PROPERTIES OF THE OPERATOR $z^{-\nu} \log z$, WHERE $z = d/dx$*

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1. *Introduction.* A considerable bibliography with modern increments has been built up around the interpretation and use of the operators z^{ν} and $z^{-\nu}$, $z = d/dx$, where ν may assume fractional as well as integral values.† The peculiar efficacy of these symbols in the solution of integral equations of Volterra type on the one hand and in the resolution of difficulties in certain types of electrical transmission problems on the other has made a study of their properties of more than passing interest.

The essential peculiarity of the operators z^{ν} and $z^{-\nu}$ regarded as analytic functions is found in the fact that they possess branch points at the origin. It becomes of interest, therefore, to inquire into the existence of other operators with branch points at $z = 0$, as for example $\log z$. Wiener, in a paper which employs the Fourier transform of a function as its basis of definition, has established a rigorous foundation for the discussion of such branch point operators.‡ He illustrates his method by applying it to the operator $z^{1/2}$, but fails to give explicit consideration to $\log z$. In a subsequent paper F. Sbrana§ has supplied this deficiency by means of a method similar to that of Wiener in its use of the Fourier transform.

The original suggestion, however, is due to V. Volterra¶ who formulated it in a theory of *logarithms of composition*, which he applied effectively in the solution of the integral equation

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* Presented to the Society, April 3, 1931.

† See for example, J. D. Tamarkin, *On integrable solutions of Abel's integral equation*, Annals of Mathematics, vol. 31 (1930), pp. 218–229.

‡ *The operational calculus*, Mathematische Annalen, vol. 95 (1925–26), pp. 557–584.

§ *Sull'operazione infinitesimale nel gruppo delle derivazione*, Atti dei Lincei, (6), vol. 11 (1930), pp. 364–368.

¶ See V. Volterra and J. Pérès, *Leçons sur la Composition et les Fonctions Permutables*, Paris, 1924, Chap. 8. See also H. T. Davis, *The Inversion of Integrals of Volterra Type*, Indiana University, 1927, pp. 62–66.

$$f(x) = \int_0^x [\log (x - t) + C]u(t)dt, \quad f(0) = 0,$$

where $C = \lim_{\nu=1} - \Gamma'(\nu)/\Gamma(\nu) = 0.5772157 \cdots$ (Euler's constant).

It is some consequences of this suggestion, cast into the notation of derivatives, which we shall employ in this note. The present investigation will be limited to the formal aspects of the problem since we shall consider only what might be termed the *operational-theoretic* properties of the symbols.

2. *The Definition of* $z^{-\nu} \log z$ *and its Inverse.* As is well known, generalized integration is defined by means of the formula

$$z^{-\nu} \rightarrow u(x) = \int_c^x \{(x - t)^{\nu-1}/\Gamma(\nu)\} u(t)dt,$$

where the symbol $\rightarrow$ is used to denote "operating upon." This becomes the definition of Riemann for $c = 0$ and the definition of Liouville for $c = -\infty$. Adopting the former for our present purposes, we replace $u(t)$ by $e^{(t-x)z} \rightarrow u(x)$ and thus attain the operational identity

$$z^{-\nu} = \int_0^x \{(x - t)^{\nu-1} e^{(t-x)z}/\Gamma(\nu)\} dt = \int_0^x \{s^{\nu-1} e^{-sz}/\Gamma(\nu)\} ds.$$

Let us now regard both terms of this equation as functions of the *continuous variable* ν and differentiate with respect to it. We thus obtain

$$(1) \quad - dz^{-\nu}/d\nu = z^{-\nu} \log z = \int_0^x [s^{\nu-1} e^{-sz} \{\psi(\nu) - \log s\}/\Gamma(\nu)] ds,$$

where $\psi(\nu) = \Gamma'(\nu)/\Gamma(\nu)$.

The formal reciprocal (the inverse operator) of this function is obtained by integrating $z^{-\mu+\nu}$ with respect to μ from 0 to ∞. We thus find

$$(2) \qquad z^{\nu+1} \int_0^\infty z^{-\mu-1} d\mu = z^\nu/\log z = z^{\nu+1} \int_0^\infty \lambda(s) e^{-sz} ds,$$

where we use the abbreviation

$$(3) \qquad \lambda(s) = \int_0^\infty \{s^\mu/\Gamma(\mu + 1)\} d\mu.$$

It is easily proved that the function $\lambda(s)$ is asymptotic to e^s in the positive interval. To see this we apply the Maclaurin integral test* for convergence to the series $e^s=1+s+s^2/2!+s^3/3!+\cdots$, and thus obtain the inequality

$$e^s - 1 \leqq \lambda(s) \leqq e^s.$$

Dividing by e^s, we have

$$1 - e^{-s} \leqq e^{-s}\lambda(s) \leqq 1,$$

which, for large positive values of s, establishes the desired property.

3. *Generalizations.* The definitions of the preceding section may be generalized in a useful way as follows. Let us denote by p the derivative operator $p=d/d\nu$, and by $\phi(p)$ a power series in p. We may then write

$$\phi(p) \rightarrow z^{-\nu} = \phi(p) \rightarrow \int_0^x \{(x-t)^{\nu-1}/\Gamma(\nu)\} e^{(t-x)z} dt,$$

from which we derive

$$z^{-\nu}\phi(-\log z) = \int_0^x \{\phi(p) \rightarrow (x-t)^{\nu-1}/\Gamma(\nu)\} e^{(t-x)z} dt. \tag{4}$$

For example, if $\phi(p)=p^2$, we obtain from (4) the formula

$$\begin{aligned} z^{-\nu}\log^2 z &= \int_0^x s^{\nu-1}e^{-sz}\{\log^2 s - 2\psi(\nu)\log s + \psi^2(\nu) - \psi'(\nu)\} ds/\Gamma(\nu), \\ &= \psi(\nu) z^{-\nu}\log z + \int_0^x s^{\nu-1}e^{-sz}\{\log^2 s - \psi'(\nu)\} ds/\Gamma(\nu). \end{aligned} \tag{5}$$

Similarly for the generalization of formula (2) we multiply $z^{-\mu+\nu}$ by $\theta(\mu)$ and integrate from 0 to ∞. We thus obtain

$$z^{\nu+1}\int_0^\infty z^{-\mu-1}\theta(\mu)d\mu = z^{\nu+1}\int_0^x e^{(t-x)z} I(x-t) dt, \tag{6}$$

where we abbreviate by means of the formula

$$I(s) = \int_0^\infty \{\theta(\mu)s^\mu/\Gamma(\mu+1)\} d\mu.$$

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* T. J. Bromwich, *Infinite Series*, 2d ed. (1926), p. 33.

An example useful to us later is derived from the function $\theta(\mu)=e^{a\mu}$. We are thus able to derive

$$
(7)\qquad
\begin{aligned}
z^{\nu+1}\int_0^\infty z^{-\mu-1}\theta(\mu)d\mu &= z^\nu\int_0^\infty e^{(a-\log z)\mu}d\mu = z^\nu/(a-\log z)\\
&= z^{\nu+1}\int_0^x e^{(t-x)z}I(x-t)dt
\end{aligned}
$$

where we write $I(s)=\int_0^\infty\{e^{a\mu}s^\mu/\Gamma(\mu+1)\}d\mu$.

4. *Properties of the Operators.* In the subsequent use of the operators defined in the preceding sections it will be important to know the order in which they may be applied. This question is answered in the following theorems.

THEOREM 1.

$$
(8)\qquad z^{-\mu}\longrightarrow(z^{-\nu}\log z) = z^{-\mu-\nu}\log z.
$$

PROOF. By definition we have

$$
\begin{aligned}
z^{-\mu}\longrightarrow(z^{-\nu}\log z) &= \int_0^x\{(x-t)^{\mu-1}/\Gamma(\mu)\}dt\int_0^t\{\psi(\nu)\\
&\qquad -\log(t-s)\}(t-s)^{\nu-1}e^{(s-x)z}ds,\\
&= \int_0^x e^{(s-x)z}ds\int_s^x\{\psi(\nu)\\
&\qquad -\log(t-s)\}(x-t)^{\mu-1}(t-s)^{\nu-1}dt/[\Gamma(\mu)\Gamma(\nu)].
\end{aligned}
$$

Making the transformation $y=(t-s)/(x-s)$, we can write this in the form

$$
(9)\qquad z^{-\mu}\longrightarrow(z^{-\nu}\log z) = \int_0^x e^{(s-x)z}J(x-s)ds/[\Gamma(\mu)\Gamma(\nu)],
$$

where we have

$$
\begin{aligned}
J(x-s) = (x-s)^{\mu+\nu-1}\{&\psi(\nu)\Gamma(\mu)\Gamma(\nu)/\Gamma(\mu+\nu)\\
&-\int_0^1\log[(x-s)y](1-y)^{\mu-1}y^{\nu-1}dy\}.
\end{aligned}
$$

From the identity

$$\int_0^1 \log y(1 - y)^{\mu-1}y^{\nu-1}dy = \{\psi(\nu) - \psi(\mu + \nu)\}B(\mu, \nu),*$$

where $B(\mu, \nu)$ is the Eulerian integral of first kind, we can then write

$$J(x - s) = (x - s)^{\mu+\nu-1}B(\mu, \nu)\{\psi(\mu + \nu) - \log (x - s)\}.$$

When this function is replaced in (9) we see that the integral reduces to the definition of $z^{-\mu-\nu} \log z$ which establishes the theorem.

THEOREM 2.

$$z^{-\nu} \log z \to z^{-\mu} = z^{-\mu} \to z^{-\nu} \log z = z^{-\mu-\nu} \log z. \tag{10}$$

PROOF. Giving our attention to the left member we have, by definition,

$$\begin{aligned} z^{-\nu} \log z \to z^{-\mu} &= z^{-\nu} \log z \to \int_0^x (x - s)^{\mu-1}e^{(s-x)z}ds/\Gamma(\mu) \\ &= \int_0^x (x - t)^{\nu-1}\{\psi(\nu) - \log (x - t)\}dt \\ &\quad \cdot \int_0^t (t - s)^{\mu-1}e^{(s-x)z}ds/[\Gamma(\mu)\Gamma(\nu)] \\ &= \int_0^x e^{(s-x)z}ds \\ &\quad \cdot \int_s^x \{\psi(\nu) - \log (x - t)\}(x - t)^{\nu-1}(t - s)^{\mu-1}dt/[\Gamma(\mu)\Gamma(\nu)]. \end{aligned}$$

Making the transformation $y=(t-s)/(x-s)$, we can then write

$$z^{-\nu} \log z \to z^{-\mu} = \int_0^x e^{(s-x)z}K(x - s)ds/[\Gamma(\mu)\Gamma(\nu)], \tag{11}$$

where we abbreviate as follows:

$$K(x - s) = (x - s)^{\nu+\mu-1}\int_0^1 \{\psi(\nu)(1 - y)^{\nu-1}y^{\mu-1}$$

* See N. Nielsen, *Handbuch der Gammafunktion*, Leipzig, 1906, p. 172.

$$-\log(x-s)(1-y)^{\nu-1}y^{\mu-1}-\log(1-y)(1-y)^{\nu-1}y^{\mu-1}\}\,dy$$
$$= (x-s)^{\nu+\mu-1}\Gamma(\mu)\Gamma(\nu)\left[\psi(\mu+\nu)-\log(x-s)\right].$$

When this is substituted in (11), the integral becomes the definition of $z^{-\mu-\nu}\log z$. But, by Theorem 1, this is equivalent to $z^{-\mu}\rightarrow z^{-\nu}\log z$, which completes the proof.

THEOREM 3.

$$z^{-\nu}\log z \rightarrow z^{-\mu}\log z = z^{-\mu-\nu}\log^2 z. \tag{12}$$

PROOF. We have, by definition,

$$\begin{aligned} &z^{-\nu}\log z \rightarrow z^{-\mu}\log z \\ &= \int_0^x (x-t)^{\nu-1}[\psi(\nu)-\log(x-t)] \\ &\quad\cdot\int_0^t (t-s)^{\mu-1}[\psi(\mu)-\log(t-s)]e^{(s-x)z}ds, \\ &= \int_0^x e^{(s-x)z}ds \\ &\quad\cdot\int_s^x (x-t)^{\nu-1}(t-s)^{\mu-1}[\psi(\nu)-\log(x-t)][\psi(\mu)-\log(t-s)]dt. \end{aligned}$$

Employing the transformation $y=(t-s)/(x-s)$, we find

$$z^{-\nu}\log z \rightarrow z^{-\mu}\log z = \int_0^x e^{(s-x)z}L(x-s)ds,$$

where we abbreviate thus:

$$\begin{aligned} L(x-s) = (x-s)^{\mu+\nu-1}\int_0^1 y^{\mu-1}(1-y)^{\nu-1}[\psi(\nu)-\log(x-s) \\ -\log(1-y)][\psi(\mu)-\log(x-s)-\log y]dy. \end{aligned} \tag{13}$$

Computing the partial derivatives $\partial B(\mu,\nu)/\partial\mu$, $\partial B/\partial\nu$, and $\partial^2 B/\partial\mu\partial\nu$, where $B(\mu,\nu)=\int_0^1 y^{\mu-1}(1-y)^{\nu-1}dy$, we get the integrals

$$(14)\quad \begin{cases} \displaystyle\int_0^1 \log y(1-y)^{\nu-1}y^{\mu-1}dy = \{\psi(\nu)-\psi(\mu+\nu)\}B(\mu,\nu), \\ \displaystyle\int_0^1 \log(1-y)(1-y)^{\nu-1}y^{\mu-1}dy = \{\psi(\nu)-\psi(\mu+\nu)\}B(\mu,\nu), \\ \displaystyle\int_0^1 \log y \log(1-y)(1-y)^{\nu-1}y^{\mu-1}dy = \{\psi(\mu)\psi(\nu) \\ \qquad -[\psi(\mu)+\psi(\nu)]\psi(\mu+\nu)+\psi^2(\mu+\nu) \\ \qquad -\psi'(\mu+\nu)\}B(\mu,\nu). \end{cases}$$

When the integrand of (13) is expanded and the values of (14) substituted, we get

$$L(x-s) = (x-s)^{\mu+\nu-1}B(\mu,\nu)[\log^2(x-s) \\ -2\psi(\mu+\nu)\log(x-s)+\psi^2(\mu+\nu)-\psi'(\mu+\nu)].$$

From this, taking note of (5), we derive the proof of the theorem.

THEOREM 4.

$$z^{-\nu}\log^n z \to z^{-\mu}\log^m z = z^{-\mu-\nu}\log^{n+m} z,$$

where n and m are positive integers.

PROOF. In the proof of this theorem, which is seen to include the preceding as a special case, we shall employ a slightly different argument. We have, by definition,

$$z^{-\nu}\log^n z \to z^{-\mu}\log^m z$$

$$= (-1)^{n+m}\frac{\partial^n}{\partial\nu^n}\int_0^x \{(x-t)^{\nu-1}/\Gamma(\nu)\}dt$$

$$\cdot\frac{\partial^m}{\partial\mu^m}\int_0^t \{(t-s)^{\mu-1}/\Gamma(\mu)\}e^{(s-x)z}ds,$$

$$= (-1)^{n+m}\int_0^x e^{(s-x)z}ds$$

$$\cdot\int_s^x \frac{\partial^{n+m}}{\partial\nu^n\partial\mu^m}\{(x-t)^{\nu-1}(t-s)^{\mu-1}/[\Gamma(\mu)\Gamma(\nu)]\}dt.$$

Making the customary transformation, we find

$$z^{-\nu}\log^n z \to z^{-\mu}\log^m z$$

$$= (-1)^{n+m}\int_0^x e^{(s-x)z}ds$$

$$\cdot\int_0^1 \frac{\partial^{n+m}}{\partial\nu^n\partial\mu^m}\{(x-s)^{\mu+\nu-1}(1-y)^{\nu-1}y^{\mu-1}/[\Gamma(\mu)\Gamma(\nu)]\}dt$$

$$= (-1)^{n+m}\frac{\partial^{n+m}}{\partial\nu^n\partial\mu^m}\int_0^x e^{(s-x)z}\{(x-s)^{\mu+\nu-1}/\Gamma(\mu+\nu)\}ds$$

$$= z^{-\nu-\mu}\log^{n+m} z,$$

which was to be proved.

5. *The Rule of Leibnitz.* In the application of operators the following generalization of the formula of Leibnitz for the nth derivative of a product is often of great importance:

$$(15)\quad F(x, z)\to uv = vF(x, z)\to u + v'F_z'(x, z)\to u/1! + v''F_z''(x, z)\to u/2! + \cdots,$$

where we employ the abbreviation $F_z^{(n)}(x, z) = \partial^n F(x, z)/\partial z^n$.*

Although it has been proved that this formula applies when $F(x, z)\equiv F(z)$ is a rational function of z, existing discussions of its validity do not include operators of the form $z^{-\nu}\log z$. It therefore becomes of interest to prove the following theorem.

THEOREM 5. *The generalized Leibnitz formula* (15) *applies when* $F(x, z) = z^{-\nu}\log z$.

PROOF. By definition, we have

$$z^{-\nu}\log z\to uv = \int_0^x (x-t)^{\mu-1}\{\psi(\nu) - \log(x-t)\}u(t)v(t)dt/\Gamma(\nu)$$

$$= \int_0^x \{\psi(\nu) - \log(x-t)\}u(t)$$

$$\cdot\left\{\sum_{n=0}^{\infty} v^{(n)}(x)(x-t)^{\nu+n-1}(-1)^n/n!\right\}dt/\Gamma(\nu).$$

* This formula was originally due to C. J. Hargreave, London Philosophical Transactions, vol. 138 (1848), p. 31. See also S. Pincherle, *Opérations fonctionnelles*, Encyclopédie des Sciences Mathématiques, part 2, vol. 4, No. 26, in particular p. 10.

But since $\psi(\nu)=\psi(\nu+1)-1/\nu$, $\Gamma(\nu)=\Gamma(\nu+1)/\nu$, we can write

$$
\begin{aligned}
z^{-\nu}\log z \to u\, v &= v(x)z^{-\nu}\log z \to u - \int_0^x \sum_{n=1}^{\infty}\Bigg\{\log(x-t) \\
&\quad - \psi(\nu+n) + \frac{1}{\nu} + \frac{1}{(\nu+1)} + \cdots + \frac{1}{(\nu+n-1)}\Bigg\} v^{(n)}(x) \\
&\quad \cdot (x-t)^{\nu+n-1}(-1)^n u(t)dt \frac{\nu(\nu+1)\cdots(\nu+n-1)}{\Gamma(\nu+n)n!} \\
&= \Bigg[v(x)z^{-\nu}\log z + \sum_{n=1}^{\infty}(-1)^n v^{(n)}(x) \\
&\quad \cdot \frac{\{\nu(\nu+1)\cdots(\nu+n-1)z^{-\nu-n}\log z - P_n(\nu)z^{-\nu-n-1}\}}{n!}\Bigg] \to u(x),
\end{aligned}
$$

where we write

$$
\begin{aligned}
P_1(\nu) &= 1,\ P_n(\nu) = (\nu+1)\cdots(\nu+n-1) + \nu(\nu+2)\cdots(\nu+n-1) \\
&\quad + \nu(\nu+1)(\nu+3)\cdots(\nu+n-1) + \cdots \\
&\quad + \nu(\nu+1)\cdots(\nu+n-2).
\end{aligned}
$$

But the coefficient of $v^{(n)}(x)/n!$ is precisely the nth derivative of $z^{-\nu}\log z$, which was to be proved.

The following more general result follows as a consequence of the preceding theorem.

THEOREM 6. *The generalized Leibnitz formula applies when* $F(x, z)=z^{-\nu}\log^n z$, *where n is a positive integer.*

PROOF. We employ induction. In Theorem 5 we have shown that the theorem is true for $n=1$. Let us now assume that it also holds for $n=k$. Making the abbreviations $\phi(z)=z^{-\nu}\log^k z$ and $\rho(z)=\log z$, we shall then immediately obtain

$$
\rho(z) \to (\phi \to uv) = \rho \to \left\{ [v\phi + v'\phi'/1! + v''\phi''/2! + \cdots] \to u(x) \right\}.
$$

But since we have shown above that the theorem is true for $\rho(z)$, we at once deduce

$$
\begin{aligned}
\rho(z) \to (\phi \to u\, v) &= \{ v\rho\phi + v'(\rho'\phi + \phi'\rho)/1! \\
&\quad + v''(\rho''\phi + 2\rho'\phi' + \rho\phi'')/2! + \cdots \} \to u(x).
\end{aligned}
$$

But, by Theorem 4, we have $\rho(z)\to\phi(z)=\rho\phi$. This fact combined with the preceding equation shows that the proposition is also true for $n=k+1$. The proof is then completed by induction in the customary manner.

6. *The Bourlet Operational Product.* For the application of the operator $z^{-\nu}\log z$ to problems in the theory of functional equations it is necessary to know whether it belongs to the class of operators the symbolic multiplication of which is determined by the Bourlet product:

$$(16)\quad S(x, z) \rightarrow T(x, z) = [S\cdot T](x, z) = ST + (\partial T/\partial x)(\partial S/\partial z)/1! + (\partial^2 T/\partial x^2)(\partial^2 S/\partial z^2)/2! + \cdots .$$

The following theorem will now be proved.

THEOREM 7. *If $S(x, z)$ and $T(x, z)$ are operators the symbolic multiplication of which is given by the Bourlet product, then $z^{-\nu}\log^n zS(x, z)$ and $z^{-\mu}\log^m z\, T(x, z)$ are also such operators.*

PROOF. If we abbreviate $\phi(z)=z^{-\nu}\log^n z$ and $\rho(z)=z^{-\mu}\log^m z$, our problem is to show that

$$\phi(z)S \rightarrow \rho T = \rho[ST + (\partial T/\partial x)(\partial\phi S/\partial z)/1! + (\partial^2 T/\partial x^2)(\partial^2\phi S/\partial z^2)/2! + \cdots].$$

To prove this we expand $S(x, z)$ and $T(x, z)$ in the series

$$S(x, z) = \sum_{n=-\infty}^{\infty} S_n(x)z^n, \text{ and } \quad T(x, z) = \sum_{n=-\infty}^{\infty} T_n(x)z^n.*$$

We can then write

$$A(x, z) \equiv (\phi S) \rightarrow \sum_{n=-\infty}^{\infty} \rho T_n(x)z^n = \sum_{n=-\infty}^{\infty} (\phi S) \rightarrow \rho T_n(x)z^n.$$

But since we have shown in Theorem 6 the validity of the Leibnitz rule for ϕz^n, in particular, and hence formally for $\sum_{n=-\infty}^{\infty} S_n(x)\phi z^n$, in general, we are able to write

$$\begin{aligned} A(x, z) &= \sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty}(d^m T_n/dx^m)(\partial^m\phi S/\partial z^m)\rho z^n/m! \\ &= \sum_{m=0}^{\infty}(\partial^m\phi S/\partial z^m)\sum_{n=-\infty}^{\infty}(d^m T_n/dx^m)\rho z^n/m! \\ &= \rho\sum_{m=0}^{\infty}(\partial^m S/\partial z^m)(\partial^m T/\partial x^m)/m!. \end{aligned}$$

We thus attain the desired equation and the theorem is proved.

* For the generality of these expansions, see C. Bourlet, Annales de l'École Normale Supérieure, (3), vol. 14 (1897), p. 150.

7. *Applications*. Three simple examples will now be given to show the natural way in which the logarithmic operator may be expected to appear in the solution of functional equations.

EXAMPLE 1. (Volterra's problem.) Solve the integral equation

$$f(x) = \int_0^x [\log (x - t) + A]u(t)dt, \qquad f(0) = 0.$$

Writing this equation symbolically, we have

$f(x) = (- z^{-1} \log z + az^{-1}) \rightarrow u(x)$, $a = A - C$ (Euler's constant),

and, by Theorem 7, we find

$$u(x) = z/(a - \log z) \rightarrow f(x).$$

The interpretation of this symbol follows at once from (7):

$$u(x) = z^2 \rightarrow \int_0^x f(t)I(x - t)dt,$$

where

$$I(s) = \int_0^\infty \{e^{a\mu}s^\mu/\Gamma(\mu + 1)\} d\mu.$$

EXAMPLE 2. (Volterra.) Solve the integral equation

$$f(x) = \int_0^x [\log^2 (x - t) + A \log (x - t) + B]u(t)dt.$$

This equation can be written symbolically in the form

$$f(x) = z^{-1}(\log^2 z + \alpha \log z + \beta) \rightarrow u(x),$$

where we abbreviate,

$$\alpha = - A - 2\psi(1), \quad \beta = B + A\psi(1) + \psi^2(1) + \psi'(1).$$

The solution then appears in the form

$$\begin{aligned} u(x) &= z/(\log^2 z + \alpha \log z + \beta) \rightarrow f(x) \\ &= z[\phi_1/(\log z - \lambda_1) + \phi_2/(\log z - \lambda_2)] \rightarrow f(x), \end{aligned}$$

where $\phi_1 = -\phi_2 = 1/(\lambda_1 - \lambda_2)$, and where λ_1 and λ_2 are roots of the equation $\lambda^2 + \alpha\lambda + \beta = 0$. The solution, when λ_1 and λ_2 are distinct, is thus seen to be attainable by means of the operator of Example 1.

When $\lambda_1 = \lambda_2 = \lambda$, the preceding solution is replaced by

$$u(x) = [z/(\log z - \lambda)^2] \to f(x)$$
$$= z^2 \to \int_0^x R(x - t)f(t)dt,$$

where

$$R(s) = \int_0^\infty \{ s^\mu \mu e^{\lambda\mu}/\Gamma(\mu + 1) \} d\mu .$$

EXAMPLE 3. Discuss the operator inverse to $X(D) = 1 - D$, $D = xz$. This operator is found to be $Y(D) = 1 - De^{-D}\mathrm{li}(e^D)$, where we abbreviate

$$\mathrm{li}(e^D) = \int_{-D}^\infty e^{-t} dt/t .$$

To verify this we compute the Bourlet product (16) and thus obtain

$$[X \cdot Y] = (1 - D)[1 - De^{-D}\mathrm{li}(e^D)]$$
$$+ D[e^{-D}\mathrm{li}(e^D) - De^{-D}\mathrm{li}(e^D) + 1] \equiv 1.$$

Employing a well known expansion for the function $\mathrm{li}(e^D)$, we can write $Y(D)$ in the form,

$$Y(D) = 1 - De^{-D}[C + \log D + D + D^2/(2 \cdot 2!) + D^3/(3 \cdot 3!) + \cdots],$$

where C is Euler's constant.* If we note that

$$e^{-D}F(z) \to f(x) = e^{-D} \to [F(z) \to f(x)] = \lim_{x=0} [F(z) \to f(x)],$$

where $F(z)$ is a rational function of z, and from our definition that $e^{-D} \log z \to f(x) = k$ (a constant), we may then write

$$Y(D) \to f(x) = Ax + 1 - x \log x - \sum_{n=2}^\infty x^n f^{(n)}(0)/[(n-1) \cdot (n-1)!],$$

where A is a constant. It can be verified without difficulty that this is the operational form of the solution of the differential equation $u(x) - xu'(x) = f(x)$.

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* See Bromwich, *Infinite Series*, 2d ed., 1926, p. 334.