SOME INVOLUTORIAL LINE TRANSFORMATIONS INTERPRETED AS POINTS OF V_2 OF S_5*

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1. *Introduction.* Consider the identity

$$\text{I} \qquad V_2(x) \equiv x_1x_4 + x_2x_5 + x_3x_6 \equiv 0$$

existing among the Plücker coordinates $x_1, x_2, \cdots, x_6$ of a line in S_3 as the equation of a quadratic hypersurface in S_5. The existence of a (1,1) correspondence between the lines of S_3 and the points of V_2 is well known, as is also the representation of ruled surfaces in S_3 by curves of the same orders and genera on V_2.†

To a bundle of lines in S_3 corresponds a plane of points on V_2, and to a plane field of lines corresponds a plane of points on V_2, called ω-planes and ρ-planes, respectively, throughout this paper. The two systems of planes are each ∞^3, and there are certain relations among them. Two planes of either system have always one and only one point in common, corresponding to the line common to the two representative bundles or plane fields; and a plane of the ω-system meets a plane of the ρ-system either in no point or in a line of points, corresponding to the flat pencil common to a bundle and a plane field of S_3 when the vertex of the bundle lies on the plane of the field.

Line transformations of S_3 are, therefore, point transformations on V_2. When the point transformations on V_2 are non-linear, their fundamental elements may be of dimension 0, 1, 2, 3, and their images, or the principal elements in the transformations, may be of dimension 1, 2, 3, 4. The transformations considered below are birational, but since the equation of V_2 does not enter into the discussion of their birationality, we conclude that they are Cremona transformations for all of S_5.

2. *Three Involutorial Transformations on V_2.*

CASE 1. J. DeVries‡ discusses synthetically several involutions

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† See, for example, W. L. Edge, *Ruled Surfaces.*

‡ *Proceedings, Koninklijke Akademie van Wetenschappen te Amsterdam,* vol. 22 (1920), pp. 478–481, 634–640.

among the rays of space. It shall be our purpose here to obtain analytic formulations of these transformations on V_2 in S_5. To this end, let us consider the four flat pencils (A_i, α_i) in the planes α_i and with vertices at A_i. Consider the planes α_i as being the faces of the tetrahedron of reference so that $\alpha_i \equiv z_i = 0$, where z_i are the variable coordinates of a point of S_3, and choose for the points A_1, A_2, A_3, A_4 the points (0,1,1,1), (1,0,1,1), (1,1,0,1), (1,1,1,0) respectively. The pencils (A_i, α_i) are therefore independent and the Plücker coordinates of a line of each are

$$(1)\qquad \begin{cases} a_1 \equiv [(\lambda_1 + \lambda_2), -\lambda_1, -\lambda_2, 0, 0, 0], \\ a_2 \equiv [-(\mu_1 + \mu_2), 0, 0, 0, \mu_2, -\mu_1], \\ a_3 \equiv [0, -(l_1 + l_2), 0, -l_2, 0, l_1], \\ a_4 \equiv [0, 0, -(m_1 + m_2), m_2, -m_1, 0], \end{cases}$$

where the λ_j, μ_j, l_j, m_j, represent parameters and the a_i lines from the pencils (A_i, α_i) respectively.

An arbitrary line (y) will determine a definite line of each pencil, and these four lines, in general mutually skew, will have another transversal line (x). Conversely, the line (x) determines the line (y), and hence if we consider the variables (x_i), (y_i) as coordinates of points in S_5, we have an involution of S_5 which leaves V_2 invariant. Such a line (y) determines in the given four pencils the lines

$$(2)\qquad \begin{cases} a_1 \equiv [(y_5 - y_6), (y_6 - y_4), (y_4 - y_5), 0, 0, 0], \\ a_2 \equiv [-(y_2 + y_3), 0, 0, 0, (y_3 + y_4), (y_4 - y_2)], \\ a_3 \equiv [0, (y_1 + y_3), 0, (y_3 - y_5), 0, (y_1 + y_5)], \\ a_4 \equiv [0, 0, -(y_1 + y_2), (y_2 + y_6), (y_6 - y_1), 0]. \end{cases}$$

Recalling that the lines meeting a given line a of S_3 are represented by the intersection with V_2 of the S_4 tangent to it at the point which represents a, we have the equations

$$(3)\qquad \begin{cases} (y_5 - y_6)x_4 + (y_6 - y_4)x_5 + (y_4 - y_5)x_6 = 0, \\ (y_3 + y_4)x_2 + (y_4 - y_2)x_3 - (y_2 + y_3)x_4 = 0, \\ (y_3 - y_5)x_1 + (y_1 + y_5)x_3 + (y_1 + y_3)x_5 = 0, \\ (y_2 + y_6)x_1 + (y_6 - y_1)x_2 - (y_1 + y_2)x_6 = 0, \end{cases}$$

of four S_4's tangent to V_2 at the points a_1, a_2, a_3, a_4 respectively.

The problem of solving four linear and one quadratic equations in six unknowns is the same as that discussed by V. Snyder,* in which he determines the two lines common to four linear line-complexes. Using his method of solution we find

$$(4)\qquad \begin{aligned} \rho x_1 &= w_1 Q\cdot R, & \rho x_2 &= w_2 S\cdot R, & \rho x_3 &= w_3 S\cdot Q,\\ \rho x_4 &= w_4 S\cdot U, & \rho x_5 &= w_5 U\cdot Q, & \rho x_6 &= w_6 U\cdot R, \end{aligned}$$

where the w_i are linear and Q, R, S, U are cubic in y_j. Thus the transformation (4) is an involution of order seven.

Since V_2 is invariant under (4), x_i must satisfy $V_2(x)=0$. But if this be true, $V_2(w)=0$ also.

If $Q(y)=0$, then in (4) $x_1=x_3=x_5=0$. Hence any point in the plane $x_1=x_3=x_5=0$ on V_2 is transformed into the intersection of the cubic hypersurface $Q(x)=0$ and $V_2(x)=0$. Likewise any point in the plane $x_1=x_2=x_6=0$ is transformed into the intersection of $R(x)=0$ and $V_2(x)=0$, any point in the plane $x_2=x_3=x_4=0$ is transformed into the intersection of $S(x)=0$ and $V_2(x)=0$, and any point in the plane $x_4=x_5=x_6=0$ is transformed into the intersection of $U(x)=0$ and $V_2(x)=0$. These planes on V_2 are therefore fundamental elements of (4). They are ρ-planes, being the representation on V_2 of the lines of the planes α_1, α_2, α_3, α_4 respectively. Dually, the ω-planes, representative of the bundles of lines of S_3 with vertices at A_1, A_2, A_3, A_4 respectively, are fundamental elements of (4).

There are ρ-planes and ω-planes which carry singular flat pencils. Consider the ρ-plane which represents the lines of the plane $A_1A_2A_3$ in S_3. Its equations are

$$(4\mathrm{a}_1)\qquad x_4+x_5-2x_6=0;\ 2x_2-x_3+x_4=0;\ 2x_1+x_3+x_5=0;$$

and the point $(0,0,1,1,-1,0)$ which lies in this plane is such that every line in the pencil with this point as vertex and $(4\mathrm{a}_1)$ as base is singular. In S_3, plane $A_1A_2A_3$ meets α_4 in a line. Every line t in $A_1A_2A_3$ meets the intersection $A_1A_2A_3$, α_4 in a point and so determines a line a_4. The line t meets each of the intersections $A_1A_2A_3$, α_1; $A_1A_2A_3$, α_2; $A_1A_2A_3$, α_3. These intersections are lines of the pencils (A_i, α_i), $(i=1, 2, 3)$ respectively, and t', the conjugate of t in (4), is any one of the lines of the pencil in the plane

* This Bulletin, vol. 3 (1897), pp. 247–250.

$A_1A_2A_3$ with vertex t, a_4. Hence, the pencil is singular. But, on V_2 in S_5, a pencil of lines of S_3 is represented by a line and since all of the pencils $(t, a_4; A_1A_2A_3)$ have in common the line $A_1A_2A_3$, α_4 whose representation on V_2 is the point (0,0,1,1,–1,0), the lines on V_2 which represent the pencils $(t, a_4; A_1A_2A_3)$ intersect in this point. In like manner, each of the planes

$$(4a_2)\ x_4 - 2x_5 + x_6 = 0;\ 2x_1 + x_2 - x_6 = 0;\ x_2 + 2x_3 + x_4 = 0;$$

$$(4a_3)\ 2x_4 - x_5 - x_6 = 0;\ x_1 + 2x_3 - x_5 = 0;\ x_1 + 2x_2 + x_6 = 0;$$

$$(4a_4)\ x_2 - x_3 - 2x_4 = 0;\ x_1 - x_3 + 2x_5 = 0;\ x_1 - x_2 - 2x_6 = 0;$$

carries a singular pencil of lines with vertices at the points (0,1,0,–1,0,1), (1,0,0,0,1,–1), (1,1,1,0,0,0) respectively. Dually, each of the ω-planes on V_2 which represent the bundles of lines of S_3 with vertices at $\alpha_1\alpha_2\alpha_3$, $\alpha_1\alpha_2\alpha_4$, $\alpha_1\alpha_3\alpha_4$, $\alpha_2\alpha_3\alpha_4$ carries a singular pencil of lines.

The involution (4) has fundamental points which are transformed into quadric surfaces lying on V_2. Consider the point (1,0,0,0,0,0). If $U=S=0$, then application of (4) gives this point. But, if

$$(5a_1) \qquad x_3 - x_5 = 0;\ x_2 + x_6 = 0,$$

then

$$U(x) = S(x) = 0.$$

But $(5a_1)$ are the equations of a three-space common to the four-spaces $x_1=0$, $x_4=0$, both of which are tangent to V_2. Thus $(5a_1)$ are the equations of a quadric surface on V_2. In like manner we find that the points

(0,1,0,0,0,0), (0,0,1,0,0,0), (0,0,0,1,0,0), (0,0,0,0,1,0), (0,0,0,0,0,1)

are fundamental points with quadric surfaces as images. Dually, the points

$$(0,1,-1,1,1,1),\ (-1,0,1,1,1,1),\ (1,-1,0,1,1,1),$$
$$(1,1,-1,1,-1,0),\ (1,-1,1,\ -1,0,1),\ (-1,1,1,0,1,-1)$$

are fundamental with quadrics as images.

The locus of invariant points of (4) is the intersection of V_2 and a hypersurface of order four. Snyder* showed that the two lines common to four linear complexes are coincident if, and only if, the combinant

* Loc. cit.

$$(6)\qquad \Delta \equiv \begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{vmatrix}$$

of the system of complexes having these four as base vanishes. Equations (3) are the equations of four special linear complexes, and the combinant of the system with (3) as base is Δ, where

$$A_{ik} \equiv a_{i1}a_{k4} + a_{i2}a_{k5} + a_{i3}a_{k6} + a_{i4}a_{k1} + a_{i5}a_{k2} + a_{i6}a_{k3},$$

where now a_{ij} represents the coefficient of x_j in the ith equation of (3).

After making the substitution indicated above and using the fact that $V_2(w)=0$, we obtain

$$(7)\qquad \begin{aligned} \Delta(x) &\equiv [w_2(x)w_5(x)x_3x_6 - w_3(x)w_6(x)x_2x_5]^2 \\ &\equiv [w_1(x)w_4(x)x_3x_6 - w_3(x)w_6(x)x_1x_4]^2 \\ &\equiv [w_1(x)w_4(x)x_2x_5 - w_2(x)w_5(x)x_1x_4]^2. \end{aligned}$$

Thus, the locus of invariant points of (4) is

$$(8)\qquad w_1w_4x_2x_5 - w_2w_5x_1x_4 = 0;\ V_2(x) = 0.$$

Among the lines of S_3, (4) transforms a pencil (T, τ) into a ruled surface R_7 of order seven. Four lines of the pencil (T, τ) are generators of R_7, for (8) is the equation of a line-complex of order four, of which *any* pencil carries four lines. On R_7 there is a double curve of order 15 and genus 6.* The plane τ intersects R_7 in four lines t_1, t_2, t_3, t_4 of (T, τ) and a curve of order 3 which must pass through T since every generator of R_7 meets five others. Hence there are two generators of R_7 not in τ met by each of the lines t_i. This cubic curve is rational, since the surface R_7 is rational, and has a double point which is not on any of the lines t_i. The double curve of order 15 meets τ in 15 points, of which 8 are the pairs of points in which t_i meet the residual cubic $[R_7, \tau]$. The six pairs of generators of R_7 among t_i show that the double curve has a 6-fold point at T. Project this curve from T upon a plane. We obtain thus a plane curve ρ_9 of

* See W. L. Edge, *Ruled Surfaces*, pp. 9 and 29.

order 9 and genus 6. The line of intersection of τ and the plane of section passes through four double points of ρ_9 and hence meets ρ_9 in one other point, the point where the line of τ through T and the double point of $[R_7, \tau]$ meets the plane of section. Now, R_7 has 10 triple points,* of which the four-fold point T counts for four. Hence there are six others which are also three-fold on the double curve. These six triple points project into triple points of ρ_9, and since ρ_9 is of genus 6 and has four double points (mentioned above), these six triple points are just sufficient to complete the singularities of ρ_9.

On V_2 in S_5 there are six ω-planes with three intersections each with C_7 (representation of R_7). These are the ω-planes which represent the bundles of lines through the triple points of R_7. Likewise, there are six ρ-planes trisecant to C_7, representing the lines of the tritangent planes of R_7. There is one plane of each system having four intersections with C_7, representing the bundle T and the plane field τ respectively. There are only ∞^1 planes of V_2 which meet C_7 twice, and ∞^2 which meet C_7 once. They represent the lines of the bundles and of the bitangent planes at points of the double curve, and the lines of the bundles and of the tangent planes at the ordinary points of R_7 respectively.

A bundle of lines T is transformed into a line-congruence of order seven formed by the generators of the ruled surfaces R_7 belonging to the singly infinite system arising from the pencils of lines through T. The invariant lines of the transformation make up a quartic cone K_4 which must pass through the points A_1, A_2, A_3, A_4; $\alpha_1, \alpha_2, \alpha_3, \alpha_1, \alpha_2, \alpha_4$; $\alpha_1, \alpha_3, \alpha_4, \alpha_2, \alpha_3, \alpha_4$, since each line t of the bundles having these points as vertices is transformed into either a quadric regulus to which t belongs or into a pencil of lines containing t.

On V_2 in S_5 the ω-plane which represents the bundle T is transformed into a surface of order seven which meets the ω-plane in a curve of order four, the representation on V_2 of K_4 of S_3.

Again, a linear complex of lines in S_3 is transformed into a line-complex of order seven, and the two complexes have in common a congruence of order four. On V_2 in S_5 the intersection of

* The number of triple points of a ruled surface of order n, genus, p, is $(n-4)$ $[(n-2)(n-3)/6-p]$. See Edge, loc. cit., p. 31.

V_2 with a general hyperplane is transformed into the intersection of V_2 with a hypersurface of order seven. The invariant points form a surface of order four.

CASE 2. Let us now consider the involution defined among the lines of S_3 (and hence among the points of V_2 in S_5) by the transversals of one regulus of a quadric surface and the lines of two flat pencils whose bases and whose vertices are not conjugate as to the quadric. The bases are not tangent to, nor do the vertices lie on the quadric.

Choose as the quadric surface

$$z_1^2 - z_2^2 + z_3^2 - z_4^2 = 0,$$

and on this the regulus

$$\frac{z_1 + z_2}{z_3 + z_4} = -\frac{z_3 - z_4}{z_1 - z_2} = k,$$

and as the bases of the pencils choose the planes

$$z_1 - 2z_2 = 0; \; z_2 - 2z_3 = 0,$$

and as their vertices the points (2,1,0,0); (0,2,1,0) respectively.

The involution defined by this system is

$$x_i = \phi_i(y), \qquad (i = 1, \cdots, 6), \tag{9}$$

where the ϕ_i are of degree five in y_j. The involution is rational.

In S_5, (9) defines a point transformation, involutorial and of order five. The ρ-planes which represent the plane fields α_i of S_3 and the ω-planes representing the bundles A_i are transformed into the intersection of V_2 with certain quadric hypersurfaces, and the points representative of the lines $\alpha_1\alpha_2$, A_1A_2 of S_3 are transformed into ordinary quadric surfaces on V_2. Also, the conic which represents the second regulus of the quadric surface with which we started in S_3 is transformed into the intersection of V_2 with a quartic hypersurface.

By the same method employed in Case 1, the locus of invariant points is found to be of order three and to contain the conic representative of the second regulus.

A general line on V_2 is transformed into a rational curve of order five meeting the line in three points. A general plane of either system is transformed into a surface of order five on V_2

meeting the plane in a cubic curve, and a general linear three-dimensional locus on V_2 is transformed into a three-dimensional locus of order five meeting the linear locus in a cubic surface.

CASE 3. The involution defined by the transversals of two generators from one regulus of each of two quadric surfaces which have no generator in common is of order three with an invariant complex of order two. Among the invariant lines are the generators of the other regulus on each quadric.

The transformation on V_2 has for fundamental elements the two conics representative of the second reguli of the quadrics in S_3 and a ruled surface R_4 of order four, each of whose points is transformed into the entire generator of R_4 on which it lies. The surface R_4 is the representation in S_5 of the (4,4) congruence of lines of S_3 formed by ∞^1 pencils whose vertices are on the curve of intersection of the two quadrics and whose bases are the planes determined by the intersecting generators of the reguli considered on the two quadrics.

A general line on V_2 is transformed into a rational curve of order three meeting the line in two points. A general plane of either system is transformed into a cubic surface which meets the plane in a conic, and a general linear three-dimensional locus is transformed into a three-dimensional locus of order three meeting the linear locus in a quadric surface.

Other properties of this transformation have also been obtained by A. R. Williams* by a partly different method; they will not be repeated here.

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* Williams, *The transformation of lines of space by means of two quadratic reguli*, the present issue of this Bulletin, vol. 38 (1932), p. 554.