MAPS OF CERTAIN CYCLIC INVOLUTIONS ON TWO-DIMENSIONAL CARRIERS

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1. *Introduction.* The following paper derives the fundamental properties of the involutions on an algebraic surface which have but a finite number of invariant points. Except for a few particular cases, they cannot be regarded as subcases of those having a curve of invariant points; they require one more equation for their definition, analogous to the singular correspondences on algebraic curves. They exist only on surfaces having particular moduli.

2. *Discussion of I_n.* Consider two surfaces $F(x)=0$ and $\Phi(x')=0$ with the property that any point P on $F(x)=0$ uniquely fixes a point P' on $\Phi(x')=0$ and, conversely, the point P' fixes n points $P_1\equiv P, P_2, \cdots, P_n$ on F. There is thus set up an $(n, 1)$ correspondence between the points of $F=0$ and $\Phi=0$. Now any one of the n points $P_1, \cdots, P_n$ on $F=0$ definitely determines the whole group of n points to which it belongs. Hence, it will be said that F *contains an involution* I_n of order n, and that this I_n *belongs to* $\Phi(x')=0$.

There are two kinds of involutions; F may contain one or more curves, each point of which contains two or more coincidences of these n points $P_1, \cdots, P_n, P_i=P_k$. Such curves are called *curves of coincidences.* The surface $\Phi(x')=0$ then contains a locus of branch points in (1, 1) correspondence with the curve of coincidences on F. The other kind of involution is such that F has only a finite number of coincident points. Thus, $\Phi(x')$ has in this case exactly the same number of branch points.

If $\Phi(x')=0$ is a rational surface, or a plane, I_n is said to be *rational.* If $F(x)=0$ is rational, $\Phi(x')=0$ must be rational.* The converse is not true.

In this paper only I_n on $F(x)=0$ with a finite number of coincident points will be considered. Such an I_n can be gener-

* Castelnuovo, Mathematische Annalen, vol. 44 (1894), pp. 125–155.

ated by a group of birational transformations of the surface $F(x)=0$ into itself. The involution is cyclic, abelian, etc. according as it is generated by a cyclic, abelian, etc. group of transformations. This group is, of course, cyclic if n is prime. However, it may or may not be birational (Cremonian) for the whole space in which $F(x)=0$ lies.*

3. *Space of r Dimensions*. In a space S_r of r dimensions, it is possible to reduce $F(x)=0$ and $\Phi(x')=0$ to their normal forms. A surface is said to be *normal* in a linear S_r when it can not be obtained as the projection of a surface of the same order from a space S_m, $m>r$.

Call T the cyclic transformation which generates I_n (where n is a prime) on $F(x)=0$, and P a point of coincidence. Let C be any curve on $F(x)=0$, through P. The image of C under T is another curve on F through P. The point P is called a *perfect point of coincidence* if C and every image of C touches the same line at P for every tangent to F at P. Otherwise P is *non-perfect*. The corresponding branch point on $\Phi(x')=0$ is said to be perfect or non-perfect according as P is a perfect or non-perfect point of coincidence.

When F is reduced to its normal form, the operations of I_p can be represented by a collineation of period p, under which F is invariant in S_r. Since $p<r$, there exist not more than p spaces of invariant points. The form of each transformation is $x_i=\theta^{s_i}x_i'$, where $\theta^p=1$.

4. *Discussion of I_2*. THEOREM 1. *If $n=2$, every point P of coincidence on $F(x)=0$ must be a perfect point.*

PROOF. It has been proved that, given an I_p, it can always be represented by a collineation in S_R, having p axes, or spaces of invariant points, only one of which meets the surface F which contains the involution.

When $p=2$ there are then only two axes. The transformation T may be written

$$X_1':X_2':\cdots:X_{r+1}':X_{r+2}':\cdots:X_{r+s+2}'$$
$$= X_1:X_2:\cdots: X_{r+1}:\epsilon X_{r+2}:\cdots:\epsilon X_{r+s+2},$$

where $\epsilon=-1$.

* F. Enriques, Bologna Rendiconti, (2), vol. 14 (1910), pp. 71–75.

Consider an invariant point A on a surface F in S_R, where $R=r+s+1$. Call the hyperplanes

$$\sum_{i=1}^{r+1} s_i x_i' = 0,\ \Sigma^{(0)},\ \text{and} \sum_{k=r+2}^{r+s+2} r_k x_k' = 0,\ \Sigma^{(1)}.$$

Now $\Sigma^{(0)}$ contains one axis $S^{(1)}$ and $\Sigma^{(1)}$ the other $S^{(0)}$. Require $\Sigma^{(0)}$ to pass through point A in axis $S^{(0)}$. It cuts F in a system of invariant curves $|C|$, on each of which point A is the only coincident point. The tangents to $|C|$ at point A on F all lie in one tangent plane. $\Sigma^{(0)}$ contains this plane. There are two invariant directions at A in this plane or all are invariant. Call two invariant directions a_1 and a_2; then a_1 must have another invariant point A_1 on it. The point A_1 must lie in the other axis $S^{(1)}$ for A is the only point of $S^{(0)}$ in the tangent plane. Likewise a_2 must have another invariant point A_2. It must also lie in $S^{(1)}$. A linear combination of the coordinates of A_1 and A_2 gives ∞ points on line A_1A_2, each of which is a coincident point (invariant point). Line A_1A_2 lies in axis $S^{(1)}$. Hence, by joining A and every point on A_1A_2, we obtain ∞ invariant directions. Therefore point A is a perfect point. This holds for a rational and an irrational surface.

Two illustrations of I_2 are given. The first one uses the quadratic transformation $x_i = 1/x_i'$ in S_2 with four fixed points $(\pm 1, \pm 1, 1)$. Nets* of cubic curves are mapped on the Cayley cubic surface with four nodes.

The second illustration maps a surface of Enriques of order six upon a double plane. The Cremona involution in S_3, $y_i = 1/y_i'$, has eight fixed points $(1, \pm 1, \pm 1, \pm 1)$ on the surface F.

Several theorems follow, the proofs of which are omitted.†

THEOREM 2. *Consider on an algebraic surface F a point A, which is non-perfect in a cyclic involution of third order under*

* A. Emch, *On the invariant net of cubics in the Steinerian transformation*, this Bulletin, vol. 24 (1918), pp. 327–330. *On plane algebraic curves which are invariant under a quadratic Cremona transformation*, Tôhoku Mathematical Journal, vol. 21(1922), pp. 310–326.

† Earlier proofs of these theorems were also given by Godeaux. See Sisam, *Involutions of irrational surfaces*, Bulletin of the National Research Council, No. 63, vol. 14 (1928), pp. 295–309. Godeaux, Brussells Bulletin, 1927, pp. 524–543.

which F is invariant; it follows that in the domain of the first order of A, there are two distinct fixed points. These points are perfect coincidences of the involution.

THEOREM 3. *Consider a cyclic involution of prime order, having only a finite number of fixed points, belonging to an algebraic surface F, and possessing a non-perfect fixed point adjacent to which are two perfect fixed points; then this involution is of order three.*

A map of I_3 belonging to an irrational quintic surface upon an irrational surface in S_7, has been discussed. There are two perfect and five non-perfect coincidences.

5. *Discussion of I_5 belonging to F_3 in S_3.* Consider the surface

$$F_3(x_1x_2x_3x_4) \equiv ax_1^2x_3 + bx_2^2x_1 + cx_3^2x_4 + dx_4^2x_2 = 0,$$

in S_3, invariant under the cyclic collineation T of order five

$$x_1' : x_2' : x_3' : x_4' = x_1 : \epsilon^1 x_2 : \epsilon^2 x_3 : \epsilon^3 x_4, \qquad (\epsilon^5 = 1).$$

There are four invariant axes $S^{(0)}$, $S^{(1)}$, $S^{(2)}$, $S^{(3)}$ each consisting of a point: $P_1 \equiv (1, 0, 0, 0)$, $P_2 \equiv (0, 1, 0, 0)$, $P_3 \equiv (0, 0, 1, 0)$, and $P_4 \equiv (0, 0, 0, 1)$. Each lies on the surface F, and since these are the only possible invariant axes, the surface F has only four points of coincidence.

Consider a curve C, not transformed into itself by T, and passing through P_1. Take a plane $x_3 + \lambda x_4 = 0$ of the pencil passing through P_1 and P_2, tangent to C. This plane $x_3 + \epsilon\lambda x_4 = 0$ by T and hence is non-invariant. The curve cut out on F by $x_3 + \lambda x_4 = 0$ is therefore non-invariant. The common tangent to the two curves is not transformed into itself. Hence, the two curves do not touch each other at P_1. Since C was a variable curve through P_1 satisfying the non-invariant property, it follows that P_1 is a non-perfect coincidence point. A similar argument shows that P_2, P_3, and P_4 are also non-perfect coincidence points. The following theorem is proved.

THEOREM 4. *The I_5 belonging to F_3 in S_3 has four non-perfect points of coincidence.*

Consider the complete system cut out on F by the quintic surfaces. Let $|A|$ be the system. Its dimension is 55, its genus is 31, and the number of variable intersections of two members of the system is 75. A curve A of this system is not in general transformed into itself by T. There are, however, five partial systems in $|A|$ which are transformed into themselves. Call these $|A_1|$, $|A_2|$, $|A_3|$, $|A_4|$, and $|A_5|$. By use of $|A_1|$, we find

$$\begin{aligned} a_{11111}x_1^5 &+ a_{22222}x_2^5 + a_{33333}x_3^5 + a_{44444}x_4^5 + a_{12444}x_1x_2x_4^3 \\ &+ a_{12223}x_1x_2^3x_3 + a_{13344}x_1x_3^2x_4^2 + a_{11233}x_1^2x_2x_3^2 \\ &+ a_{11224}x_1^2x_2^2x_4 + a_{11134}x_1^3x_3x_4 + a_{23334}x_2x_3^3x_4 \\ &+ a_{22344}x_2^2x_3x_4^2 = 0. \end{aligned}$$

We refer the curves A_1 projectively to the hyperplanes of a linear space of eleven dimensions. We obtain a surface Φ, of order 15, with hyperplane sections of genus 7, as the image of I_5.

The equations of the transformation for mapping I_5 upon Φ in S_{11} are

$$\begin{array}{llll} \rho X_1 = x_1^5, & \rho X_4 = x_4^5, & \rho X_7 = x_1x_3^2x_4^2, & \rho X_{10} = x_1^3x_3x_4, \\ \rho X_2 = x_2^5, & \rho X_5 = x_1x_2x_4^3, & \rho X_8 = x_1^2x_2x_3^2, & \rho X_{11} = x_2x_3^3x_4, \\ \rho X_3 = x_3^5, & \rho X_6 = x_1x_2^3x_3, & \rho X_9 = x_1^2x_2^2x_4, & \rho X_{12} = x_2^2x_3x_4^2. \end{array}$$

By eliminating ρ, x_1, x_2, x_3, x_4 from these twelve equations and from $F_3(x_1x_2x_3x_4)=0$, we get as the nine equations defining the surface Φ,

$$\left\| \begin{matrix} X_4 & X_5 & X_{12} \\ X_5 & X_9 & X_6 \end{matrix} \right\| = 0, \quad \left\| \begin{matrix} X_2 & X_9 & X_{12} \\ X_6 & X_{10} & X_7 \end{matrix} \right\| = 0, \quad \left\| \begin{matrix} X_1 & X_{10} & X_8 \\ X_{10} & X_7 & X_{11} \end{matrix} \right\| = 0,$$

$$\left\| \begin{matrix} X_3 & X_{11} & X_7 \\ X_{11} & X_{12} & X_5 \end{matrix} \right\| = 0,$$

and $aX_8+bX_6+cX_{11}+dX_{12}=0$. Designate by P_1' the branch point of Φ corresponding to the point P_1 on F. The coordinates of P_1' are all zero except X_1.

The curves A_1 on F pass through P_1 if $a_{11111}=0$. The tangent plane at P_1 to F is $x_3=0$. Now, the system of quintic surfaces passing through P_1 cuts $x_3=0$ in the curves $x_3=0$, $a_{22222}x_2^5 + a_{44444}x_4^5 + a_{12444}x_1x_2x_4^3 + a_{11224}x_1^2x_2^2x_4 = 0$. For general values of the constants this is a quintic curve with a triple point at P_1, two branches being tangent to the line $x_2=x_3=0$ and one to the line $x_3=x_4=0$. When $a_{12444}=a_{11224}=0$, the plane quintic curve breaks up into five lines through P_1. These are all distinct except when either $a_{22222}=0$ or $a_{44444}=0$, when they coincide with $x_3=x_4=0$ or $x_2=x_3=0$, respectively. Since P_1 is non-perfect, the $|A_1|$ through P_1 must have five distinct branches, unless each branch touches one of the two invariant directions.

In the plane $x_3=0$ the involution I_5 is generated by the homography T_1, which is $x_1':x_2':x_4'=x_1:\epsilon x_2:\epsilon^3 x_4$. By use of the plane quadratic transformation S, which is $x_1:x_2:x_4=z_1^2:z_1z_2:z_2z_4$ and its inverse $z_1:z_2:z_4=x_1x_2:x_2^2:x_1x_4$, we can investigate the character of the adjacent invariant points along the two invariant directions at P_1. By the application of $ST_1S^{-1}\equiv T_1'$,

$$(z_1,z_2,z_4)\overset{S_{-1}}{\sim}(x_1x_2,\ x_2,\ x_1x_4)\overset{T_1}{\sim}(\epsilon x_1x_2,\ \epsilon^2x_2^2,\ \epsilon^3x_1x_4),$$

or

$$(x_1x_2,\ \epsilon x_2^2,\ \epsilon^2x_1x_4)\overset{S}{\sim}(z_1,\ \epsilon z_2,\ \epsilon^2 z_4).$$

Thus the new transformation T_1' is $x_1':x_2':x_4'=x_1:\epsilon x_2:\epsilon^2x_4$. The invariant point adjacent to P_1 along the line $x_3=x_4=0$ is still a non-perfect coincidence point. Investigate the next point by use of $ST_1'S^{-1}\equiv T_1''(z_1,\ z_2,\ z_4)\sim(z_1,\ \epsilon z_2,\ \epsilon z_4)$. This point is a perfect point of coincidence.

By use of another quadratic transformation R, namely $x_1:x_2:x_4=z_1^2:z_2z_4:z_1z_4$ and its inverse $z_1:z_2:z_4=x_1x_4:x_1x_2:x_4^2$, the adjacent point to P_1 along $x_2=x_3=0$ can be investigated. Applying RT_1R^{-1} as above, we have $(z_1,\ z_2,\ z_4)\sim(\epsilon^2z_1,\ z_2,\ z_4)$. Hence we get a perfect coincidence point. The following theorem is proved.

THEOREM 5. *The non-perfect coincidence point P_1 on F has one adjacent perfect point along the line $x_2=x_3=0$, a non-perfect one along the line $x_3=x_4=0$, with a perfect one adjacent to this.*

The tangent plane to F at $P_2(0, 1, 0, 0)$ is $x_1=0$. The homography T_2 in $x_1=0$ is $x_2':x_3':x_4'=x_2:\epsilon x_3:\epsilon^2x_4$. Apply ST_2S^{-1} and proceed as above. We find $x_2':x_3':x_4'=x_2:\epsilon x_3:\epsilon x_4$. Hence, the adjacent point along $x_1=x_3=0$ is perfect. By use of RT_2R^{-1} we obtain $(z_2,\ z_3,\ z_4)\sim(z_2,\ z_3,\ \epsilon_3z_4)$. This indicates a non-perfect point. By use of $RT_2'R^{-1}$, we get $x_2':x_3':x_4'=\epsilon^3x_2:x_3:x_4$. This gives a perfect point. Hence we may state the following theorem.

THEOREM 6. *The non-perfect coincidence point P_2 on F has one adjacent point along the line $x_1=x_4=0$, a non-perfect adjacent one along $x_1=x_3=0$, with a perfect one adjacent to this.*

The point $P_3(0,\ 0,\ 1,\ 0)$ has $x_4=0$ for its tangent plane. The homography becomes, in this tangent plane, T_3: $x_1':x_2':x_3'$

$=x_1:\epsilon x_2:\epsilon^2 x_3$. Introduce two quadratic transformations for use in discovering the nature of $P_3(0, 0, 1, 0)$. Calling the first one U and the second V, we have

$$U: \quad y_1:y_2:y_3 = w_2w_3:w_1w_2:w_3^2,$$
$$U^{-1}: \quad w_1:w_2:w_3 = y_2y_3:y_1^2:y_1y_3,$$
$$V: \quad y_1:y_2:y_3 = w_1w_2:w_1w_3:w_2^2,$$
$$V^{-1}: \quad w_1:w_2:w_3 = y_1^2:y_1y_3:y_2y_3.$$

The adjacent points along the line $x_4=x_2=0$ compel the use of $UT_3U^{-1}\equiv T_3'$, then $VT_3'V^{-1}$ or T_3'', and then $UT_3''U^{-1}$ before a perfect point is found. We have $(w_1, w_2, w_3)\sim(\epsilon_3 w_1, w_2, \epsilon^2 w_3)$. This point is non-perfect. Consider $(w_1, w_2, w_3)\sim(\epsilon^3 w_1, w_2, \epsilon^2 w_3)$. This also is non-perfect.

Consider $(w_1, w_2, w_3)\sim(\epsilon^2 w_1, \epsilon^2 w_2, \epsilon^2 w_3)$. This is a perfect point in the neighborhood of P_3 of the third order.

Now consider the possibilities along $x_4=x_1=0$, an invariant direction. Consider $(w_1, w_2, w_3)\sim(w_1, \epsilon^2 w_2, \epsilon^3 w_3)$. This is a non-perfect point. We have $(w_1, w_2, w_3)\sim(\epsilon^2 w_1, \epsilon^2 w_2, \epsilon^3 w_3)$. Hence we may state the following theorem.

THEOREM 7. *The non-perfect coincidence point P_3 on F has no adjacent perfect point of coincidence. There is one perfect point in the domain of the second order of P_3 and another in the domain of the third order of P_3.*

The point $P_4(0, 0, 0, 1)$ has the plane $x_2=0$ for its tangent plane. The homography T in this plane becomes T_4: $x_1':x_3':x_4'=x_1:\epsilon^2 x_3:\epsilon^3 x_4$. Consider the direction $x_3=x_2=0$ at P_4. We have $(w_1, w_3, w_4)\sim(w_1, w_3, \epsilon^3 w_4)$. Hence, the adjacent point is perfect along $x_2=x_3=0$. Now $(w_1, w_3, w_4)\sim(w_1, \epsilon^3 w_3, w_4)$ and $(w_1, w_3, w_4)\sim(w_1, w_3, \epsilon^3 w_4)$. Hence we have the following theorems.

THEOREM 8. *The non-perfect coincidence point P_4 has an adjacent perfect point along the line $x_2=x_3=0$, a non-perfect one along the line $x_1=x_2=0$, with a perfect one adjacent to this.*

THEOREM 9. *The system of invariant curves cut upon F by surfaces of degree lower than five all pass through the four coincidence points along the invariant directions. The number of branches through each point is less than five.*

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