Since $a \leqq 374930473917097$, we have in each case $k \leqq 39111579$. Thus the problem of representing N as the difference of squares was split into 8 parts. The first two parts were covered by the machine without any result. On the third run, however, the machine stopped almost at once at $x=58088$. This gives

$$a = 556846584735, \quad b = 556644555032.$$

Hence we have the factorization

$$2^{79} - 1 = 2687 \cdot 202029703 \cdot 1113491139767.$$

It is not difficult to show that the factors are primes. This is the 13th composite Mersenne number to be completely factored. The author's recent report* on Mersenne numbers should be changed accordingly.

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MATRICES WHOSE sTH COMPOUNDS ARE EQUAL

BY JOHN WILLIAMSON

If A is a matrix of m rows and n columns and s is any positive integer less than or equal to the smaller of n and m, from A can be formed a new matrix A_s of ${}_mC_s$ rows and ${}_nC_s$ columns, the elements in the tth row of A_s being the ${}_nC_s$ determinants of order s that can be formed from the t_1th, $\cdots$, t_sth rows of A, and the elements in the tth column being the ${}_mC_s$ determinants of order s that can be formed from the t_1th, $\cdots$, t_sth columns of A. The matrix A_s, so defined, is called the sth compound matrix of A. In the following note we discuss the necessary and sufficient conditions under which the sth compounds of two matrices are equal. We shall require the following lemmas.

LEMMA I. *The rank of the sth compound of a matrix A, whose rank is r, is ${}_rC_s$ if $r \geqq s$ and is zero if $s > r$.*†

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* This Bulletin, vol. 38 (1932), p. 384. Dr. N. G. W. H. Beeger has kindly called my attention to the fact that $2^{233}-1$ has two known prime factors and should be classified accordingly.

† Cullis, *Matrices and Determinoids*, vol. 1, p. 289.

LEMMA II. *The sth compound of the product of two matrices is the product of the sth compounds of the two matrices, or, in symbols,**

$$(AB)_s = A_sB_s. \tag{1}$$

THEOREM. *If A is a matrix of rank r, the necessary and sufficient condition that $A_s=B_s$ is that*
(a) *the rank of B be less than s when $r< s$;*
(b) *there exist two non-singular matrices C and D such that*

$$CAD = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}, \quad CBD = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix},$$

where T and S are two non-singular matrices of r rows and columns such that $|T| = |S|$, when $r=s$;
(c) *$A=\omega B$, where ω is an sth root of unity, when $r>s$.*

In case (a) if $A_s=B_s$, then $B_s=0$ and by Lemma I the rank of B is less than r. On the other hand if the rank of B is less than s, then $B_s=0=A_s$. In case (b) the sufficiency of the condition follows from (1) and the fact that

$$\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}_s = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}_s.$$

We now proceed to prove that the condition stated above is necessary. Since A has rank r there exist two non-singular matrices C and D such that

$$CAD = R = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix},$$

where T is any non-singular r-rowed square matrix. If

$$CBD = F = \begin{pmatrix} S & G \\ H & K \end{pmatrix},$$

where S is an r-rowed square matrix, G an r by $n-r$ matrix, H an $m-r$ by r matrix, and K an $m-r$ by $n-r$ matrix, then, since $A_s=B_s$, it follows that $R_s=F_s$ and $|S| = |T| \neq 0$. Since R_s contains only one element different from zero, every determinant of order s that can be formed from $s-1$ columns of S and one of G is zero. If

$$S = (s_{ij}), \quad G = (g_{iq}), \quad (i, j = 1, 2, \cdots r; \quad q = 1, 2, \cdots, n-r),$$

* H. W. Turnbull, *Determinants, Matrices and Invariants*, pp. 81–82.

and S_{ij} is the cofactor of s_{ij} in S, then

$$(2)\qquad \sum_{i=1}^{r} S_{ij} g_{iq} = 0, \quad (j = 1, 2, \cdots, r; q = 1, 2, \cdots, n - r).$$

For a fixed q, the equation (2) represents a set of r homogeneous equations in the r unknowns g_{iq}, and since $|S_{ij}| = |S|^{r-1} \neq 0$, it follows that $g_{iq} = 0$. Accordingly $G = 0$ and by a similar argument $H = 0$, so that F has the form

$$\begin{pmatrix} S & 0 \\ 0 & K \end{pmatrix}.$$

But, since S is non-singular, at least one of the quantities $S_{ij} \neq 0$. If k is any element of K, we observe that kS_{ij} is an element of F_s which must be zero, and therefore $K = 0$.

In case (c), the sufficiency of the condition is an immediate consequence of (1). If the rank r of A is greater than s, there must exist in A a submatrix T of $s+1$ rows and columns, which is non-singular. Without any loss of generality we may suppose that

$$A = \begin{pmatrix} T & K \\ L & M \end{pmatrix}, \quad B = \begin{pmatrix} S & H \\ P & Q \end{pmatrix},$$

where S is an $(s+1)$-rowed square matrix. From $A_s = B_s$, we deduce that $T_s = S_s$ and

$$|T_s| = |T|^s = |S_s| = |S|^s,$$

so that

$$(3)\qquad |S| = \omega |T|,$$

where ω is an sth root of unity. Moreover*

$$(T_s)_s = |T|^{s-1} T = (S_s)_s = |S|^{s-1} S,$$

so that, by (3), $S = \omega T$. Since T is non-singular, there must exist in T a non-singular submatrix T' of s rows and columns. If A' denote a matrix obtained from A by a rearrangement of rows and columns, so that T' occurs in the top left-hand corner of A', and B' is the matrix obtained from B by exactly the same rearrangement, then

* $(T_s)_s$ denotes the sth compound of T_s. That $(T_s)_s = |T|^{s-1} T$ is simply the well known theorem on the adjugate of the adjugate of a matrix.

$$A' = \begin{pmatrix} T' & K' \\ L' & M' \end{pmatrix}, \quad B' = \begin{pmatrix} \omega T' & H' \\ P' & Q' \end{pmatrix},$$

and from $A_s = B_s$ it follows that $A_s' = B_s'$. If

$$T' = (t_{ij}), \quad K' = (k_{iq}), \quad H' = (h_{iq}),$$
$$(i, j = 1, 2, \cdots, s;\ q = 1, 2, \cdots, n - s),$$

and T_{ij} denote the cofactor of t_{ij} in T', then

$$\sum_{i=1}^{s} T_{ij} k_{iq} = \sum_{i=1}^{s} \omega^{s-1} T_{ij} h_{iq}, \qquad \text{or} \qquad \sum_{i=1}^{s} T_{ij}(k_{iq} - \omega^{s-1} h_{iq}) = 0.$$

But, since $|T_{ij}| \neq 0$, $k_{iq} - \omega^{s-1} h_{iq} = 0$ or $H' = \omega K'$. Similarly it may be shown that $P' = \omega L'$. Let T'' be a submatrix of T' of order $s-1$ which is non-singular. If m_{ij} is any element of M' and q_{ij} the corresponding element of Q', the determinant of order s formed from A' of the $s-1$ rows and columns of which T'' is composed and the row and column in which m_{ij} lies is equal to the corresponding determinant formed from B'. But from the equality of these two determinants it follows that $m_{ij}|T''| = \omega^{s-1} q_{ij} |T''|$ and therefore, since $|T''| \neq 0$, it follows that $Q' = \omega M'$, $A' = \omega B'$, and $A = \omega B$. This completes the proof of the theorem.

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REMARKS ON PROPOSITIONS $*1 \cdot 1$ AND $*3 \cdot 35$ OF PRINCIPIA MATHEMATICA†

BY B. A. BERNSTEIN

1. *Object.* Among the propositions of the theory of deduction underlying Whitehead and Russell's *Principia Mathematica* are the two following:

$*1 \cdot 1$. *Anything implied by a true elementary proposition is true.*

$*3 \cdot 35$. $\vdash : p \cdot p \supset q \cdot \supset \cdot q$.

The authors interpret $*3 \cdot 35$ as "if p is true, and q follows from it, then q is true," and they remark that $*3 \cdot 35$ "differs

† Presented to the Society, September 2, 1932.