# NOTE ON RULED SURFACES AND THEIR DEVELOPABLES*

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If a plane $\pi$, whose coordinates $\kappa$, $\lambda$, $\mu$, $\nu$, are functions of the independent variable $x$ in the system of differential equations

$$
(1) \qquad \begin{aligned} y'' + p_{12}z' + q_{11}y + q_{12}z &= 0, \\ z'' + p_{21}y' + q_{21}y + q_{22}z &= 0, \end{aligned}
$$

defining a ruled surface $R$, is to be fixed relative to $R$, then must these coordinates satisfy the relations

$$
(2) \qquad \begin{aligned} 2\kappa' &= \eta\kappa - p_{12}\lambda + \mu, \\ 2\lambda' &= - p_{21}\kappa + \eta\lambda + \nu, \\ 2\mu' &= (p_{12}p_{21} - 4q_{11})\kappa + \eta\mu - p_{12}\nu, \\ 2\nu' &= (p_{12}p_{21} - 4q_{22})\lambda - p_{21}\mu + \eta\nu, \end{aligned}
$$

where $\eta$ is an arbitrary function of $x$.†

The pole of this plane with respect to the quadric $Q$,

$$
(3) \qquad x_1x_4 - x_2x_3 = 0,
$$

which osculates $R$ along a line element $l_{yz}$ is given by the expression

$$
(4) \qquad \theta = \nu y - \mu z - \lambda\rho + \kappa\sigma
$$

where

$$
(5) \qquad \rho = 2y' + p_{12}z, \qquad \sigma = 2z' + p_{21}y,
$$

and the point which corresponds to $\pi$ in the null-system of the linear complex which osculates $R$ along $l_{yz}$ is given by the expression

$$
(6) \qquad \phi = - p_{21}\mu y + p_{12}\nu z + p_{21}\kappa\rho - p_{12}\lambda\sigma .
$$

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† Carpenter, *Some fundamental relations in the projective differential geometry of ruled surfaces*, Annali di Matematica, (3), vol. 26.

The equations of the quadrics $Q_1$, $Q_2$, which osculate the two branches of $R$'s flecnode surface along those line elements which intersect $l_{yz}$, are

$$p_{12}(x_1x_4 - x_2x_3) - 2qx_4^2 = 0, \tag{7}$$

$$p_{21}(x_1x_4 - x_2x_3) - 2qx_3^2 = 0, \tag{8}$$

where $q = q_{11} - q_{22}$.

The poles of $\pi$ with respect to these two quadrics are given respectively by the expressions

$$\theta_1 = (4q\kappa + p_{12}\nu)y - p_{12}\mu z - p_{12}\lambda\rho + p_{12}\kappa\sigma = 4q\kappa y + p_{12}\theta, \tag{9}$$

$$\theta_2 = p_{21}\nu y + (4q\lambda - p_{21}\mu)z - p_{21}\lambda\rho + p_{21}\kappa\sigma = 4q\lambda z + p_{21}\theta. \tag{10}$$

From (9) and (10) it results that the lines $l_{\theta\theta_1}$, $l_{\theta\theta_2}$ pass through the respective points $y$, $z$ and hence that the plane $\pi$, determined by these three points, contains $l_{yz}$. Its equation is found to be

$$\kappa x_3 + \lambda x_4 = 0. \tag{11}$$

The one-parameter family of planes $\pi_1$, one for each line element of $R$, determines a developable surface. The generators of this surface are the lines $l_{\alpha\theta}$ where

$$\alpha = \lambda y - \kappa z \tag{12}$$

is the point of intersection of plane $\pi$ with $l_{yz}$. This we show by finding the characteristic line of plane $\pi_1$.

By making use of conditions (2) the equation of the plane determined by the points $\theta + d\theta$, $\theta_1 + d\theta_1$, $\theta_2 + d\theta_2$ is found to be

$$\begin{aligned} (13)\quad & q[\kappa^2\lambda x_1 + \kappa\lambda^2 x_2 - (2\kappa\lambda\mu + \kappa^2\nu)x_3 - (2\kappa\lambda\nu + \lambda^2\mu)x_4]dx \\ & - [2q\kappa\lambda + (3q\kappa\lambda\eta + 4q'\kappa\lambda - p_{12}q\lambda^2 - p_{21}q\kappa^2)dx](\kappa x_3 + \lambda x_4) = 0. \end{aligned}$$

From (11) and (13) we find that this characteristic line is determined by the pair of planes

$$\kappa^2\lambda x_1 + \kappa\lambda^2 x_2 - \kappa(2\lambda\mu + \kappa\nu)x_3 - \lambda(2k\nu + \lambda\mu)x_4 = 0,$$
$$\kappa x_3 + \lambda x_4 = 0.$$

The points $\alpha$, $\theta$ are seen to lie in both planes.

Again, by making use of (2), we find

(14) $2pA\theta' + 2qB\phi' - p(q\kappa\lambda + A\eta)\theta - q(B\eta + 4C)\phi = 0,$

(15) $2A\theta' - 4q\kappa\lambda\alpha' - (A\eta + 2q\kappa\lambda)\theta + 2q(\kappa\lambda\eta + \kappa\nu + \lambda\mu)\alpha = 0,$

where

$$p = p_{12}q_{21} - p_{21}q_{12}, \quad A = \kappa\nu - \lambda\mu, \quad B = p_{21}\kappa^2 - p_{12}\lambda^2,$$
$$C = q_{21}\kappa^2 - q_{12}\lambda^2.$$

Equation (14) expresses the condition that the line $l_{\theta\phi}$ shall generate a developable. Equation (15) is the similar condition for $l_{\alpha\theta}$.

The focal points of these respective lines are seen to be

(16) $$\beta = pA\theta + qB\phi, \quad \gamma = A\theta - 2q\kappa\lambda\alpha.$$

Combining the above results we may state the following theorem.

THEOREM 1. *Each plane $\pi$ fixed in position with respect to a ruled surface $R$ determines with $R$ two developable surfaces. One cuts $\pi$ in the curve of intersection of $\pi$ and $R$, the other cuts $\pi$ in a curve whose points are those which correspond to $\pi$ in the null-systems of the osculating linear complexes of $R$, and they intersect each other in a curve whose points are the poles of $\pi$ with respect to the quadrics which osculate $R$ along its line elements.*

The plane $\pi_1$ is tangent to $Q$ at the point $\alpha$, and its null-point as determined by the linear complex osculating $R$ along $lyz$ is given by the expression

(17) $$\kappa p_{21}y - \lambda p_{12}z.$$

This point is on the line $l_{yz}$ and is in fact the point of intersection of $l_{yz}$ and $l_{\theta_1\theta_2}$, since

$$4q(\kappa p_{21}y - \lambda p_{12}z) = p_{21}\theta_1 - p_{12}\theta_2.$$

The equation of the plane of the points $\theta$, $\phi$, $\theta_1$ is

$$Bx_2 - p_{12}Ax_3 + Dx_4 = 0,$$

where $D = p_{21}\kappa\mu - p_{12}\lambda\nu$. Its pole with respect to $Q$ is given by

(18) $$Dy + p_{12}Az - B\rho$$

and its null-point by

(19) $$p_{21}Ay + Dz - B\sigma.$$

The equation of the plane of the points $\theta$, $\phi$, $\theta_2$ is

$$Bx_1 + Dx_3 - p_{21}Ax_4 = 0.$$

Its pole with respect to $Q$ is given by

$$p_{21}Ay + Dz - B\sigma \tag{20}$$

and its null-point by

$$Dy + p_{12}Az - B\rho. \tag{21}$$

In view of (18), (19), (20), (21) we may state the following theorem.

THEOREM 2. *The polar reciprocal of the line $l_{\theta\phi}$ with respect to $Q$ is identical with its polar reciprocal as determined by the linear complex osculating $R$ along $l_{yz}$.*

The equation of the plane of the points $\phi$, $\theta_1$, $\theta_2$, is

$$B(p_{12}\lambda x_1 + p_{21}\kappa x_2) - p_{12}[(p_{21}\kappa^2 + p_{12}\lambda^2)\nu - 2p_{21}\kappa\lambda\mu + 4q\kappa\lambda^2]x_3$$
$$+ p_{21}[(p_{21}\kappa^2 + p_{12}\lambda^2)\mu - 2p_{12}\kappa\lambda\nu - 4q\kappa^2\lambda]x_4 = 0,$$

the null-point of this plane is given by

$$[(p_{21}\kappa^2 + p_{12}\lambda^2)\nu - 2p_{21}\kappa\lambda\mu + 4q\kappa\lambda^2]y$$
$$+ [(p_{21}\kappa^2 + p_{12}\lambda^2)\mu - 2p_{12}\kappa\lambda\nu - 4q\kappa^2\lambda]z + B(\lambda\rho - \kappa\sigma),$$

and this expression, when multiplied by $2q$, becomes

$$(2q\lambda^2 + p_{21}A)\theta_1 - (2q\kappa^2 + p_{12}A)\theta_2.$$

Moreover the coordinates of this null-point satisfy the equation

$$\kappa x_1 + \lambda x_2 + \mu x_3 + \nu x_4 = 0,$$

of the plane $\pi$. We have thus the following theorem.

THEOREM 3. *The null-point of the plane determined by the null-point of $\pi$ and $\pi$'s two poles with respect $Q_1$ and $Q_2$, lies on the line $l_{\theta_1\theta_2}$ at the point where it cuts $\pi$.*

Many other interesting properties of this tetrahedron $\theta$, $\phi$, $\theta_1$, $\theta_2$, determined by $\pi$ and $R$, can be obtained by methods similar to the above.

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