HAUSDORFF TRANSFORMATIONS FOR DOUBLE SEQUENCES

BY C. R. ADAMS

1. *Introduction.* The purpose of this note is to extend to double sequences some of the results of Hausdorff's notable papers on methods of summability and moment-sequences.†

Let $\lambda = \|\lambda_{pqmn}\|$, $(p, q, m, n = 0, 1, 2, \cdots)$, be a four-dimensional matrix of real or complex numbers. Then the system of equations

$$A_{pq} = \sum_{m,n=0}^{\infty} \lambda_{pqmn} a_{mn}, \tag{1}$$

if the series all converge, transforms a double sequence $\{a_{mn}\}$ into a new double sequence $\{A_{pq}\}$. Necessary and sufficient conditions‡ that this transformation be convergence-preserving for bounded sequences are the following:

$$\text{(A)} \qquad \sum_{m,n=0}^{\infty} |\lambda_{pqmn}| \leqq M, \qquad (p, q = 0, 1, 2, \cdots),$$

$$\text{(B)} \qquad \lim_{p,q\to\infty} \sum_{m,n=0}^{\infty} \lambda_{pqmn} = l,$$

$$\text{(C)} \qquad \lim_{p,q\to\infty} \lambda_{pqmn} = l_{mn}, \qquad (m, n = 0, 1, 2, \cdots),$$

$$\text{(D)} \qquad \lim_{p,q\to\infty} \sum_{m=0}^{\infty} |\lambda_{pqmn} - l_{mn}| = 0, \qquad (n = 0, 1, 2, \cdots),$$

† Hausdorff, *Summationsmethoden und Momentfolgen*, I, II, Mathematische Zeitschrift, vol. 9 (1921), pp. 74–109, 280–299.

‡ See Robison, *Divergent double sequences and series*, Transactions of this Society, vol. 28 (1926), pp. 50–73, especially pp. 71–72. Such transformations are sometimes, if not always, convergence-preserving for certain unbounded sequences; see Adams, *Transformations of double sequences, with application to Cesàro summability of double series*, this Bulletin, vol. 37 (1931), pp. 741–748; Lösch, *Über den Permanenzsatz gewisser Limitierungsverfahren für Doppelfolgen*, Mathematische Zeitschrift, vol. 34 (1931), pp. 281–290; Adams, *On summability of double series*, Transactions of this Society, vol. 34 (1932), pp. 215–230, hereafter referred to as *A*; and Agnew, *On summability of double sequences*, American Journal of Mathematics, vol. 54 (1932), pp. 648–656.

$$\text{(E)} \qquad \lim_{p,q\to\infty} \sum_{n=0}^{\infty} | \lambda_{pqmn} - l_{mn} | = 0, \quad (m = 0, 1, 2, \cdots).$$

A matrix λ satisfying these conditions we call, with Hausdorff, a *C-matrix*. Such a matrix defines a transformation (1) which carries a bounded sequence $\{a_{mn}\}$ convergent to α into a bounded sequence $\{A_{pq}\}$ convergent to

$$l\alpha + \sum_{m,n=0}^{\infty} l_{mn}(a_{mn} - \alpha),$$

this double series being always absolutely convergent. For a C-matrix to define a transformation regular for bounded *null* sequences it is necessary and sufficient that all the l_{mn} vanish; in such a case the C-matrix will be called *pure*. For a pure C-matrix to define a transformation regular for *all* bounded sequences it is necessary and sufficient that $l=1$; in this event the C-matrix will be called *normalized*.

Now let ρ be a *fixed* matrix having an inverse and $\mu = \|\mu_{mnkl}\|$ an *arbitrary "diagonal"* matrix; that is,

$$\mu_{mnkl} = 0,$$

for $k\neq m$ or $l\neq n$ or both, so that the only elements $\neq 0$ are $\mu_{mnmn}(m, n=0, 1, 2, \cdots)$. For simplicity we write

$$\mu_{mnmn} = \mu_{mn}.$$

Henceforth we restrict ourselves to matrices λ of the form†

$$\text{(2)} \qquad \lambda = \rho^{-1}\cdot\mu\cdot\rho,$$

which by means of the fixed ρ are transformable into diagonal matrices. It is seen at once that *any two such matrices are permutable*; moreover, the proof is immediate that if $\lambda^* = \rho^{-1}\cdot\mu^*\cdot\rho$ is one of the matrices (2) for which the corresponding diagonal matrix μ^* has elements μ^*_{mn} no two of which are equal, then *all* matrices λ permutable with λ^* are of the form (2).

The system of equations (1) may be written in matrix form as

$$\text{(3)} \qquad A = \lambda\cdot a,$$

where $a = \|a_{mnkl}\|$ and $A = \|A_{pqmn}\|$, with

† By a product such as $\lambda\cdot\mu$ we mean the matrix whose element with indices $p, q, k; l$ is $\sum_{m,n=0}^{\infty}\lambda_{pqmn}\mu_{mnkl}$.

$$a_{mnkl} = \begin{cases} a_{mn} \text{ for } k = l = 0, \\ 0 \text{ otherwise,} \end{cases} \qquad A_{pqmn} = \begin{cases} A_{pq} \text{ for } m = n = 0, \\ 0 \text{ otherwise.} \end{cases}$$

If we set $b = \rho \cdot a$, $B = \rho \cdot A$, the matrix equation (3) becomes

$$B = \mu \cdot b, \qquad \text{or} \qquad B_{mn} = \mu_{mn} b_{mn}, \tag{4}$$

a *multiplication*. To the matrix λ there corresponds a unique diagonal matrix μ, or factor sequence $\{\mu_{mn}\}$, and conversely. If λ is a C-matrix, we call $\{\mu_{mn}\}$ a *C-sequence*; a C-sequence will be said to be pure or normalized according as λ is pure or normalized.

2. *Difference Sequences*. From a double sequence $\{a_{mn}\}$ one may form the quadruple sequence of double differences of various orders,

$$\Delta_{ij} a_{mn} = \sum_{k,l=0}^{i,j} (-1)^{k+l} \binom{i}{k} \binom{j}{l} a_{m+k,n+l}. \tag{5}$$

The recursion formulas

$$\begin{aligned} \Delta_{ij} a_{mn} &= \Delta_{ij} a_{m+1,n} + \Delta_{i+1,j} a_{mn} \\ &= \Delta_{ij} a_{m,n+1} + \Delta_{i,j+1} a_{mn} \end{aligned} \tag{6}$$

may at once be derived. For brevity we call the double sequence of numbers

$$\Delta_{ij} a_{00} = b_{ij} = \Delta_{00} b_{ij}$$

the difference sequence of the a_{mn}, and from (6) readily follows the relation

$$\Delta_{ij} a_{mn} = \Delta_{mn} b_{ij},$$

so that the $a_{mn} = \Delta_{00}\, a_{mn}$ also constitute the difference sequence of the b_{ij}. The matrix associated with (5) is the matrix of the Euler transformation for double sequences,

$$E = E' \odot E',$$

where E' denotes the Euler matrix for simple sequence transformations and a notation employed in A is used.† By Theorem

† That is, if each element a_{mnkl} of a four-dimensional matrix a is equal to $a'_{mk} \cdot a'_{nl}$, where these factors are the elements of two-dimensional matrices a' and a'', we write $a = a' \odot a''$.

7 of A, E is its own inverse. We now choose E to be the fixed matrix ρ introduced above, and hereafter consider only such systems of equations (1) as have their difference sequences b_{mn}, B_{mn} in the multiplicative form (4).

We may express the equations (4) in terms of a_{mn}, A_{pq} and obtain without difficulty from $A = E \cdot B$ the relation

$$A_{pq} = \sum_{i,j=0}^{p,q} \binom{p}{i}\binom{q}{j}[\Delta_{p-i,q-j}\mu_{ij}]a_{ij}.$$

Setting all the $a_{ij} = 1$, we have

$$\sum_{i,j=0}^{p,q} \binom{p}{i}\binom{q}{j}\Delta_{p-i,q-j}\mu_{ij} = \Delta_{00}\mu_{00} = \mu_{00}. \tag{7}$$

Then the conditions (A)-(E) become

$$(\alpha) \qquad \sum_{i,j=0}^{p,q} \binom{p}{i}\binom{q}{j} | \Delta_{p-i,q-j}\mu_{ij} | = M_{pq} \leqq M, \qquad (p, q = 0, 1, 2, \cdots),$$

$$(\gamma) \qquad \lim_{p,q\to\infty} \binom{p}{i}\binom{q}{j}\Delta_{p-i,q-j}\mu_{ij} = l_{ij}, \qquad (i, j = 0, 1, 2, \cdots),$$

$$(\delta) \qquad \lim_{p,q\to\infty} \sum_{i=0}^{p} \left| \binom{p}{i}\binom{q}{j}\Delta_{p-i,q-j}\mu_{ij} - l_{ij} \right| = 0, \qquad (j = 0, 1, 2, \cdots),$$

$$(\epsilon) \qquad \lim_{p,q\to\infty} \sum_{j=0}^{q} \left| \binom{p}{i}\binom{q}{j}\Delta_{p-i,q-j}\mu_{ij} - l_{ij} \right| = 0, \qquad (i = 0, 1, 2, \cdots),$$

the condition (β) corresponding to (B) being automatically satisfied because of (7). These conditions (α)-(ϵ) are thus necessary and sufficient for a C-sequence. A C-sequence is pure if and only if all the l_{ij} vanish; a pure C-sequence is normalized if and only if $\mu_{00} = 1$.

The corresponding conditions, necessary and sufficient for a *simple* sequence $\{\mu_m\}$ to be a C-sequence, are

$$(\mathfrak{A}) \qquad \sum_{i=0}^{p} \binom{p}{i} \left| \Delta_{p-i}\mu_i \right| = M_p \leqq M, \qquad (M_p \to M),$$

$$(\mathfrak{C}) \qquad \lim_{p\to\infty} \binom{p}{i} \Delta_{p-i}\mu_i = l_i, \qquad (i = 0, 1, 2, \cdots).$$

Of these the first implies the second with $l_i = 0$ for $i > 0$. It is therefore natural to inquire to what extent condition (α) implies the three remaining conditions (γ)-(ϵ). The complete answer to this question will be given in §4, although even in §3 it will become apparent that (α) does not imply (δ) and (ϵ).

3. *C-Sequences of Product Type.* An interesting class of matrices λ of the form (2) is that for which the diagonal matrix corresponding to λ is the *product* (see the last footnote above) of (simple) diagonal matrices. The importance of this class is sufficiently indicated by the fact that, by Theorem 6 of A, this class is identical with the class of matrices $\lambda = \lambda' \odot \lambda''$, in which λ' and λ'' are transformable by E' into diagonal matrices.

If $\{\mu_{mn}\} = \{\mu_m' \mu_n''\}$, we clearly have

$$\Delta_{p-i,q-j}\mu_{ij} = \Delta_{p-i}\mu_i' \Delta_{q-j}\mu_j''.$$

The following theorems may now be established easily.

THEOREM 1. *A double sequence* $\{\mu_{mn}\} = \{\mu_m' \mu_n''\}$, *where* $\{\mu_m'\}$ *and* $\{\mu_n''\}$ *are (simple) C-sequences, is a (double) C-sequence if and only if the following condition is satisfied*:

$$(8) \qquad |l_0'|\,(M'' - |l_0''|) = |l_0''|\,(M' - |l_0'|) = 0.$$

Here the M and l_0 of ($\mathfrak{A}$) and ($\mathfrak{C}$) are primed to agree with the priming of the μ's. Any double sequence thus factorable into simple C-sequences obviously satisfies (α), but it will satisfy both (δ) and (ϵ) if and only if (8) is also fulfilled.

THEOREM 2. *A double sequence* $\{\mu_{mn}\} = \{\mu_m' \mu_n''\}$, *where* $\{\mu_m'\}$ *and* $\{\mu_n''\}$ *are (simple) C-sequences, is a pure (double) C-sequence if and only if one of its factors is a pure C-sequence and the other is a pure C-sequence or satisfies the condition* $M = |l_0|$.

THEOREM 3. *If* $\{\mu_{mn}\} = \{\mu_m' \mu_n''\}$ *is a (double) C-sequence that is not pure, both of its factors are C-sequences, neither factor is pure, and* $l_0' l_0''$ *equals* l_{00}.

THEOREM 4. *If* $\{\mu_{mn}\} = \{\mu_m' \mu_n''\}$ *is a pure (double) C-sequence, not all of whose elements are zero, both of its factors are C-sequences and at least one of them is pure.*

4. *The Relation Between C-Sequences and Moment-Sequences.* Let $\chi(u, v)$ be a function which in the square $U(0 \leqq u \leqq 1, 0 \leqq v \leqq 1)$ is of bounded variation in the sense of Hardy-Krause; then the sequence $\{\mu_{mn}\}$, where

$$(9) \qquad \mu_{mn} = \int_0^1 \int_0^1 u^m v^n d_u d_v \chi(u, v), \qquad (m, n = 0, 1, 2, \cdots),$$

may be termed a (double) *moment-sequence.*† According to a theorem of Hildebrandt and Schoenberg,‡ if $\{\mu_{mn}\}$ is any double sequence satisfying condition (α) for $q = p$, there exists a function $\chi(u, v)$ which generates this sequence, and conversely; hence we have the following theorem.

THEOREM 5. *Every C-sequence is a moment-sequence.*

Clearly not every moment-sequence is a C-sequence; for otherwise, by Hildebrandt and Schoenberg's theorem, condition (α) for $q = p$ would imply both (δ) and (ϵ), and we have already seen that this is not so. We therefore seek to determine under what circumstances a moment-sequence is a C-sequence.

It should be observed first that since $\chi(u, v)$ can always be expressed as the difference between two functions $\chi_1(u, v)$ and $\chi_2(u, v)$, each of which is bounded, is non-decreasing in x alone and in y alone, and satisfies the condition $\chi_i(u_1, v_1) - \chi_i(u_1, v_2) - \chi_i(u_2, v_1) + \chi_i(u_2, v_2) \geqq 0$ for $u_2 > u_1, v_2 > v_1$; and since the difference (or sum) of two C-sequences is a C-sequence, no loss of generality results from assuming $\chi(u, v)$ to be of this monotonic character. Then we have, for all i, j, m, n,

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† Although apparently it is more general to assume only that $\chi(u, v)$ is of bounded variation in the sense of Vitali, this is not actually so; for if $\chi(u, v)$ is of bounded variation in the Vitali sense, there always exists a related function $\chi'(u, v)$, where $\chi'(u, v) = \chi(u, v) - \chi(u, 0) - \chi(0, v) + \chi(0, 0)$, such that the Riemann-Stieltjes integral (9) taken with respect to $\chi'(u, v)$ has the same value, and $\chi'(u, v)$ is of bounded variation in the sense of Hardy-Krause. Moreover, $\chi'(u, v)$ vanishes along $u = 0$ and $v = 0$; hence it would be no real restriction to assume $\chi(u, 0) \equiv \chi(0, v) \equiv 0$. We do not make this assumption, however, for reasons of symmetry.

‡ Hildebrandt and Schoenberg, *On linear functional operations and the moment problem for a finite interval in one or several dimensions*, Annals of Mathematics, (2), vol. 34 (1933), pp. 317–328, Theorem 1.

$$\Delta_{ij}\mu_{mn} = \int_0^1 \int_0^1 u^m v^n (1-u)^i (1-v)^j d_u d_v \chi(u, v) \geqq 0,$$

so that $\{\mu_{mn}\}$ is *completely monotonic*, and by (7), condition (α) is satisfied for all p, q. We shall now show that the remaining conditions (γ)-(ϵ) are fulfilled *in case* $\chi(u, v)$ *satisfies the continuity condition*

$$(10)\quad \begin{aligned} &\chi(u, +0) = \chi(u, 0), \quad \chi(u, +0) = \lim_{v \to 0} \chi(u, v), \quad 0 \leqq u \leqq 1; \\ &\chi(+0, v) = \chi(0, v), \quad \chi(+0, v) = \lim_{u \to 0} \chi(u, v), \quad 0 \leqq v \leqq 1. \end{aligned}$$

For all values of u and v in U we have

$$(11)\quad \begin{aligned} \binom{p}{i} u^i (1-u)^{p-i} &\leqq \sum_{i=0}^{p} \binom{p}{i} u^i (1-u)^{p-i} \\ &= [u + (1-u)]^p = 1, \end{aligned}$$

together with a like relation in v. Hence, for $0 < \delta < 1$, we obtain

$$\binom{p}{i}\binom{q}{j} \Delta_{p-i,q-j}\mu_{ij} \leqq \int_0^\delta \int_0^1 d_u d_v \chi(u, v) + \binom{p}{i} (1-\delta)^{p-i} \int_\delta^1 \int_0^1 d_u d_v \chi(u, v).$$

Consequently, for each δ, we have

$$\overline{\lim_{p,q \to \infty}} \binom{p}{i}\binom{q}{j} \Delta_{p-i,q-j}\mu_{ij} \leqq \chi(\delta, 1) - \chi(0, 1) - \chi(\delta, 0) + \chi(0, 0),$$

and so, by virtue of (10), the limit exists and equals zero for each i and each j. Thus condition (γ) is satisfied with all $l_{ij} = 0$. That (δ) and (ϵ) are fulfilled then follows in a precisely similar manner, proper account being taken of the relations (11).

When $\chi(u, v)$ does not satisfy condition (10), let

$$(12)\quad \begin{aligned} J_1(u) &= \chi(u, +0) - \chi(u, 0), \quad u > 0; \\ J_2(v) &= \chi(+0, v) - \chi(0, v), \quad v > 0; \\ J_1(0) = J_2(0) &= \lim_{\substack{0 \neq u \to 0 \\ 0 \neq v \to 0}} \chi(u, v) - \chi(0, 0). \end{aligned}$$

Since $\chi(u, v)$ is bounded and monotonic, all the limits involved exist, and $J_1(u)$ and $J_2(v)$ are functions of bounded variation. Moreover, condition (10) is satisfied by the function

$$\chi^*(u, v) = \chi(u, v) + J(u, v),$$

where

$$J(u, v) = \begin{cases} J_1(u) & \text{for } v = 0, \\ J_2(v) & \text{for } u = 0, \\ 0 & \text{for } u, v > 0. \end{cases}$$

Therefore it remains only for us to investigate the sequence $\{\mu_{mn}\}$ generated by $J(u, v)$.

From the definition of the integral in (9), we have at once

$$(13)\qquad \begin{aligned} \mu_{00} &= J_1(0) - J_1(1) - J_2(1), \\ \mu_{m0} &= -\int_0^1 u^m dJ_1(u), && (m = 1, 2, 3, \cdots), \\ \mu_{0n} &= -\int_0^1 v^n dJ_2(v), && (n = 1, 2, 3, \cdots), \\ \mu_{mn} &= 0, && (m, n = 1, 2, 3, \cdots). \end{aligned}$$

Thus $\{\mu_{m0}\}$, $(m=0, 1, 2, \cdots)$, and $\{\mu_{0n}\}$, $(n=0, 1, 2, \cdots)$, are both (simple) C-sequences. By (13) we have

$$M_{pq} = \sum_{i=0}^{p} \binom{p}{i} \left| \Delta_{p-i,0}\mu_{i0} \right| + \sum_{j=0}^{q} \binom{q}{j} \left| \Delta_{0,q-j}\mu_{0j} \right| - \left| \Delta_{pq}\mu_{00} \right| .$$

Both sums are bounded, and the remaining term on the right is also bounded, since

$$(14)\qquad \Delta_{pq}\mu_{00} = \Delta_{p0}\mu_{00} + \Delta_{0q}\mu_{00} - \mu_{00}$$

is the sum of a term depending upon p alone and a term depending upon q alone, each of which tends to a limit as p and q become infinite. Therefore (α) is satisfied.

Condition (γ) is obviously satisfied for $i, j > 0$, with $l_{ij} = 0$. For $i = 0, j > 0$, the quantity in question reduces to

$$(15) \qquad \binom{q}{j}\Delta_{0,q-j}\mu_{0j},$$

which tends to zero with $1/q$; for $i>0$, $j=0$, the situation is similar. For $i=j=0$, the quantity in question reduces to (14), which, as we have already observed, tends to a limit. Therefore (γ) is fulfilled, with $l_{ij}=0$ for $i,j\neq 0$, 0 and

$$l_{00} = \lim_{p\to\infty} \Delta_{p0}\mu_{00} + \lim_{q\to\infty} \Delta_{0q}\mu_{00} - \mu_{00}.$$

For $j>0$, the sum in (δ) reduces to the absolute value of (15), which tends to zero with $1/q$. For $j=0$, the sum reduces to

$$\left| \Delta_{pq}\mu_{00} - l_{00} \right| + \sum_{i=0}^{p} \binom{p}{i} \left| \Delta_{p-i,0}\mu_{i0} \right| - \left| \Delta_{p0}\mu_{00} \right|,$$

of which the first term tends to zero but the remaining part in general does not.

For $i>0$, the sum in (ϵ) reduces to

$$\binom{p}{i} \left| \Delta_{p-i,0}\mu_{i0} \right|,$$

which tends to zero with $1/p$. For $i=0$, the sum reduces to

$$\left| \Delta_{pq}\mu_{00} - l_{00} \right| + \sum_{j=0}^{q} \binom{q}{j} \left| \Delta_{0,q-j}\mu_{0j} \right| - \left| \Delta_{0q}\mu_{00} \right|,$$

of which the first term tends to zero but again the remaining part in general does not. The results of this discussion may be summed up as follows.

THEOREM 6. *Let $J_1(u)$ and $J_2(v)$ represent the jump of $\chi(u, v)$ at the sides $v=0$ and $u=0$, respectively, of the square U (for orthogonal approach except at* (0, 0), *where the approach is made in an arbitrary manner from the interior of U). $J_1(u)$ and $J_2(v)$ are functions of bounded variation and so generate (simple) moment-sequences which we may denote by $\{j_m^{(1)}\}$ and $\{j_m^{(2)}\}$, respectively. The sequences $j_0^{(1)}-J_2(1)$, $j_1^{(1)}$, $j_2^{(1)}$, $\cdots$ and $j_0^{(2)}-J_1(1)=j_0^{(1)}-J_2(1)$, $j_1^{(2)}$, $j_2^{(2)}$, $\cdots$, are also moment-sequences and therefore C-sequences; let the M of ($\mathfrak{A}$) and the l_0 of ($\mathfrak{C}$) associated with these sequences be denoted respectively by $M^{(1)}$, $l_0^{(1)}$ and $M^{(2)}$, $l_0^{(2)}$. Then a (double) moment-sequence is a (double) C-sequence if and only if we have*

$$M^{(1)} - | l_0^{(1)} | = M^{(2)} - | l_0^{(2)} | = 0. \tag{16}$$

A (double) moment-sequence is a pure (double) C-sequence if and only if in addition to (16) *we have*

$$l_0^{(1)} + l_0^{(2)} - j_0^{(1)} + J_2(1) = 0.$$

In particular we note that if $J_1(u) \equiv J_2(v) \equiv 0$, $\chi(u, v)$ generates a pure C-sequence.

THEOREM 7. *Condition* (α) *for* $q=p$ *implies the entire set of conditions* (α)-(ϵ), *including* $l_{ij}=0$ *for* $i, j \neq 0, 0$, *with the exception of* (δ) *for* $j=0$ *and* (ϵ) *for* $i=0$.

Although in this section it has been tacitly assumed that the sequences considered are real, the extension of the results to complex sequences is immediate.

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ERRATA

This Bulletin, vol. 38, No. 12 (Dec., 1932):

Page 841, first formula: inside of the large parentheses in the denominator, the numerator of the small fraction should be n instead of 1.

Page 847, equation (11): the quantity c/2 should be added to the left-handed side.

Page 847, last line: the second and third integrals should be preceded by the negative sign.

This Bulletin, vol. 39, No. 1 (Jan., 1933):

Page 18, line 8: *read* Kline *in place of* Kine.