# ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

235. Professor Clifford Bell: *Useful functions associated with rational cubic curves.*

The six parameters of the intersection and contact points of a rational plane cubic with a circumscribing triangle of reference appear in the parametric equations of the cubic. A certain function of these parameters, invariant under any linear transformation on the parameter of the cubic, is shown to have some useful properties for determining the nature of the singularity of the curve. Like results are obtained when an inscribed triangle is used for the triangle of reference. (Received May 11, 1931.)

236. Professor R. L. Wilder: *A converse of the theorem regarding the separation of $E_3$ by a closed two-dimensional manifold of genus $p$.* Second paper.

It is shown that a bounded point set which is a common boundary, in $E_3$, of two (this does not imply *only* two) uniformly connected im kleinen domains $D_1$ and $D_2$ such that the connectivity number $R^1(D_1)$ is finite, is a closed two-dimensional manifold. This theorem still holds if the condition that $R^1(D_1)$ be finite is replaced by the condition that there is a positive number $\epsilon$ such that no 1-cycle of $D_i (i=1, 2)$ of diameter less than $\epsilon$ links $K$. (Received May 11, 1931.)

237. Professor C. M. Cramlet: *An interpretation of the invariants of a system of linear homogeneous second-order differential equations.*

A complete system of tensors and invariants of the system of equations $d^2y^i/dx^2+p_\alpha{}^i(x)dy^\alpha/dx+q_\alpha{}^i(x)y^\alpha=0$ were obtained in an earlier paper. Here new tensors $q_{rs}$ are obtained which establish a metric at every point of an integral curve. The notion of parallel displacement is introduced in an invariantive manner so that comparisons of directions can be made at different points of an integral curve of the basic differential equations. (Received May 13, 1931.)

238. Professor R. M. Winger: *On certain projective trochoids.*

In this paper the author considers two pencils of rational curves of order $2n$ which are invariant under dihedral groups of order $2(2n-1)$ and $4(n-1)$ respectively. The treatment is general, properties being stated for arbitrary $n$. In the proper metrical setting, the curves are trochoidal, exhibiting multiple symmetry. While both pencils fall short of the maximum symmetry for algebraic curves, the first possesses the greatest symmetry possible for rational trochoids. (Received May 13, 1931.)

239. Professor J. P. Ballantine. *The numerical solution of linear equations by vectors.*

The notion of vectors leads to a very simple and rapid method of solving linear equations. No errors of rounding off, which by other methods result from multiplications of decimals, are possible. Errors due to mistakes can be checked at any step. Answers of increasing accuracy can be read off at each step. (Received May 14, 1931.)

240. Professor A. R. Jerbert: *A certain congruence associated with a ruled surface.*

Following Wilczynski let $y$ and $z$ denote the points $y_i(t)$, $z_i(t)$, $(i=1, 2, 3, 4)$, which generate the flecnode curves of a general ruled surface. It is evident then, that the point $y+\lambda z$ ($\lambda=$const.) generates a unique one-parameter family of curves on the ruled surface. The tangents to these curves constitute a congruence of lines whose relation to the surface is developed in detail. (Received May 14, 1931.)

241. Professor W. E. Milne: *On the numerical integration of certain differential equations of the second order.*

Differential equations of the second order in which the first derivatives do not appear are of frequent occurrence and great importance in many applied fields. For their numerical integration a special set of quadrature formulas of marked simplicity is derived. The method of solution is well adapted for use with a calculating machine, provides a check on each step, and requires a minimum of written calculation. (Received May 15, 1931.)

242. Professor R. E. Moritz: *A new theory of depreciation of physical assets.*

The various methods employed for the computation of depreciation of physical assets are based on the assumption that the life time of the asset is a known constant. In this paper it is shown that this is not true, but that the life time is a function of several variables one of which is the assumed rate of depreciation. In this paper depreciation is treated from the point of view here suggested. A method for determining the life time of an asset is developed such that the annual charge to production necessary to recover original or replacement costs and all repair charges shall be a minimum. (Received May 15, 1931.)

243. Professor R. E. Moritz: *On a totally discontinuous function.*

In this paper it is shown that every continuous function of $x$ may be defined as the limiting case of a function whose derivative fluctuates between $+\infty$ and $-\infty$ in every, however small, interval of $x$. (Received May 15, 1931.)

244. Dr. Leo Zippin: *On the Moore-Kline problem.*

By a Moore-Kline set the author denotes a closed compact point set $M$ whose maximal connected subsets are arcs or points, and such that no inner point of an arc $L$ of $M$ is a limit point of $M-L$; by a generalized continuous curve, a separable, metric, complete, connected, and locally connected space. It is proved in this paper that, in order that a generalized continuous curve $C$ contain, for every one of its Moore-Kline subsets $M$, an arc in which $M$ is embedded, *it is necessary and sufficient* that (A) if $X$ is any arc of $C$, and $t$ one of its end points, then for every $e>0$ there exists a $d>0$ such that if $x$ and $y$ are points of $S(t, d)\cdot(C-X)$, the points $x$ and $y$ may be arc-joined in $S(t, e)\cdot(C-X)$. This property A is shown to have the following non-metric equivalent: (B) if $D$ is an open connected subset of $C$, and $ab$ an arc lying in $D$ except for the point $a$ which belongs to the boundary of $D$, then $D-(ab-a)$ is connected. In this form, it will be recognized as an early plane axiom of R. L. Moore. The author gives also an extension to generalized continuous curves of a theorem recently announced by G. T. Whyburn (see Abstract No. 36-9-329), with an independent proof which is derived by the methods employed in the Moore-Kline problem. (For special cases of this problem, compare E. W. Miller, Abstract 36-5-265.) (Received May 15, 1931.)

245. Dr. I. J. Schoenberg: *Some applications of the calculus of variations to Riemannian geometry.*

In 1855 O. Bonnet, using Jacobi's variational equation for geodesics on a surface and Sturm's comparison theorem for linear second-order differential equations, established a theorem relating the curvature of an ovaloid surface to its size (see W. Blaschke, *Vorlesungen über Differentialgeometrie*, vol. 1, 3d ed.). The author gives here a comparison theorem for the second variation in the calculus of variations, which, together with Levi-Civita's system of variational equations for geodesics of a Riemannian manifold $V_n$ (see T. Levi-Civita, *Sur l'écart géodesique*, Mathematische Annalen, vol. 97), permits one to generalize O. Bonnet's theorem to a manifold $V_n$ of positive curvature, satisfying presupposed conditions concerning the existence of the absolute shortest distance between two of its points. The problem of the shortest distance in a $V_n$ is also discussed by means of a sufficiency theorem of Bliss for the general parametric problem in the calculus of variations. (Received May 16, 1931).

246. Professor G. A. Bliss and Dr. I. J. Schoenberg: *On separation, comparison and oscillation theorems for self-adjoint systems of second-order linear differential equations.*

This paper gives a generalization of the Sturm-Liouville separation, comparison and oscillation theorems for one second-order linear differential equation, which is based on what we call a conjugate system of points with respect to a self-adjoint system of second-order linear differential equations. Such a conjugate system of points is a generalization of the set of zeros of a solution of a second-order linear differential equation. Our separation theorem includes results of J. Radon on conjugate points in the calculus of variations (see J. Radon, *Zum Problem von Lagrange*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, vol. 6, pp. 298–299). The comparison theorem, proved by means of some arguments of the calculus of variations, makes it possible to extend to this more general situation the classical continuity proof for the Sturm-Liouville oscillation theorem (see L. Bieberbach, *Differentialgleichungen*, 2d ed.). (Received May 16, 1931.)

247. Professor A. D. Michal: *Concerning a mixed functional equation and an associated invariant theory.*

The object of this paper is twofold: first, to develop the properties of the solutions of the mixed functional equation (1) $\partial A_i(\gamma, x)/\partial x = L[A_{i-1}(x, s); \gamma]$ and second, to study the $A_i(\tau, x)$ that are solutions of (1) and are in addition relative invariants, with a factor $a^{i-s}$, under the mixed point and functional transformation $\bar{x} = ax + b$, $\bar{y}(\tau) = F[y(s); \tau]$. The functionals $L$ and $F$ are assumed to be linear. Linear differential expressions and linear functionals of the Fredholm type furnish interesting examples. In the latter case, the Fredholm integral equation theory plays a fundamental role. (Received May 16, 1931.)

248. Dr. H. C. Ayres: *Some theorems concerning matrices of continuous functions.*

Bliss, in an article in the Transactions of this Society in 1918, has demonstrated the following theorem: If a matrix $\|a_{\mu\nu}\|$ of continuous functions $a_{\mu\nu}(x)$ has rank $m$ at every point in an interval $x_1x_2$, then an additional $n-m$ rows of polynomials $a_{\tau\nu}(x)$ can always be selected so that the determinant $|a_{\sigma\nu}|$ secured by adding them as rows to the original matrix is non-vanishing in the interval $x_1x_2$. The method used by Bliss apparently cannot be extended to the $n$-variable case. The purpose of this paper is to attack the problem by a different method which may be extended to $n$ variables by the use of mathematical induction. (Received May 18, 1931.)

249. Professor L. H. McFarlan: *The problem of minimizing a double integral with adjoined partial differential equations.*

This paper treats the problem of giving a minimum value to a double integral $\iint f(x, y, z_1, \cdots, z_n, z_{1_x}, \cdots, z_{n_x}, z_{1_y}, \cdots, z_{n_y})dx\,dy$, where the functions $z_1(x, y), \cdots, z_n(x, y)$ take on assigned values along the boundary of the region of integration and also satisfy a system of $m<n$ partial differential equations of the first order. The first necessary condition on the minimizing functions is stated in the form of a multiplier rule similar to the well known Lagrange multiplier rule for the Problem of Lagrange in one independent variable. The multipliers are functions of both $x$ and $y$. (Received May 18, 1931.)

250. Dr. C. E. Seely: *Note on kernels of positive type.*

This note gives sufficient conditions for the existence of a characteristic constant for non-symmetric kernels of positive type. (Received May 22, 1931.)

251. Professor G. T. Whyburn: *Concerning continuous images of the interval.*

In this paper there is given a new proof for the celebrated theorem of Hahn-Mazurkiewicz to the effect that *any compact, metric, and locally connected continuum M is the image under a continuous transformation of the unit interval*, which seems to be simpler and more elementary than any previously known proof for this fundamental proposition. The new proof is effected by modifying a proof given by Hahn for the weaker theorem that *any two points of such a continuum M can be joined by a subset M′ of M which is the continuous image of the interval* in such a way as to insure that the subset $M'$ obtained will be identical with $M$. (Received June 1, 1931.)

252. Professor Karl Menger: *Remarks concerning a paper of W. L. Ayres.*

The author calls attention to some further results suggested by the paper of W. L. Ayres on the regular points of a continuum published in the Transactions of this Society (vol. 33, pp. 252–262), and makes some additions to the bibliography given in that paper. (Received June 12, 1931.)

253. Professor Edward Kasner: *Circular trajectories in dynamics.*

It is known that the $\infty^3$ circles of the plane do not constitute a family of dynamical trajectories (see Kasner, Transactions of this Society, 1906, or the Princeton Colloquium, 1913). The problem of the present paper is to find (positional) fields of force whose $\infty^3$ trajectories include $\infty^2$ circles. All central fields of this sort are determined explicitly. The result contains Maxwell's force (varying as the inverse fifth power) as a special case. (See also the writer's paper in the National Academy Proceedings, June, 1931.) The $\infty^3$ circles, though not in the dynamical type, are in both the sectional type and the curvature trajectory type. (Received June 12, 1931.)

254. Dr. E. J. McShane (National Research Fellow): *On the semi-continuity of double integrals in the calculus of variations.*

The integrals considered are of the form $F(S) = \int f(x, X)du \equiv \iint f(x^1, x^2, x^3, X^1, X^2, X^3)du^1du^2$, where $S$ is a rectifiable surface, $x^i = x^i(u^1, u^2)$, the functions $x^i(u)$ satisfying a Lipschitz condition, and where the $X^i$ are the Jacobians of the $x^i$ with respect to the $u^i$. If (1) $f(x, X) \geqq 0$ for every $x$ and every $X \neq (0,0,0)$, and (2) $E(x, X, \overline{X}) \geqq 0$ for every $x$, every $X \neq (0, 0, 0)$, and every $\overline{X} \neq (kX^1, kX^2, kX^3)$, $k \geqq 0$, then, for every constant $N$, $F(S)$ is lower semi-continuous on the class of all rectifiable surfaces of area $\leqq N$. If in either (1) or (2) the equality be excluded, then $F(S)$ is lower semi-continuous on the class of all rectifiable surfaces. Conditions sufficient for lower semi-continuity at a given surface are also obtained. (Received May 27, 1931.)

255. Professor O. E. Glenn: *Proof of a generalized form of Newton's law of gravity.*

The formula derived in this paper for central gravitational force, for astral motion, results from an hypothesis not previously used in celestial mechanics. A determinate stable central orbit, around which a mass is rotating, is assumed to have a self-restitutional property which restores it when it is perturbed by neighboring astronomical objects. Physical chemists have referred to an analogous property of atomic orbits. This hypothesis alone, developed by invariantive methods under a theory of fields of perturbations, gives, for the central force function, $F=2\gamma^2e^2[(u/v^2)/r^2+(3u^2/v^2+(1/2)e^2+1)/r^3+(3u^3/v^2+3u)/r^4+((u^2+v^2)^2/v^2)/r^5]$, $r$ being the distance and the other literal quantities being constant. The Newtonian formula is a special instance. A brief account of the theory, with astronomical interpretations, is to be published by the Indiana Academy of Science. Further investigation is in progress. (Received April 17, 1931.)

256. Mr. E. A. Kholodovsky: *On a generalized binomial series.*

The series in question, $S_{(k,q,n)}^{\infty}=1+nq+C_{n^2-(k-1)}q^2+\cdots+C_{n-l(k-1)}^{l+1}q^{l+1}+\cdots$, a generalization of the binomial series (for $k=1$ and $k=0$), plays an important role in certain problems in biology dealing with the multiplication of organic beings and geochemical constants. The author investigates this series with regard to convergence and summability (generalized Euler's method). The formula $S_{(k,q,n)}^{\infty}=e^{\phi_k(q)n+\psi_k(q)}$ is established, where $\phi_k(q)$, $e^{\phi_k(q)}$, and $e^{\psi_k(q)}$ can be written as series of the same character as $S_{(k,q,n)}^{\infty}$. An interesting relation is derived between $S_{(k,q,n)}^{\infty}$ and the trinomial equation $z^k-z^{k-1}-q=0$, which yields a very simple explicit expression for $S_{(k,q,n)}^{\infty}$. Some special cases are considered. (Received March 11, 1931).

257. Mr. E. A. Kholodovsky: *On a method of summation of infinite series and a geometrical interpretation of the method.*

This paper generalizes Euler's method of differences. It considers the summatrix series $\sum_{n=0}^{k}m^{k-n}C_k^n$; the sum is $\lim_{k\to\infty}{}_m\sigma_p^{(k)}=\lim\{1/(m+1)^k\}\sum_{n=0}^{k}m^{k-n}C_k^nS_{p+k}$. We have ${}_m\sigma_p^{(k)}=\{1/(1+m)\}\ (m_m\sigma_p^{(k-1)}+{}_m\sigma_{p+1}^{(k-1)})$. If $a_n=f_{(k-1)}(x)\ m^n$, $f_{(k-1)}(x)$ being an integral function of degree $k-1$, then ${}_m\sigma_p^{(k)}=$const. $=s$. A geometric interpretation is given similar to that for $m=1$. If $m=1$ the values of $\sigma_p^{(n)}$ can be computed successively as arithmetic means. When the general term $a_n$ of an alternating series is an integral function of degree $k-1$, $\sigma_p^{(k)}$ is constant for $p=-1, 0, 1, 2, \cdots$; this constant is the sum of the series. Hence we have a formula for computing any Bernoulli number. If $\sigma_p^{(k)}=$const. $=s$, the curves $y=S_0(x)$ and $y=S_1(x)$ passing respectively through the points $(0, S_0)$, $(2, S_2)$, $\cdots$, and $(1, S_1)$, $(3, S_3)$, $\cdots$ are symmetrical with respect to the straight line $y=s$. The same is true for the curves $y=\sigma_0^{(k)}(x)$ and $y=\sigma_1^{(k)}(x)$ passing respectively through the points $(0, \sigma_0^{(k)})$, $(2, \sigma_2^{(k)})$, $\cdots$ and $(1, \sigma_1^{(k)})$, $(3, \sigma_3^{(k)})$, $\cdots$ $(k=0, 1, 2, \cdots)$. Hence we have a graphical method of summation of divergent series. If $a_n$ is not an integral function, this method approximates the sum. (Received May 13, 1931)

258. Professor R. L. Wilder: *On Jordan continua that are the common boundaries of two or more domains in $E_n$.*

Examples are given to show that a Jordan continuum in $E_3$ may be the common boundary of any finite number or a denumerable infinity of mutually exclusive domains. No simple closed curve disconnects such a continuum, and this property is generalized in a theorem for any number of dimensions. If $M$ is a common boundary of two uniformly connected im kleinen domains in $E_3$ and there is a point $P$ on $M$ and a positive number $d$ such that all 1-cycles of $D_i \cdot S(P, d)(i=1, 2)$ bound in $D_i$, then $M$ is the common boundary of only two domains. Thus, if $M$ is the common boundary of at least three uniformly connected im kleinen domains (examples are given of such continua), all of the domains link one another in every neighborhood of any point of $M$. An example is given, in $E_3$, of a Jordan continuum whose complement has the same connectivity numbers as the complement of the sphere, and is not disconnected by the omission of any arc; however, it not only fails to satisfy the Jordan Curve Theorem internally (see L. Zippin, American Journal of Mathematics, vol. 52, pp. 340–1), but is not disconnected by the omission of any simple closed curve. (Received May 8, 1931.)

259. Dr. A. B. Brown: *Group invariants and torsion coefficients.*

The paper is largely expository. Two sets of invariants of a finite commutative group, their relation to each other, to the Poincaré numbers of an infinite group with a finite number of generators, and to torsion coefficients in analysis situs are considered. Four brief proofs of the uniqueness of the invariants, three of them developed by men working in analysis situs, are mentioned. These are equivalent to four proofs of the group-theoretic part of the proof of the invariance of the torsion coefficients in analysis situs. The question of isomorphism of quotient-groups of isomorphic sub-groups is considered as well as its bearing on a method for determining the invariants. The torsion coefficients in analysis situs are one set of invariants of a certain finite Abelian group determined topologically by the complex. The possibility of using the other set of invariants as torsion coefficients is discussed. (Received May 21, 1931.)

260. Dr. N. H. McCoy, (National Research Fellow): *On the resultant of a system of forms homogeneous in each of several sets of variables.*

The resultant of a system of general forms, homogeneous in each of several sets of variables, is here shown to have properties analogous to the well known properties of the resultant in the case of a single set of variables. Following Koenig (Algebraischen Grössen), it is shown that the resultant is determined by the two properties (i) that it is an irreducible polynomial in the coefficients of the general forms, and (ii) that it belongs to the module defined by the forms. In particular, a general expression is obtained for the degrees and weights of the resultant, thus furnishing a proof of a theorem stated by Sylvester. It is then found that Sylvester's dyalitic method of elimination may be

extended to the case of certain forms of the type here discussed. As special cases it is shown that the method may be used to obtain the resultant of multiple binary forms with certain restrictions on the degrees. The resultant of a system of forms linear in each of the sets of variables may also be expressed in determinant form by this method. (Received May 25, 1931.)

261. Dr. R P. Agnew (National Research Fellow): *On deferred Cesàro means.*

With each pair of sequences $\{p_n\}$ and $\{q_n\}$ of non-negative integers satisfying the conditions $q_n > p_n$, $n = 1, 2, 3, \cdots$, and $\lim_{n\to\infty} q_n = +\infty$, we associate a method $D(\{p_n\}, \{q_n\})$ of summability which assigns to a given sequence $\{s_n\}$ the value $\lim_{n\to\infty} (s_{p_n+1}+s_{p_n+2}+\cdots+s_{q_n})/(q_n-p_n)$ when this limit exists. In case $\lim_{n\to\infty} p_n = +\infty$, $D(\{p_n\}, \{q_n\})$ has useful properties not possessed by $(M)$, the Cesàro arithmetic mean method. Methods of the form $D(\{p_n\}, \{q_n\})$ are compared in a study of mutual consistency, relative inclusion, and equivalence. (Received June 8, 1931.)

262. Dr. L. M. Blumenthal: *Note on Volterra and Fredholm products of symmetric kernels.*

This paper establishes two theorems concerning Volterra and Fredholm products of two continuous symmetric functions. Theorem 1: If the Volterra product of two continuous symmetric functions is a symmetric function, then one of the functions is identically zero. Theorem 2: In order that the Fredholm product of two continuous symmetric kernels, admitting only a finite number of characteristic values, be symmetric, it is necessary and sufficient that the characteristic functions of the two kernels be orthogonal, or that $\phi_i(x) = c_{ij}\theta_j(x)$ where $\phi_i(x)$ and $\theta_j(x)$ are the characteristic functions of the kernels and $c_{ij}$ is a constant. The first theorem is proved by first showing that two symmetric functions have a symmetric product (Volterra) if and only if the two functions are skew permutable; that is, $K_1K_2 = -K_2K_1$. It is then shown that if a continuous function is skew permutable with a given continuous function of determined order, the function is identically zero. Though the second theorem can be discussed in terms of permutability of the second kind (as is shown), it is obtained by special methods more germane to the problem considered. (Received June 16, 1931.)

263. Dr. L. M. Blumenthal: *An application of metric geometry to determinants.*

This paper points out by means of examples that a certain theorem on determinants announced by H. W. Richmond (Proceedings of the Cambridge Philosophical Society, vol. 14 (1908), p. 475) and which formed the basis of a paper by B. Segre (Atti, Accademia dei Lincei, (6), vol. 2 (1925), p. 539) is not valid. The main interest of the paper is the establishment of the following "counter-theorem": If a symmetric determinant of the fifth order $|a_{ij}|$, $a_{ij} = a_{ji}$, $a_{1j} = 1$, $a_{ij} > 0 (i \neq j)$, $a_{ii} = 0$, $(i, j = 1, 2, 3, 4, 5)$ is different from zero, and the complementary minors of four of the elements in the principal diagonal vanish, then the complementary minor of the remaining element does not vanish. This is counter to the above-mentioned "theorem" of Richmond-Segre, for de-

terminants of order five. The determinant contains six distinct positive elements different from unity. Interpreting these numbers as mutual distances of four distinct points, it is shown that the four points are pseudo-linear (Karl Menger, *Untersuchungen über allgemeine Metrik*, Mathematische Annalen, vol. 100 (1928), p. 125) and the theorem follows from known properties of these points. Four corollaries give interesting properties of the class of determinants dealt with in the theorem. (Received June 16, 1931.)

264. Professor R. G. Lubben: *Two theorems concerning abstract spaces.*

(1) In order that in a space $H$ Fréchet each point set either contain a point of condensation of itself or be separable, it is necessary and sufficient that every point set contain a limit point of itself. (2) A space $S$ Fréchet or a Hausdorff space which satisfies the first countability axiom and is locally compact is regular. Theorem (1) may be extended to more general spaces. (Received June 19, 1931.)

265. Miss Beatrice Aitchison: *Concerning regular accessibility.*

This paper comprises a systematic treatment of the subject of regular accessibility from a new point of departure, suggested by the recently established theorem of Moore-Menger that any connected, locally connected $G_\delta$ set is arcwise connected. A general theorem on the regular accessibility of limit points of such sets is proved, and most of the known propositions on regular accessibility are developed from this point. Simpler proofs are obtained, and some of the previously known results have been generalized substantially. (Received June 23, 1931.)

266. Professor B. W. Jones: *On Selling's reduction of positive ternary quadratic forms.*

In a positive ternary quadratic form $f$, L. Charve (Annales de L'Ecole Normale, (2), vol. 9 (1880), Suppl.) replaces the variables $x, y, z$ by $x-t, y-t, z-t$ respectively to give the form $\phi = -g(y-z)^2-h(z-x)^2-k(x-y)^2-l(x-t)^2-m(y-t)^2-n(z-t)^2$. With Selling, he calls $f$ reduced if none of the coefficients of $\phi$ are negative and proves algebraically that the $\phi$ of such a reduced form is unique except for permutations of $g, h, k, l, m, n$. This paper provides further inequalities on the coefficients which result in a unique form $f$. This theory of reduction is correlated with the Eisenstein method of reduction first by showing that every Eisenstein reduced form with non-positive cross-product coefficients coincides with the reduced form of this paper and second, by finding the transformations that take the reduced forms of one type into those of the other, thus giving an algebraic proof of Eisenstein's reduction. (Received June 29, 1931.)

267. Professor A. A. Albert: *A note on cyclic algebras of order sixteen.*

In this note it is proved that there exist cyclic normal division algebras of order sixteen over a field $R(x, y)$, $x$ and $y$ indeterminates, such that the algebras do not satisfy the Wedderburn norm condition for cyclic algebras and are

direct products of generalized quaternion algebras. This example is a modification of an example of R. Brauer in that he also considered division algebras which were such direct products over such a field but these algebras are cyclic and moreover the proof that these algebras are division algebras is essentially different from his. (Received June 29, 1931.)

268. Professor A. A. Albert: *On the construction of cyclic algebras with a given exponent.*

The problem of the construction of cyclic algebras has been reduced to the case where the order of the algebra is a power of a single prime. The author then considers cyclic algebras of order $p^{2e}$, $p$ a prime, generated by a cyclic field $Z$ or order $p^e$ over the general non-modular field $F$, and a number $\gamma$ in $F$. It is proved that the least power of $\gamma$ which is the norm of an element of $Z$ is a power of $p$ and that this integer is the exponent of the algebra. The field $Z=Z_e$ contains sub-fields $Z_{e-1}, \cdots, Z_1, Z_0=F$ of orders $p^i$ respectively and it is proved that $\gamma^\alpha$, $\alpha=p^i$ is the norm of a quantity of $Z$ if and only if $\gamma$ itself is the norm $N_i(f)$ for the field $Z_i$, of a quantity of $Z_i$. This criterion is used to give a particularly simple criterion for the construction of all cyclic algebras over an algebraic field of finite order with respect to the field of all rational numbers. (Received June 29, 1931.)

269. Professor A. A. Albert: *Division algebras over an algebraic field.*

The theory of quadratic forms over an algebraic field $R(x)$ of H. Hasse, (Crelle, 1923) applied to give extensions of the theorems of the author's recent papers on rational division algebras to algebras over $R(x)$. In particular it is proved that the direct product of two generalized quaternion algebras over $R(x)$ is never a division algebra, that a necessary and sufficient condition that a normal division algebra of order sixteen over $R(x)$ be a cyclic (Dickson) algebra is that the algebra contain a quantity $u$ not in $R(x)$ but such that $u^2=a^2+b^2$ where $a$ and $b$ are in $R(x)$. The theorems of an as yet unpublished Transactions paper of Hasse, *The theory of cyclic algebras over an algebraic number field*, are applied to give an alternative proof of the above theorem on generalized quaternion algebras and it is in fact proved that a necessary and sufficient condition that a direct product of two normal division algebras over $R(x)$ be a division algebra is that their orders be relatively prime. (Received June 29, 1931.)

270. Mr. C. H. Harry: *An examination of some cut sets of space.*

This paper treats some cut sets of a connected, locally connected, locally compact and metric space $S$. Using the property that any collection $(X)$ of closed and non-separated cuttings of $S$ which separate any two fixed points $a$ and $b$ is ordered, the paper proves that any infinite monotonic collection $(X_i)$ of sets $X$ is convergent and has a non-vacuous limit $L$ which separates $a$ and $b$ if $L\cdot(a+b)=0$. The second part of the paper deals with the collection $L$ of all points $x$ which together with some point $y$ separate two fixed points $a$ and $b$. The theorem established is that $L+a+b$ is closed and compact. (Received July 1, 1931.)

271. Professor Morris Marden: *On Stieltjes polynomials* (Second Paper.)

The author has simplified his previous treatment (see Abstract No. 36-9-341) through introducing the idea of "covering function" of a convex region, by which he means a function $k(\lambda)$ such that the inequality $|z-\lambda| \leqq k(\lambda)$ is satisfied by and only by points $z$ in $K$. The result that, if the $a_j$ lie in a circle $C$ of radius $r$, the zeros of every characteristic and Stieltjes polynomial lie in a concentric circle $C'$ of radius $r\sec\gamma$ has been thereby extended to read that, if the $a_j$ lie in a convex region $K$ with covering function $k(\lambda)$, these zeros lie in a convex region $K'$ with covering function $k(\lambda)\sec\gamma$. If, for instance, $K$ is an ellipse with major axis $A$, $K'$ is a confocal ellipse of major axis $A\sec\gamma$. The paper also considers the case that the coefficient of $dw/dz$ in the differential equation is the ratio of two real polynomials. The results are somewhat more extensive and the proof much simpler than the corresponding in the doctoral thesis of Charles Vuille, of Zurich. (Received July 3, 1931.)

272. Dr. M. G. Scherberg: *The Degree of Convergence of a Series of Bessel Functions.*

The author finds the degree of convergence; i.e., the order of magnitude of the difference between a function $f(x)$ and the first $n$ terms of a series representing it, of the Bessel Series known as the Fourier-Bessel Series and also the Dini Series of Bessel Functions. In the first part of the paper $f'(x)$ is restricted to limited variation and permitted of having a finite number of finite jumps. The results are reported in the set of five theorems which show the influence of the conditions at the end points of the interval of convergence. The second part allows $f(x)$ $(p-1)$ continuous derivatives and a $p$th derivative like that of $f'(x)$ above. A set of theorems analogous to those in the previous section are found. The third and final part of the paper deals with the order of magnitude of the constants which are found in formulas employed in the development of the first two sections of the paper. Thus the results of the paper may readily be carried through in numerical detail for any particular problem in hand. (Received July 6, 1931.)

273. Mr. Max Coral: *The Euler-Lagrange multiplier rule for double integrals.*

Let $z_i=Z_i(x, y)(i=1, 2)$ be a surface of class $C''$ furnishing a minimum for $I=\iint f(x, y, z_i, p_i, q_i)\,dxdy$ $(p_i=\delta z_i/\delta x, q_i=\delta z_i/\delta y)$ in a class of surfaces $z_i=z_i(x, y)$ which have the same boundary above the unit square $B$ in the $(x, y)$-plane and which satisfy the differential equation $\phi(x, y, z_1, z_2, p_1, p_2)=0$. Gross (Monatshefte für Mathematik und Physik, vol. 27 (1916), pp. 114-120) showed that if the minimizing surface is "normal," there exists a unique function $\lambda(x, y)$ of class $C'$ in $B$ such that along the minimizing surface the Euler-Lagrange differential equations $F_{z_i}-(\delta/\delta x)F_{p_i}-(\delta/\delta y)F_{q_i}=0$, formed for the function $F=f+\lambda\phi$, are satisfied. In the present paper a somewhat more general region $B$ is considered, an improved definition of normality is given, and a much simpler proof is made of the multiplier rule for the above problem of Lagrange for double integrals by introducing an auxiliary problem of Lagrange for simple integrals. Furthermore, similar methods are found to be effective in

proving the multiplier rule for the case when the equation of condition has the more general form $\phi(x, y, z_1, z_2, p_1, p_2, q_2)=0$. (Received July 7, 1931.)

274. Dr. I. J. Schoenberg (International Research Fellow): *A supplement to the law of inertia for real quadratic forms with an application to the projective geometry of hyperquadrics.*

In this paper the following results are established: 1. Let the real quadratic form $Q=\sum_1^n a_{ik}x_ix_k$, whose positive and negative type numbers are $p$ and $q$, be transformed by the real linear transformation (1) $x_i=\sum_{\alpha=1}^m c_{i\alpha}y_\alpha (i=1, 2, \cdots, n)$ of rank $r$ into the new quadratic form $R=\sum_1^m b_{\alpha\beta}y_\alpha y_\beta$ with the type numbers $p'$ and $q'$. Then the following inequalities hold (2) $p'+q'\leqq r$, $p+q\leqq n$, $0\leqq p-p'\leqq n-r$, $0\leqq q-q'\leqq n-r$. 2. Conversely: If $Q=\sum_1^n a_{ik}x_ix_k$ and $R=\sum_1^m b_{\alpha\beta}y_\alpha y_\beta$ are two real quadratic forms with $p\geqq p'$ and $q\geqq q'$, and if $r$ satisfies the conditions $p'+q'\leqq r\leqq Min(n-p+p', n-q+q', m)$, then there exists $\infty^\varepsilon$ real transformation (1) of rank $r$ carrying $Q$ into $R$. Theorem 1 yields as a special case $m=n=r$ the law of inertia and theorem 2 gives the conditions of real equivalence of real quadratic forms ($p=p'$, $q=q'$, $n=m$ give $p+q\leqq r\leqq n$). Similar theorems are established for hermitian forms and for the complex field. The equation $Q=0$ represents a hyperquadric in projective coordinates in a $(n-1)$-dimensional space. Making $m=r\leqq n-1$, the theorems stated above permit us to characterize completely from the real projective point of view all the possible different sections of our hyperquadric by all the real linear subspaces of dimensions $r-1=1,2,3, \cdots, n-2$. (Received July 8, 1931.)

275. Dr. T. H. Gronwall: *On the theory of potentiometric titration.*

The end point of the titration is determined by an algebraic equation of the fifth degree with coefficients dependent on two parameters. According to the values of the latter, there are either one or three real and positive roots, and the present paper deals with their separation and approximate computation. (Received July 10, 1931.)

276. Dr. T. H. Gronwall: *An inequality for the Bessel functions of the first kind with imaginary argument.*

In a problem in wave mechanics, it was found necessary to obtain fairly close lower and upper bounds for the expression $y_\gamma(z)=zI_\gamma'(z)/I_\gamma(z)$, the relative order of magnitude of $z$ and $\gamma$ being unrestricted. The present note accomplishes this by means of the Riccati equation satisfied by $y_\gamma(z)$. (Received July 10, 1931.)

277. Dr. T. H. Gronwall: *A special conformal space of three dimensions occurring in wave mechanics.*

The $ds^2=g_{ij}dx^idx_j$ for which $g^{11}=g^{22}=1$, $g^{33}=2$, $g^{12}=g^{23}=0$, $g^{13}=[(x^1)^2-(x^2)^2+(x^3)^2]/2x^1x^3$, $g^{23}=[(x^2)^2-(x^1)^2+(x^3)^2]/2x^2x^3$, occurs in connection with the wave equation of the helium atom (Hylleraas, Zeitschrift für Physik, vol. 54 (1929), pp. 347–366). This $ds^2$ is shown to be conformal to a flat $ds^2$, and the transformation formulas are worked out. The physical application of this transformation forms the subject of another paper. (Received July 10, 1931.)