$\{n_2\}$ of $\{n\}$ such that $n=n_1+n_2+2$ is an $\{n-1\}$, the V_{T-1}^{N-k+2} is a $\{T-1\}$ in $\{T\}$, for $T=T'+T''+2$. Hence, we have

$$N - k + 2 = 1,$$

or

$$N = k - 1,$$

which was to be proved.

It is about impossible to state generally any proposition concerning the properties of the hypersurfaces of this type. The properties of some of them can be obtained readily.

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A NECESSARY AND SUFFICIENT CONDITION FOR THE NON-EQUIVALENCE OF ANY TWO RATIONAL GENERALIZED QUATERNION DIVISION ALGEBRAS*

BY A. A. ALBERT

Two algebras $\mathfrak{A}$ and $\mathfrak{B}$ over the same field F are called *equivalent* (or *simply isomorphic*) if it is possible to establish between their quantities a (1-1) correspondence such that if any quantities x and y of $\mathfrak{A}$ correspond to X and Y of $\mathfrak{B}$, then $x+y$, xy and αx correspond to $X+Y$, XY, αX respectively for every α of F. We shall consider two generalized quaternion division algebras† $\mathfrak{A}$ and $\mathfrak{B}$ over the field of all rational numbers, R. Let $\mathfrak{A}$ be given by

$$(1)\quad \mathfrak{A} = (e, i_1, j_1, i_1j_1),\ i_1^2 = \rho_1 e,\ j_1^2 = \sigma_1 e,\ j_1i_1 = -i_1j_1,$$

where e is the modulus of $\mathfrak{A}$, and ρ_1 and σ_1 are in R. Without loss of generality ρ_1 and σ_1 may be taken to be each products of distinct rational prime integers.

Similarly let

$$(2)\quad \mathfrak{B} = (E, I_1, J_1, I_1J_1),\ I_1^2 = \rho_2 E,\ J_2^2 = \sigma_2 E,\ J_1I_1 = -I_1J_1,$$

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† For the definition and properties of these algebras see L. E. Dickson, *Algebren und ihre Zahlentheorie*, pp. 46–49.

where ρ_2 and σ_2 are each products of distinct rational prime integers, and E is the modulus of $\mathfrak{B}$. Without loss of generality we may take $\sigma_2<0$ since one of $\rho_2,\sigma_2,-\rho_2\sigma_2$ is negative and by changing the roles of I_1, J_1, $I_1 J_1$ we may take σ_2 to be the negative number. The rank equation of $\mathfrak{A}$ is known (loc. cit.) to be

$$\omega^2 - 2\alpha_4\omega + (\alpha_4^2 - \alpha_1^2\rho_1 - \alpha_2^2\sigma_1 + \alpha_3^2\sigma_1\rho_1) = 0, \tag{3}$$

where the general quantity of $\mathfrak{A}$ is $x=\alpha_4+\alpha_1 i_1+\alpha_2 j_1+\alpha_3 i_1 j_1$. We have obviously the following property

LEMMA 1. *A quantity x which is not a rational multiple of e has a rational multiple of e as its square if and only if*

$$x = \alpha_1 i_1 + \alpha_2 j_1 + \alpha_3 i_1 j_1, \tag{4}$$

in which case

$$x^2 = (\alpha_1^2\rho_1 + \alpha_2^2\sigma_1 - \alpha_3^2\sigma_1\rho_1)e. \tag{5}$$

Consider the form

$$q = \alpha_1^2\rho_1 + \alpha_2^2\sigma_1 - \alpha_3^2\sigma_1\rho_1 - \alpha_4^2\rho_2 - \alpha_5^2\sigma_2. \tag{6}$$

The signs of ρ_1, σ_1 and $-\rho_1\sigma_1$ are the same if and only if $\rho_1<0$, $\sigma_1<0$. But we have taken $\sigma_2<0$, whence $-\sigma_2>0$ and has a sign different from that of $\rho_1<0$. The signs of the coefficients of q are therefore not all alike and they are all different from zero when $\mathfrak{A}$ and $\mathfrak{B}$ are division algebras. But every indefinite quadratic form in five variables is a null form* so that there exist integers $\alpha_1, \cdots, \alpha_5$, not all zero, for which $q=0$. Let $\alpha_{12}, \alpha_{22}, \cdots, \alpha_{52}$ be *any* set of integer solutions of $q=0$. The quantity $\rho'=\alpha_{42}^2\rho_2+\alpha_{52}^2\sigma_2$ is a rational integer equal to $\alpha_{12}^2\rho_1+\alpha_{22}^2\sigma_1-\alpha_{43}^2\sigma_1\rho_1$.

The quantities

$$i' = \alpha_{12}i_1 + \alpha_{22}j_1 + \alpha_{32}i_1j_1, \quad I' = \alpha_{42}I_1 + \alpha_{52}J_1$$

have the properties that

$$(i')^2 = \rho' e, \qquad (I')^2 = \rho' E.$$

Since $\alpha_{12}, \cdots, \alpha_{52}$ are not all zero, i' and I' are not both zero quantities. Hence $\rho'\neq 0$. But $\mathfrak{A}$ and $\mathfrak{B}$ are division algebras

* A theorem of A. Meyer, Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich, vol. 29 (1884), pp. 209–222; see also Bachmann, *Zahlentheorie*, vol. IV$_1$, p. 266.

so that $i' \neq 0e$, $I' \neq 0E$. If we write $\rho' = \rho\pi^2$, where ρ and π are integers and ρ is a product of distinct primes, then, by replacing the $\alpha_{r2} (r = 1, \cdots, 5)$ by $\alpha_{r1} = \alpha_{r2}/\pi$, we replace i' by $i = i'/\pi$, I' by $I = I'/\pi$ and ρ' by ρ.

LEMMA 2. *There exist rational numbers* $\alpha_{11}, \alpha_{21}, \cdots, \alpha_{51}$ *with* $\alpha_{11}, \alpha_{21}, \alpha_{31}$ *not all zero and* α_{41}, α_{51} *not both zero, such that if*

$$i = \alpha_{11} i_1 + \alpha_{21} j_1 + \alpha_{31} i_1 j_1, \quad I = \alpha_{41} I_1 + \alpha_{51} J_1, \tag{7}$$

then

$$i^2 = \rho e, \quad I^2 = \rho E, \quad \rho = \alpha_{41}^2 \rho_2 + \alpha_{51}^2 \sigma_2, \tag{8}$$

where ρ *is a product of distinct rational prime integers.*

Write $J = I_1 J_1$. Then obviously $JI = -IJ$ and $J^2 = \delta E$, δ in R. By replacing J by a rational multiple of itself we may obviously take δ a product of distinct primes. If $\alpha_{21} = \alpha_{31} = 0$, then by writing $j = j_1$, we have $ji = -ij$ and $j^2 = \gamma e$ with γ in R. If α_{21} and α_{31} are not both zero, then let

$$j = \alpha_{31} \rho_1 j_1 + \alpha_{21} i_1 j_1 \neq 0e, \tag{9}$$

whence $j^2 = -\sigma_1 \rho_1 (\alpha_{21}^2 - \alpha_{31}^2 \rho_1) e = \gamma e$ with γ in R.

Using the multiplication table of $\mathfrak{A}$, we find

$$\begin{aligned} ji &= (\alpha_{31}\rho_1 j_1 + \alpha_{21} i_1 j_1)(\alpha_{11} i_1 + \alpha_{21} j_1 + \alpha_{31} i_1 j_1) \\ &= -\alpha_{31}\alpha_{11}\rho_1 i_1 j_1 + \alpha_{31}\alpha_{21}\rho_1\sigma_1 - \alpha_1 \rho_{31}^2 i_1 \\ &\quad - \alpha_{21}\alpha_{11}\rho_1 j_1 + \alpha_{21}^2 \sigma_1 i_1 - \alpha_{21}\alpha_{31}\rho_1\sigma_1 \\ &= (\alpha_{21}^2\sigma_1 - \alpha_{31}^2\rho_1) i_1 - \alpha_{11}\alpha_{21}\rho_1 j_1 - \alpha_{31}\alpha_{11}\rho_1 i_1 j_1, \end{aligned} \tag{10}$$

while

$$\begin{aligned} -ij &= -(\alpha_{11} i_1 + \alpha_{21} j_1 + \alpha_{31} i_1 j_1)(\alpha_{31}\rho_1 j_1 + \alpha_{21} i_1 j_1) \\ &= -(\alpha_{11}\alpha_{31}\rho_1 i_1 j_1 + \alpha_{11}\alpha_{21}\rho_1 j_1 - \alpha_{21}^2\sigma_1 i_1 + \alpha_{31}^2\rho_1\sigma_1 i_1) \ . \end{aligned} \tag{11}$$

Hence $ji = -ij$ and in all cases j is not a polynomial in i with rational coefficients. It follows that e, i, j, ij are linearly independent with respect to R and form a basis of $\mathfrak{A}$. Similarly E, I, J, IJ are linearly independent with respect to R and form a basis of algebra $\mathfrak{B}$.

THEOREM 1. *By finding a single solution of a solvable diophantine equation we may represent any pair of generalized quaternion division algebras in the canonical form*

$$(12) \quad \mathfrak{A} = (e,i,j,ij),\ i^2 = \rho e,\ j^2 = \gamma e,\ ji = -ij,$$

$$(13) \quad \mathfrak{B} = (E,I,J,IJ),\ I^2 = \rho E,\ J^2 = \delta E,\ JI = -IJ,$$

with e and E respectively the moduli of $\mathfrak{A}$ and $\mathfrak{B}$, where ρ, γ and δ are multiplication constants expressed in terms of the original multiplication constants of $\mathfrak{A}$ and $\mathfrak{B}$ and the above solution, and where, without loss of generality, ρ, γ, δ may be taken to be products of distinct rational primes.

We shall now discuss a necessary and sufficient condition that any two generalized quaternion division algebras $\mathfrak{A}$ and $\mathfrak{B}$ be equivalent. We take the pair in the canonical form (12), (13). Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are equivalent so that there exists a (1-1) correspondence between the quantities of $\mathfrak{A}$ and $\mathfrak{B}$ which is preserved under addition, multiplication and scalar multiplication. The modulus e of $\mathfrak{A}$ will correspond to the modulus E of $\mathfrak{B}$. Let s in $\mathfrak{A}$ correspond to I, and t correspond to J. Then $s^2 = \rho e$, $t^2 = \delta e$ and $ts = -st$. But, by Lemma 1,

$$(14) \qquad s = \lambda_1 i + \lambda_2 j + \lambda_3 ij$$

so that

$$(15) \qquad s^2 = (\lambda_1^2\rho + \lambda_2^2\gamma - \lambda_3^2\gamma\rho)e = \rho e,$$

and

$$(16) \qquad \lambda_1^2\rho + \lambda_2^2\gamma - \lambda_3^2\gamma\rho = \rho,$$

for rational $\lambda_1, \lambda_2, \lambda_3$. Let first λ_2 and λ_3 be not both zero. Then if

$$(17) \qquad t_1 = \lambda_3\rho j + \lambda_2 ij,$$

we have $t_1 s = -st_1$, $t_1 \neq 0e$, and

$$(18) \qquad t_1^2 = \gamma[(\lambda_3\rho)^2 - \lambda_2^2\rho].$$

For we may evidently use here the proof (10), (11) by which we showed that $ji = -ij$. But $ts = -st$, so that $tst^{-1} = t_1 s t_1^{-1}$ and $(t_1^{-1}t)s(t_1^{-1}t)^{-1} = s$. It follows that $t_1^{-1}t$ is a polynomial in s and we may write

$$t = (\xi_1 e + \xi_2 s)t_1, \qquad (\xi_1 \text{ and } \xi_2 \text{ in } R). \tag{19}$$

Then $\delta e = t^2 = (\xi_1^2 - \xi_2^2\rho)t_1^2 = (\xi_1^2 - \xi_2^2\rho)[(\lambda_3\delta)^2 - \lambda_2^2\rho]\gamma e$. Write $\eta_1 = \xi_1(\lambda_3\rho) + \xi_2\lambda_2\rho, \eta_2 = \xi_1\lambda_2 + \xi_2(\lambda_3\rho)$, so that η_1 and η_2 are rational numbers. Then we have proved that, when λ_2 and λ_3 are not both zero,

$$\delta = (\eta_1^2 - \eta_2^2\rho)\gamma. \tag{20}$$

Next let $\lambda_2 = \lambda_3 = 0$. We have $s = \lambda_1 i$ and $\lambda_1^2\rho e = \rho e$ so that $\lambda_1 = 1$, $s = i$. We then take $t_1 = j$ and have $t_1^2 = \gamma e$. As before $t = (\xi_1 e + \xi_2 s)t_1$ and, if we write $\eta_1 = \xi_1$, $\eta_2 = \xi_2$, we again have (20).

Conversely let (20) be true. Then $\mathfrak{A}$ and $\mathfrak{B}$ are equivalent under the correspondence

$$\mu_1 e + \mu_2 i + \mu_3 t + \mu_4 it \sim \mu_1 E + \mu_2 I + \mu_3 J + \mu_4 IJ, \tag{21}$$

where $\mu_1, \cdots, \mu_4$ are independent variables in R and $t = (\eta_1 e + \eta_2 i)j$. For $i^2 = \rho e$, $I^2 = \rho E$, $t^2 = \delta e$, $J^2 = \delta E$, $ti = -it$, $JI = -IJ$.

THEOREM 2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be any two generalized quaternion division algebras over R. Then they are equivalent* if and only if, when they are put in a canonical form* (12), (13), *we have*

$$\delta = (\eta_1^2 - \eta_2^2\rho)\gamma \tag{20}$$

for rational η_1 and η_2.

Necessary and sufficient conditions that (20) be true are known. Let the greatest common divisor of γ and δ be ν so that $\gamma\delta = \nu^2\epsilon$, where ϵ is a product of distinct primes. Then (20) is true if and only if

$$\epsilon = \eta_3^2 - \eta_4^2\rho = 0 \tag{22}$$

for rational η_3, η_4. Let the greatest common divisor of ϵ and ρ be π so that $\epsilon = \epsilon'\pi$, $\rho = \rho'\pi$. Then (20) is true if and only if

$$\epsilon' = \pi\eta_5^2 - \rho'\eta_4^2 \tag{23}$$

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* Any generalized quaternion algebra $\mathfrak{A}$ is self-reciprocal under the correspondence given by $i \sim i$, $j \sim j$, $ij \sim ji$. Hence we may add the word "reciprocal" to the above.

for rational η_5 and η_4, where now ϵ', π, ρ' and $\epsilon'\pi\rho'$ are each products of distinct primes. Then (20) is true if and only if

$$\pi\theta_1^2 - \rho'\theta_2^2 - \epsilon'\theta_3^2 = 0 \tag{24}$$

for integer θ_1, θ_2, θ_3 not all zero. For when (20) is satisfied so is (24) when we write $\eta_5 = \theta_1/\theta_3$, $\eta_4 = \theta_2/\theta_3$ with integer θ_1, θ_2, θ_3. Conversely let (24) be satisfied for integer θ_1, θ_2, θ_3 not all zero. If $\theta_3 = 0$ then $\pi\theta_1^2 - \rho'\theta_2^2 = 0$, $\pi\rho' = \rho$, so that $(\pi\theta_1)^2 - \rho\theta_2^2 = 0$ contrary to the hypothesis that $\mathfrak{A}$ is a division algebra and hence ρ is not a rational square. It follows from our definitions of ϵ', π, ρ' that when (23) is satisfied so is (20). But (23) is satisfied by $\eta_5 = \theta_1/\theta_3$, $\eta_4 = \theta_2/\theta_3$. Hence (20) can be satisfied by rational η_1, η_2 if and only if the form in (24) is a null form. Using a known result of the theory of numbers*, we have the theorem:

THEOREM 3. *Let $\gamma\delta = \nu^2\epsilon$ where ν and ϵ are integers and ϵ is a product of distinct primes. Let the greatest common divisor of ϵ and ρ be π so that $\rho = \rho'\pi$, $\epsilon = \epsilon'\pi$. Then two division algebras $\mathfrak{A}$ and $\mathfrak{B}$ in a canonical form* (12), (13) *are equivalent if and only if*

$$-\epsilon'\rho' \text{ is a quadratic residue of } \pi, \tag{25}$$

$$\epsilon \text{ is a quadratic residue of } \rho', \tag{26}$$

$$\rho \text{ is a quadratic residue of } \epsilon'. \tag{27}$$

As a corollary of Theorem 2 we shall establish *the non-equivalence of any two of the D_τ algebras* of L. E. Dickson (loc. cit., Chapter IX) *which have different t's.* The D_τ algebras have $\rho = -1$ and $\gamma = \tau$ taken to be a product of distinct primes of the form $4n+3$. Let D_{τ_1} and D_{τ_2} be two such algebras with $\tau_1 \neq \tau_2$. Then $\tau_2 \neq (\eta_1^2 + \eta_2^2)\tau_1$ since otherwise we would have $\tau_2\tau_1$ expressible as a sum of two rational squares which is impossible when $\tau_2\tau_1$ contains a prime factor $4n+3$ to an odd power. Hence, by Theorem 2, D_{τ_1} and D_{τ_2} are non-equivalent.

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* See P. Bachmann, *Arithmetik der Quadratischen Formen*, Chapter 8.