instance, if A is a closed domain with

a smooth boundary, we can extend the definition of the $f_{k_1 \dots k_n}(x_1, \dots, x_n)$ throughout E so that $(\partial/\partial x_i)f_{k_1 \dots k_n} = f_{k_1 \dots k_{i+1} \dots k_n}$ for $k_1 + \dots + k_n < m$. This is a strengthening of a previous result of the author (see this Bulletin, January, 1933, p. 31). (Received March 22, 1933.)

164. Professor I. M. Sheffer: *Certain linear functional equations, and their relation to a regular singular point of a differential equation.*

Taking the operator (1) $J[y] = \sum_1^\infty a_n y^{(n)}(x)$ as fundamental element, we form the new operator (2) $L[y] \equiv \sum_0^\infty L_n(x) J^{(n)}[y]$, where $J^{(n)}[y] = J[J^{(n-1)}[y]]$, and $L_n(x)$ is a polynomial of degree $\leq n$. The corresponding equation (3) $L[y] = \lambda y$ has a sequence of characteristic numbers λ_n for each of which there is a polynomial solution. We have elsewhere introduced the notion of *dual equation*. The dual of (3) is a linear differential equation (usually of infinite order). Let $H[y] = 0$ be a linear differential equation with either a regular singular point or an ordinary point at the origin. If we write (4) $H[y] = \lambda y$, an equation of type (3) is determined, of which (4) is the dual, and it is shown that the number of solutions of (4), analytic at the origin, is related to the number of polynomial solutions of (3), and the characteristic numbers of (3) and (4) are the same. (Received March 22, 1933.)

165. Dr. Jacob Sherman: *On the numerators of the convergents of the Stieltjes continued fractions.*

In this paper, which extends some of the results obtained by the author in a previous one (Transactions of this Society, vol. 35 (1933), pp. 65-84) are given (1) an explicit expression for the characteristic function $p_1(x)$ for the numerators of the convergents of the continued fraction "associated" with the classical orthogonal polynomials (Jacobi, Laguerre, Hermite); (2) a second-order non-homogeneous differential equation satisfied by the numerators of the odd convergents of the continued fraction "corresponding" to the polynomials of Jacobi and Laguerre; (3) the evaluation (obtained in the course of the computation) of certain singular definite integrals of type $J(x) = \int_a^b [(y-a)^{\alpha-1}(b-y)^{\beta-1}/(x-y)] dy$, for example, where $\alpha + \beta = 1$, $\alpha, \beta > 0$, $(a, b) \equiv (-1, 1)$, $J(x) = \pi(-\cot(2n+1)\alpha\pi + i)(1+x)^{\alpha-1}(1-x)^{-\alpha}(-1 < x < 1)$, where n is chosen to obtain a particular α th root of -1 . (Received March 21, 1933.)

166. Dr. Clement Winston: *On the mechanical quadratures formulas related to Hermite and Laguerre polynomials.*

The author considers the coefficients $H_{i,n}$ in the mechanical quadratures formula $\int_a^\infty f(x)p(x)dx = \sum_{i=1}^n H_{i,n}f(x_{i,n}) + R_n(f)$, $H_{i,n} = \int_a^\infty \{ \phi_n(x)p(x)/[(x-x_{i,n})\phi_n'(x_{i,n})] \} dx$, $\phi_n(x_{i,n}) = 0$, where $\phi_n(x)$ ($n=0, 1, 2, \dots$) are the polynomials of Hermite and Laguerre with $p(x) = e^{-x^2}$, $e^{-x}x^{\alpha-1}$ ($\alpha > 0$), and $a = -\infty, 0$ respectively. In the case of Hermite polynomials it is shown, in a very simple way, that $K_n(x) = \sum_{i=0}^n \phi_i^2(x)$ is always increasing for $x > 0$, decreasing for $x < 0$, and hence is minimum at $x=0$. This yields at once upper and lower bounds for the coefficients $H_{i,n}$ (see an article by the author to appear soon in the Annals of Mathematics, where similar results have been ob-

tained in a different way). Moreover, the same considerations lead to the following interesting property of the coefficients: $\{H_{i,n}\}$ separate $\{H_{i,n+1}\}$. Similar results, although not so complete, are obtained for Laguerre polynomials. (Received March 20, 1933.)

167. Dr. Hillel Poritsky: *Application of analytic function theory to two-dimensional elasticity problems.*

In this paper analytic function theory is applied to the solution of two-dimensional elasticity problems that reduce to finding functions satisfying the equation $(1) \nabla^4 F = 0$, $\nabla^4 = (\partial^2/\partial x^2 + \partial^2/\partial y^2)^2$, and proper boundary conditions. The general solution of (1) in the form $F = R[f(z) + \bar{z}g(z)]$, where $z = x + iy$, $\bar{z} = x - iy$, f and g are analytic, and R denotes "the real part of," is employed. The boundary conditions are expressed in terms of f and g , and solutions are given for several boundaries. In particular, Green's functions and "reflections" of applied point forces in plane and circular boundaries are discussed. (Received March 24, 1933.)

168. Dr. Selby Robinson (National Research Fellow): *The χ^2 test of goodness of fit.*

In 1922 R. A. Fisher proposed a modification in the use of the χ^2 test in case the hypothesis being tested was secured by a method which involved the computation of c statistics from the data. A satisfactory proof of Fisher's theory was given by J. Neyman and E. S. Pearson in the special case in which the sample is assumed to be indefinitely large, the c statistics are efficient, and a third condition is also satisfied. The present paper deals with a coin tossing experiment in which there are seventy samples of 128 items each. Seventy χ^2 's were computed and found to be distributed in accordance with Fisher's theory. (Received March 23, 1933.)

169. Professor C. R. Adams: *On non-factorable transformations of double sequences.*

The matrix $\|a_{mnkl}\|$ ($m, n, k, l = 1, 2, 3, \dots$, $a_{mnkl} = 0$ for $k > m$ or $l > n$ or both) defines a transformation of double sequences (A) : $y_{mn} = \sum_{k,l=1}^{m,n} a_{mnkl} x_{kl}$, which is called factorable when and only when $a_{mnkl} = a_{mk} \cdot a_{nl}$ ($m, n, k, l = 1, 2, 3, \dots$). Löscher (Mathematische Zeitschrift, vol. 34 (1931), pp. 281-290) has proved a theorem which, after a slight extension by the present writer, may be given the following form: A factorable transformation (A) , regular for bounded sequences, is regular for all sequences having bounded A -transforms. Agnew (American Journal, vol. 54 (1932), pp. 648-656) has shown that such a transformation is regular for all sequences whose A -transforms are ultimately bounded (that is, $\limsup_{m,n \rightarrow \infty} |y_{mn}| < \infty$), thus enlarging the class to its maximum. The purpose of the present note is to show that *neither theorem remains valid when the hypothesis of factorability is deleted*. This is done by exhibiting the example $a_{mnkl} = \delta_{mn} \delta_{nl} / n$, $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$; $a_{mnkl} = 1/(mn)$ ($k = 2, 3, 4, \dots$; $m, n, l = 1, 2, 3, \dots$): the transformation (A) thus defined carries the null sequence $x_{kl} = l$, $x_{kl} = 0$ ($k = 2, 3, 4, \dots$; $l = 1, 2, 3, \dots$) into the sequence $y_{mn} = \delta_{mn}$, bounded but divergent. This example shows also that a non-

factorable transformation (A) may satisfy conditions far more stringent than those which are necessary and sufficient for regularity for bounded sequences and still not be regular for all sequences having bounded A -transforms. (Received April 14, 1933.)

170. Professor J. F. Ritt: *Integral functions obtained by compounding polynomials.*

Let $P_1(z), \dots, P_n(z)$ be a sequence of polynomials whose degrees do not exceed a fixed integer m and which, ordered in ascending powers of z , start with the term z . This paper considers the sequence of polynomials $Q_n(z)$ defined by $Q_1(z) = P_1(z)$; $Q_{n+1}(z) = Q_n[P_{n+1}(z)]$, $n = 1, 2, \dots$, and also the sequence of $R_n(z)$ defined by $R_1(z) = P_1(z)$; $R_{n+1}(z) = P_{n+1}[R_n(z)]$, $n = 1, 2, \dots$. Conditions are given under which these sequences converge to integral functions. (Received April 14, 1933.)

171. Dr. B. F. Kimball: *Application of Bernoulli polynomials of negative order to differencing.* Second paper.

The author extends the results obtained in his previous paper to the case of unequal difference intervals, real and complex. The following theorem is proved. Let w_1, w_2, \dots, w_n denote the n real positive difference intervals of $B_{2m}^{-n}(x, w_1 w_2 \dots w_n)$. Set $t_n = (w_1 + w_2 + \dots + w_n)/2$. Let $\sum^r w_k$ represent the sum of r difference intervals w_k taken from the above group, none being repeated. If r is an integer (or zero) such that $t_n - \sum^r w_k \geq 0$ for all possible sums $\sum^r w_k$ taken from the above group; then $B_{2m}^{-n}(-t_n, w_1 w_2 \dots w_n) \leq \{n(n-1) \dots (n-r)/[(n+2m)(n-1+2m) \dots (n-r+2m)]\} i^{2m}$, the equality sign not applying when $n > 2$. This theorem enables us to test the rapidity of convergence of a series expansion of a difference in terms of the Bernoulli polynomials. If all the difference intervals are pure imaginaries so that $\omega_k = i w_k$, $B_{2m}^{-n}(-\tau_n, \omega_1 \omega_2 \dots \omega_n) = (-1)^m B_{2m}^{-n}(-t_n, w_1 w_2 \dots w_n)$, $\tau_n = (\omega_1 + \omega_2 + \dots + \omega_n)/2$. In the case of complex difference intervals, the following relation simplifies matters: $|B_{2m}^{-n}(-\tau_n, \omega_1 \omega_2 \dots \omega_n)| \leq B_{2m}^{-n}(-t_n, w_1 w_2 \dots w_n)$, where $w_k = |\omega_k|$, $\tau_n = (\omega_1 + \omega_2 + \dots + \omega_n)/2$. (Received March 3, 1933.)

172. Professor J. A. Shohat: *On some applications of the Tchebycheff inequality for definite integrals.*

Consider Taylor's formula for a given function $f(x)$, and write its remainder in the well known form of a definite integral. To the latter, where under the integral sign we find a product of two functions, we apply the Tchebycheff inequality for definite integrals, assuming that a certain derivative $f^{(v)}(x)$ is monotonic in the interval under consideration. Thus we obtain various estimates of the said remainder leading to some interesting inequalities for the classical cases (e^x , $\sin x$, etc.). (Received March 7, 1933.)

173. Dr. S. S. Wilks (International Research Fellow): *A criterion for testing the hypothesis of mutual independence of a set of normal multivariate populations.*

A criterion of the Neyman-Pearson λ type is found by the method of maximum likelihood for testing the hypothesis H that an n -variate sample Σ of N

individuals comes from a population which is composed of a set of p mutually independent sub-populations $\{\pi_i\}$ when it is assumed that Σ has been drawn from some n -variate normal population. The criterion is expressible in the form $\lambda_H = D/\Pi_{i=1}^p d_i$, where D is the determinant of variances and covariances of all n variates of Σ and d_i is the principal minor of D of order n_i ($\Sigma_{i=1}^p n_i = n$) whose elements are the sample values of the variances and covariances belonging to π_i . An expression is found for the sampling distribution of λ_H when H is true, which yields relatively simple probability integrals for certain values of p and the n_i . The limiting form of the distribution of λ_H for large samples is shown to be a Pearson Type III frequency function. (Received March 21, 1933.)

174. Professor J. L. Walsh: *Line integrals as a measure of polynomial approximation to analytic functions.*

Let C be a point set of the z -plane not separating the plane but consisting of a finite number of rectifiable Jordan arcs and Jordan regions bounded by rectifiable curves, let $w = \phi(z)$ map (conformally but not necessarily uniformly) the complement of C onto $|w| > 1$ so that the points at infinity correspond to each other, and let C_ρ denote the curve or curves $|\phi(z)| = \rho > 1$. If $f(z)$ is analytic on C , there is a largest finite or infinite $R > 1$ such that $f(z)$ (or its analytic extension) is single-valued and analytic interior to C_R . Then any sequence of polynomials of best approximation to $f(z)$ in the sense of least p th powers ($p > 0$) measured by a line integral over the boundary of C with a positive continuous (this restriction may be lightened) norm function converges to $f(z)$ interior to C_R , uniformly on any closed point set interior to C_R , and converges uniformly in no region containing in its interior a point of C_R . (Received March 27, 1933.)

175. Dr. Stanislaw Saks: *Addition to the note "On some functionals."*

In the present note the author extends to completely additive functions of sets in abstract spaces some results of the paper mentioned in the title. (Received April 7, 1933.)

176. Mr. L. B. Robinson: *On equations in mixed differences.* Part IV.

Consider the determinant $|\Sigma_{i-j} + a_{ij}|$. The sum of the a 's is a converging double series. When $i-j$ is negative or positive and greater than m , Σ vanishes. When $i-j$ is zero, $\Sigma = 1$. In all other cases the sigmas represent the symmetric functions of the roots of a polynomial, all of whose roots are absolutely less than unity. Under the above assumptions, by addition and subtraction of rows or columns we can reduce the above determinant to a form whose convergence has been demonstrated by Poincaré. Consider now an equation previously studied by the author (see this Bulletin, vol. 32, p. 316). When all the a 's are inferior in absolute value to unity, then with the help of the above determinant we show that a general solution of this equation exists, that is, it depends on a variable parameter. If one or more of the a 's are greater in ab-

solute value than unity, the existence of a singular solution can be proved. Our equation is a normal form to which other equations are reducible. (Received April 1, 1933.)

177. Professor R. L. Jeffery: *Derivatives with respect to functions of bounded variation.*

In a previous paper by the author (Transactions of this Society, vol. 34, p. 645) the derived numbers of $F(x)$ with respect to a function of bounded variation $\alpha(x)$ were defined for functions F with discontinuities of the first kind only. The present paper obtains the distribution of the values of these derived numbers for any finite function F , whether it be measurable relative to α or not. Theorems on approximate derivatives with respect to α are obtained, also on λ -derivatives with respect to α analogous to those obtained by Burhill, Haslam-Jones, and Besicovitch (Proceedings of the London Mathematical Society, (2), vol. 32, No. 5) for ordinary derivatives. Further results are the following: a method for determining F when a derived number of F with respect to α is given and finite at each point; the necessary and sufficient conditions that F be an indefinite integral with respect to α , or a total indefinite integral with respect to α . (Received March 14, 1933.)

178. Professor C. R. Adams and Mr. J. A. Clarkson: *On definitions of bounded variation for functions of two variables.*

Several definitions have been given of conditions under which a function of two real variables shall be said to be of bounded variation. Of these definitions six are usually associated with the names of Vitali, Hardy, Arzelà, Pierpont, Fréchet, and Tonelli, respectively. A seventh, formulated by Hahn and attributed by him to Pierpont, we readily prove equivalent to Pierpont's definition. For convenience, let the classes of functions satisfying the respective definitions be denoted by V , H , A , P , F , and T ; in addition, let the class of continuous functions be designated by C , and let a product (such as $A \cdot T$) stand for the subclass of functions common to the two or more classes named. The fundamental relations of inclusiveness between these classes are as follows: (1) $P > A > H$, $F > V > H$, $T > H$. We easily show that any two classes not named in the same part of (1) overlap, and we investigate the extent of the common part of two or more classes (such as $A \cdot T$). The same questions are studied when only continuous functions are admitted to consideration, in which case we have, in contrast with (1), $T \cdot C > P \cdot C > A \cdot C > H \cdot C$, $F \cdot C > V \cdot C > H \cdot C$. (Received February 16, 1933.)

179. Professor J. A. Shohat: *On the transfinite diameter for a linear interval.*

It is known (Stieltjes) that the zeros of Jacobi polynomials $\phi_n(x; \alpha, \beta)$ possess certain extremal properties (resulting from the differential equation which they satisfy). We let α and β approach zero, and thus get an explicit expression for the polynomial of degree n , whose zeros maximize the absolute value of the Vandermonde determinant $D_n(x_1, \dots, x_n)$, the points x_i belonging to a finite interval (which, for $n \rightarrow \infty$, leads to the "transfinite diameter"

(Fekete) for the said interval). This expression is simpler and exhibits more plainly the properties of the said x_i than that given by I. Schur (Mathematische Zeitschrift, vol. 1 (1918)). We further show how these x_i can be applied to interpolation and mechanical quadratures, improving a result due to Fekete. (Received March 7, 1933.)

180. Professors Einar Hille and J. D. Tamarkin: *Questions of relative inclusion in the domain of Hausdorff matrices.*

In the present paper we investigate the question of when $[H, q_1(u)] \supset [H, q_2(u)]$, i.e., when a Hausdorff definition of summability includes another. In particular, we have found necessary and sufficient conditions for $[H, q(u)] \supset (C, \alpha)$ and for $[H, q(u)] \supset (E_p)$. It turns out that if these relations hold simultaneously, then $[H, q(u)] \supset (C, \alpha) (E_p)$ as well. We have found some sufficient conditions ensuring that an analytic function of a Hausdorff matrix be another such matrix. As a particular case we obtain a generalization of a theorem of I. Schur (Mathematische Annalen, vol. 74 (1913), pp. 453–456) for all Hausdorff matrices. In addition we have found a number of new proofs for already known inclusion theorems; thus, for instance, still another proof for the equivalence of the definitions of Cesàro and Hölder is obtained. (Received March 25, 1933.)

181. Professors Einar Hille and J. D. Tamarkin: *On the summability of Fourier series.* IV.

In the present paper we extend our investigations on the summability of Fourier series to the general class of "radial" matrices introduced by R. Schmidt. To each matrix A of this type there corresponds a "mass-function" $q_A(u)$ of bounded variation over $(0, \infty)$, such that $q_A(0) = 0$. For such matrices the question of their (F) -effectiveness is answered completely by the following theorem. In order that the matrix A should sum to $f(x)$ the Fourier series of an arbitrary integrable function $f(x)$, at each point x such that $f(x+t) + f(x-t) - 2f(x) \rightarrow 0$ as $t \rightarrow 0$, it is necessary and sufficient that the cosine Fourier transform of $q(u)$ be integrable over $(0, \infty)$. The continuity of $q(u)$ for $u > 0$ results as a necessary condition of the (F) -effectiveness. The class of matrices in question embraces practically all regular definitions of summability known in the literature and the method used allows us to decide with great ease the question of the (F) -effectiveness of various important transformations, such, for instance, as those of Hausdorff, Le Roy, de la Vallée Poussin, and others, which in many cases caused considerable difficulties. (Received March 25, 1933.)

182. Professor J. L. Walsh: *On the location of the critical points of Green's function for a plane region.*

Any finite number of mutually exterior Jordan curves C_k can be simultaneously uniformly approximated by a lemniscate. Green's function G_1 , with pole at infinity for the region C exterior to these curves, is thus approximated by Green's function G_2 with pole at infinity for the exterior of the lemniscate; G_2 is of the form $a \log |p(z)| + b$, where $p(z)$ is a polynomial. The critical points

of G_2 are the roots of the derivative of $p(z)$, and the roots of $p(z)$ can be chosen interior to the C_k . Well known theorems (Lucas, Jensen, Walsh) on the roots of the derivative of a polynomial thus lead to results on the critical points of G_1 . For example, if C is an infinite region whose boundary is limited, then all critical points of the (assumed existent) Green's function for C with pole at infinity lie in the smallest convex region containing the boundary of C . (Received April 15, 1933.)

183. Mr. R. L. Peek, Jr.: *Some new theorems on limits of variation.*

The theorems presented in this paper permit the evaluation of upper limits to the relative frequency of occurrence of members of a frequency distribution differing from the arithmetic mean by more than a specified amount. They are therefore similar to the Tchebycheff and Camp-Meidell inequalities, but differ from them in involving not only the standard deviation, but the mean deviation as well, and consequently afford, in general, much closer limits. A preliminary lemma shows that the standard deviation of the distribution of absolute deviations from the mean (σ_1), is equal to the square root of the difference between the squares of the standard and mean deviations ($\sigma_1^2 = \sigma^2 - \bar{x}_1^2$). If ρ is the ratio of the mean and standard deviations (\bar{x}_1/σ), it is shown that the relative frequencies of values differing from the mean by more than $t\sigma$ is less than $t\rho^2/t^2 - 2t\rho + 1$. For a continuous frequency distribution, subject to the same limitations as apply to the Camp-Meidell inequality, the relative frequency of values differing from the mean by more than $t\sigma$ is shown to be less than $4(1 - \rho^2)/9(t - \rho)^2$. A further theorem gives another expression for such upper limits in terms of the mean and standard deviations and certain additional parameters. (Received April 28, 1933.)

184. Professor W. D. Baten: *The probability law for the sum of n independent variables when each is subject to the law $(1/2h) \operatorname{sech}(\pi x/2h)$.*

The probability law for the sum of the n independent variables is found by Dodd's method, which leads to certain definite integrals involving the hyperbolic secant raised to the n th power. These definite integrals are evaluated for $n=2, 3, 4, 5, 6$ by integrating over certain contours in the complex plane and allowing two parallel boundaries to approach infinity. Mathematical induction is employed to evaluate the integrals for any values of n , the proof of which depends on proving that certain terms in a Laurent expansion are integers and on the evaluation of a certain definite integral, which is implied by ideas relating to ${}_nC_r$. An approximation for the law is obtained when n approaches infinity. Wallis' theorem is obtained as a by-product together with explicit expressions for the derivatives $d^{2s}(hw/\sin hw)^{2n}/dw^{2s}$ and $d^{2s}(w/\sin w)^{2n}/dw^{2s}$. (Received April 27, 1933.)

185. Professor M. S. Knebelman: *A canonical form for a set of vectors.*

In this paper it is shown how a coordinate system may be obtained in which the components of a given set of vectors take a particularly simple form. If $S(\leq n)$ independent contravariant vectors in a V_n are given, such that the completed set contains $r(\geq s, \leq n)$ independent ones, then a coordinate system may be found in which all components after the r th are zero and the K th components of the m th vector for $K > m$ vanishes on a given subspace V_{n-m+1} . If a complete set of r independent covariant vectors in V_n is given, then a coordinate system is obtained in which all components after the r th are zero, the remaining ones being functions of at most r variables such that the K th component of the m th vector for $K < m$ vanishes on a given subspace V_{n-m+1} . (Received April 27, 1933.)

186. Professor M. S. Knebelman: *On Finsler spaces.*

In a recent paper (Comptes Rendus, Feb., 1933) E. Cartan showed how the geometry of Finsler spaces may be brought more closely into line with Riemannian geometry by changing the law of parallelism. He points out the advantages of his definition of parallel displacement over that of L. Berwald. This note shows that the advantage is not entirely on the side of Cartan and that some of the tensor invariants obtained by him do not have the geometrical significance usually associated with them. (Received April 27, 1933.)

187. Mr. W. O. Gordon: *A neighborhood treatment of general topological spaces.*

In this paper it is shown that the most general topological space (in the sense of Fréchet) can be treated as a neighborhood space by using a definition of limit point in terms of neighborhood which is slightly different from the usual ones. Necessary and sufficient conditions on neighborhoods in order that the space have certain well known properties are then derived. (Received May 1, 1933.)

188. Dr. D. H. Lehmer: *On imaginary quadratic fields whose class number is unity.*

This paper gives the results of an investigation of the magnitude of the class number $h(\sqrt{-D})$ of the quadratic field $K(\sqrt{-D})$ with special reference to the cases in which $h=1$. Gauss noted that $h=1$ for $D=1, 2, 3, 7, 11, 19, 43, 67$, and 163, and conjectured that only a finite number of such D 's exist, after an examination of all D 's less than 3000. By showing that $x^2+x+(D+1)/4$ is composite for some $x < 15$, Dickson was able to prove that $h > 1$ for $163 < D < 1,500,000$. The present investigation enables one to make this assertion for $163 < D < 5,000,000,000$. The method differs from that of Dickson, Rainich, and Nagel, and depends on the calculation of the least number N_p of the form $8n+3$ for which $-N_p$ is a quadratic non-residue of all odd primes $\leq p$. The method not only shows that $h > 1$, but also gives a lower limit for h . The examination of this large range of values of D was made with the author's number sieve, which sifted the determinants at a rate of 100,000 a second. (Received May 6, 1933.)

189. Professor M. A. Basoco: *On the element of decomposition of a doubly periodic function of the second kind.*

The element of decomposition for a doubly periodic function of the second kind, in the sense of Hermite, may be written in terms of the Weierstrass sigma function in the form $\sigma(u+v)e^{\rho u}/\sigma(u)\sigma(v)$. It may be shown that it satisfies a differential equation of the Lamé type. For suitable values of ρ special functions of importance in the applications are obtained. In particular, the expansion in powers of u of this function for $\rho = x - \zeta(v)$, where $\zeta(v)$ is the Weierstrass zeta function, is discussed. The coefficients in the development are Appell polynomials in x , which satisfy a certain differential equation and can be constructed by recurrence. Certain other Appell polynomials associated with these are also obtained and discussed. (Received May 5, 1933.)

190. Mr. G. D. Nichols: *On the arithmetized Fourier developments of certain doubly periodic functions of the second kind.*

This paper derives the explicit arithmetized Fourier series expansions for the functions $\Phi_{\alpha\beta\gamma}(x, y) = \theta_{\alpha}^{13}[x+y]/\theta_{\beta}^3(x) \cdot \theta_{\gamma}^3(y)$, where the θ 's are the elliptic theta functions of Jacobi, x and y are independent complex variables, and (α, β, γ) are certain sixteen triads out of the possible sixty-four which can be selected from the numbers 0, 1, 2, 3. The method used follows closely that of M. A. Basoco (this Bulletin, vol. 38 (1932), p. 560) and is a generalization of that due to Teixeira (Journal für Mathematik, vol. 125 (1901), p. 301). The developments have applications in the theory of quadratic forms. The expressions $\theta_i^{12}(y)/\theta_i^2(y)$ and $\theta_i^{11}(y)/\theta_i^2(y)$ ($i=0, 1, 2, 3$) occur in the above developments. The explicit arithmetized expansions are also worked out for these, and it is believed they are an addition to the lists of such expansions given by Bell (Messenger of Mathematics, vol. 53 (1924), p. 166) and others. (Received May 4, 1933.)

191. Mr. Nathan Jacobson: *Non-commutative polynomials and cyclic algebras.*

Let $F'(x)$ denote the domain of polynomials in x with coefficients in a cyclic field $F' = F(a)$ of order $r > 1$, for which $xa = \bar{a}x$ where $a = \theta(a)$ gives a generating automorphism of the Galois group of F' over F . Consider the cyclic algebra $F(\Pi)$ as the algebra of residues of the polynomials of $F'[x]$ modulo $\Pi = x^r - \gamma$ ($\gamma \neq 0$). With any polynomial, P , associate a residue algebra $F(P)$ restricted to the sub-domain $F(P)$ of $F'(x)$ of polynomials A for which $PA = A'P$. If $\Pi = PP^*$, P irreducible, then $F(P)$ is isomorphic to the division algebra part of $F(\Pi)$ in the Wedderburn decomposition. If $\Pi = P_1P_2 \cdots P_s$ is a decomposition of Π into irreducible factors, each P has the same degree t and the order of the division algebra part of $F(\Pi)$ is t^2 . These results reduce the problem of determining the structure of $F(\Pi)$ to a factorization problem in $F'(x)$. They give immediately the theorem of Hasse, that a necessary and sufficient condition that $F(\Pi)$ be matric is that γ be the norm of an element of F' . From Wedderburn's norm conditions is obtained the generalization that if q is the smallest integer such that γ^q is the norm of an element of F' , then the division algebra part of $F(\Pi)$ has order $\geq q^2$. The theorems of Brauer concerning the exponent of a cyclic algebra then follow. (Received May 3, 1933.)