THE MONTESANO QUINTIC SURFACE*

BY A. R. WILLIAMS

In a previous paper in this Bulletin (vol. 34 (1928), pp. 631–638), I have mentioned the fact that Montesano† has shown how to obtain a quintic surface with two consecutive skew double lines by quadratic transformation of a rational sextic. The latter has a quadruple point Q, at which are concurrent three coplanar double lines d_1, d_2, d_3, and a triple line k, which of course does not lie in the plane of the double lines. The plane system for the rational sextic and the derived quintic are given by Montesano. That of the sextic consists of the web of curves of order nine having in common 8 triple points $A_1, A_2, \cdots, A_8$ and 3 simple points B_1, B_2, B_3. The image of the triple line is the sextic with double points at $A_1, A_2, \cdots, A_8$ and passing through the points B. The image of a double line d_i is the cubic determined by the 8 points A and B_i. To B_i corresponds the residual line of the surface in the plane of the triple line and d_i. If we now apply a quadratic transformation whose fundamental system is an isolated point O on one of the double lines, say d_3, and the degenerate conic kd_1, the resulting surface will be of degree five. For it loses the plane kd_3 three times by reason of the triple line, the plane d_1d_3 twice by reason of the double line d_1, and the plane kd_1 twice on account of the double point at O. The transform of any ray through Q is another ray through Q. Therefore the quintic has a triple point at Q. A generic plane of the pencil k becomes a plane of the pencil d_1, and the relation between these planes is a collineation. Since a section of the sextic by a plane through k is a cubic which meets the plane d_1d_3 three times at Q, the section of the new surface by the corresponding plane through d_1 is a cubic which has d_1 for inflectional tangent at Q. Moreover,

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† Montesano, Rendiconti di Napoli, vol. 46 (1907), p. 66.

the plane kd_2 intersects the sextic in a residual line. The transform of this plane, which we will call π, meets the quintic, therefore, in d_1 and one other line; that is, d_1 counts for two coincident double lines on the quintic. Two sheets of the surface are tangent along d_1, the tangent plane being π. The trace of the quintic in the plane kd_1, which we will call σ, is the double line d_1, the line k, which is a simple line on the quintic corresponding to the residual intersection of the plane kd_3 with the sextic, and two lines passing through Q which correspond to the tangent planes to the sextic at O. They are in fact simply the intersections of those tangent planes with the plane kd_1. The tangent cone to the quintic at Q is the plane π and the plane σ taken twice. The equation of the surface is of course easily obtained and will be used later.

When the quadratic transformation is applied to the sextic the intersections of the homoloidal quadrics with the sextic become the plane sections of the quintic. The image of the intersection of a general quadric with the sextic is a curve of degree 18 passing 6 times through each of the 8 points A and twice through each point B. Since the homoloidal quadrics contain the triple line and the double line d_1 and have in common the point O on d_3, the image of the intersection of such a quadric with the sextic is of degree 9, having triple points at the 8 points A, passing through B_2 and B_3, and through the two points on the cubic determined by B_3 which correspond to O on d_3. To directions at these points correspond directions at O in the two tangent planes to the sextic, which we have seen give two lines through the node on the quintic. Therefore the plane system for the quintic consists of the web of curves of order nine having in common 8 triple points $A_1, A_2, \cdots, A_8$ and four simple points, say B, and E_1, E_2, E_3, where the three latter are the residual intersections of a cubic of the pencil with any proper nonic which has triple points at the 8 points A and passes through B. Let ψ be the cubic of the pencil determined by B, and ϕ the cubic that contains the points E. Then it is easily seen that ϕ corresponds to the node (triple point). To the nodal sections correspond the sextics of the net B, that is, the sextics that have double points at $A_1, A_2, \cdots, A_8$ and pass through B and B', where B' corresponds to B in I_{17}, the plane involution determined by the sextics that have double points at the 8 points A. Therefore B' lies on ψ and the complete image of the node is ϕ and the point B'. To directions at B', which corresponded on the sextic to directions at the node in the plane of the triple line and d_2, correspond on the quintic directions at the node in the plane π, the transform of that plane. To B there corresponds the line b, or residual intersection of π with the quintic. To E_1, E_2, E_3 correspond e_1, e_2, e_3, the simple lines of the quintic passing through the node in the plane σ. The cubic ψ is the image of the double line $\pi\sigma$. The points of ϕ are paired by I_{17}. A pair correspond to a direction at the node in σ. Through such a pair pass a pencil of sextics of the net B corresponding to the pencil of plane sections whose axis is the associated direction in σ. To a pencil of nodal sections having its axis in π correspond a pencil of sextics having a common tangent at B'. To a pencil of such sextics having a common tangent at B correspond a pencil of nodal sections whose axis joins the node to a point on b. No proper sextic of the net B, and no proper nonic of the fundamental system can pass through A_0, the ninth base point of the pencil of cubics. To A_0 corresponds at the node the direction of the double line. The images of the sections by planes through the double line are the cubics of the pencil. Twelve of these have a node; that is, the corresponding planes are tangent to the surface.

The number of fundamental curves of a pencil that have an additional node, that is, the class of the surface, is 30. For we have in the first place $3(9-1)^2$ less 20 for each of the eight common triple points, that is, 32. But a general pencil of planes contains one nodal section, and the corresponding pencil of fundamental curves has one degenerate curve made up of ϕ and a sextic of the net B. The two points where the latter meets ϕ are to be deducted, and the class of the surface is thus 30. Further, it is obvious geometrically that the Jacobian of a general net, which is the image of the curve of contact of the corresponding tangent cone, must contain ϕ as a factor. For through any point of space go a pencil of nodal sections; that is, in the corresponding net of fundamental curves there is a pencil of degenerate curves one of which has a double point at any given point of ϕ. The image of the curve of contact of a general tangent cone is therefore of degree 21, and has a seven-fold point at each of the 8 points A, a double point at B and simple points at E_1, E_2, E_3. Of the intersections of two such curves, 399 take place at the fundamental points. The class of the surface accounts for 30 more, leaving 12 to be found. We must, in fact, account for 5 more intersections of such a curve with ψ (the image of the double line), 4 more with ϕ, and 7 more with a general cubic of the pencil, or image of a plane section through the double line. Such a cubic meets ψ only at the fundamental points and A_0. This shows, as is otherwise obvious, that the curve of contact of a tangent cone can meet the double line only at the node and at pinch points, and therefore its image (of degree 21) meets ψ only at the fundamental points and at points which correspond to pinch points or to the node. Since in any net there is a pencil of degenerate curves corresponding to nodal sections, the net may be determined by three curves of the form $U\phi$, $V\phi$, and N, where U and V are sextics of the net B, and N is a proper fundamental nonic. We now find easily that $zJ=9\phi[\phi R+NS]$, where R is of degree 19 and $S=U(\phi_x V_y-\phi_y V_x)+V(\phi_y U_x-\phi_x U_y)$. Now remembering that B is a point of ψ, any sextic of the net B can be written $\alpha(\phi^2 f'-\phi'^2 f)+\beta\psi^2 f'+\gamma\phi\psi f'=0$, where f is any proper sextic having double points at $A_1, A_2, \cdots, A_8$ and f' and ϕ' are the values of f and ϕ at B*. Expressing U and V in this way we find that S vanishes for all points common to ϕ and ψ, that is, the curve of order 21 passes through A_0, a point corresponding to the node. It is im-

* Snyder, American Journal of Mathematics, vol. 33, p. 328.

mediately seen by use of R that it does not pass through the other point on ψ corresponding to the node, B', where U and V are 0. Hence there are just four pinch points on the double line. The remaining three intersections of the curve of order 21 with ϕ are of interest. The locus of self-corresponding points in the involution I_{17} is a curve of order 9, having a triple point at each of the 8 points A and meeting a cubic of the pencil at 3 other points. At such a point on ϕ the pencil of sextics of the net B that pass through it are tangent to ϕ, one of them to be sure having a double point. Now in a general net of fundamental curves there is one degenerate curve composed of a sextic of this pencil and ϕ. Since any three linearly independent curves of the net determine it we may let U be this sextic. Then at the point in question we have $U=\phi=0$ and $\phi_y U_x - \phi_x U_y = 0$, since U and ϕ are tangent there. Hence for such a point S vanishes, and it is a point on the curve of order 21. Since the points on ϕ correspond to directions at the node in the plane σ, the curve of contact of a general tangent cone passes four times through the node, three times in fixed directions in σ and once tangent to the double line. Therefore of the 12 intersections of two curves of degree 21 not accounted for by the fundamental points, either 8 take place at the images of the four pinch points on ψ, with simple intersections at A_0 and the other 3 points on ϕ just considered, or vice versa. It can be shown definitely by use of the above equation that the two curves are not tangent at A_0, so they must be tangent at the images of the four pinch points. That, of course, might be assumed. For at a pinch point on an ordinary double line the curves of contact of any two tangent cones have two points in common, while (if the surface is rational) the corresponding Jacobians have a simple intersection at the image of the pinch point. In the case of two coincident double lines the former number is 4. Hence we should expect the two Jacobians to be tangent. Moreover, one of our curves of degree 21 meets any cubic of the pencil six times, aside from the 8 points $A_1, A_2, \cdots, A_8$ and A_0.

That is, on any plane section through the double line are six points at which the tangent plane passes through a given point. We have seen that the image of the curve of contact of a general tangent cone is of degree 21. Its genus is also 21, and therefore the tangent cone (of order 16) has 42 cuspidal edges and 42 double edges. The quadruple edge at the node counts for 6 of the latter. In fact it may be shown by use of the equation of the surface that the tangent cone has a quadruple edge, 42 cuspidal edges, and 36 double edges with distinct points of contact.*

The sextics having double points at the 8 points A, but which do not pass through B, give on the surface skew sextics of genus 2. One of these plane sextics meets ϕ in two associated points, that is, two points corresponding to the same direction at the node in the plane σ. Hence the skew sextic has a tacnode at the node of the surface and meets the double curve in two other generally distinct points. The plane sextic may be composed of the line joining two of the 8 points A and the quintic having double points at the other 6 and passing through the first two. The line and the quintic meet ϕ in two associated points, and meet each other in 3 points on the locus of self-corresponding points in I_{17}. Thus we get on the surface two skew cubics, intersecting at 3 points, tangent to each other at the node, and each meeting the double line at one other point. There are obviously 28 such pairs. Likewise the plane sextic may be composed of the conic through 5 points A and the quartic having double points at the other 3 and containing the first 5. This gives a similar pair of space cubics. There are 56 of these pairs. The cubics corresponding to the 8 fundamental points A_i are of this type. With the cubic corresponding to one of them, say A_i, is paired the cubic whose image is the sextic corresponding to A_i in I_{17}, that is, the sextic having a triple point at A_i and double points at the other 7 points A. There are thus 92 pairs, or 184 such skew cubics on the surface. To the locus of self corresponding points

* Salmon, *Solid Geometry*, vol. II, Chap. XVIIa.

in I_{17} corresponds on the surface a curve of degree 9 and genus 4, which passes through the 3 points in which any one of these cubics meets the cubic with which it is paired.

The genus of a curve of degree n on the surface does not exceed the greatest integer in $(n^2+n+14)/10$.

A few slightly specialized types will now be mentioned. In setting up the plane system we may choose the 8 points A, which determine A_0, and a point B, which determines B' and the cubic ψ. We take any other cubic of the pencil for ϕ and then any proper curve of degree 9 having triple points at the 8 points A and passing through B will meet ϕ in 3 other points E_1, E_2, E_3. Two of these may fall together. In fact we may make the ninth degree curve tangent to ϕ at any point. Then two of the lines through the triple point will coincide; the class will be reduced by two, and the surface will acquire correspondingly a conic node on the two coincident lines, but in general distinct from the triple point. The Jacobian will have a triple point where the ninth degree curve is tangent to ϕ; but one of the 3 branches is tangent to ϕ and the removal of ϕ leaves the curve of degree 21 with a double point. Two such curves have thus four intersections at the coincident points instead of two, corresponding to the fall in the class of two; and everything else takes place as before. The images of the plane sections through the new node are the ∞^2 proper fundamental curves that have a double point at the two coincident base points on ϕ. The only points on ψ, the image of the double line, that correspond to the node are A_0 and B'; and a proper fundamental nonic can pass through neither. Therefore a general plane section through the new node does not contain the triple point, and the two singular points are distinct. The one pencil of plane sections that contain both nodes have for images the pencil of sextics of the net B that pass through the point of tangency of the ninth degree curve and ϕ. Similarly it might happen that E_1, E_2, E_3 fall together. Then the 3 simple lines through the triple point coincide; the class of the surface will be reduced by 3; and the new node on the coincident lines will be a binode, B_3. The plane of the coincident lines and the double line osculates the surface along the former. The equation of the surface, to be given later, shows that the new node, whether conic node or binode, may be thrown to the origin by suppression of a single coefficient. Apparently that can be accomplished only by allowing B to approach the coincident points E on ϕ. Also if ϕ is fixed we can not select 3 coincident points E arbitrarily. It appears that such a point is a possible ninth triple point for the nonics having triple points at the 8 points A.

If B is taken on the line joining two of the 8 points A, this line with the residual quintic and ϕ constitute the image of a plane section. To the line corresponds a conic on the surface cutting b and passing through the triple point. The quintic must pass through B', and the corresponding curve on the surface is a plane cubic with a double point at the node. If the join of two points A contains B and one of the points E, the corresponding conic consists of the line e and one other line. Finally, if three of the 8 points A lie on a line the surface has a new conic node. A_0 then lies on the conic determined by the other five; and to this conic corresponds a plane rational cubic that lies in a plane through the double line, passes through the original node (triple point) of the surface, and has its own double point at the new node. The images of the plane sections through the new node are of course octic curves which with the line in question constitute fundamental curves. One pencil of these octics consists of ϕ and the quintics which have double points at the remaining 5 points A, simple points at the 3 that are collinear, and which contain B. These are the images of the sections that pass through both nodes. Similarly if 6 of the 8 points A lie on a conic we get a new conic node. To the line joining the other two, and containing of course A_0, corresponds a plane rational cubic like that just mentioned.

It seems clear that the rational sextic described by Montesano is the most general sextic that has those singularities. If we assume this, to obtain the equation of the quintic we have merely to write the general equation of a sextic having xy for triple line and yz; $y+mx$, z; and xz for double lines, and apply the quadratic transformation

$$x':y':z':w' = xy:xz:yz:yw \text{ or } x:y:z:w = x'y':x'z':y'z':y'w'.$$

This gives the quintic

$$\begin{aligned} a_0y^2(z + my)w^2 + wy[a_1xy(z + my) + a_2xz(z + my) \\ + a_3y^3 + a_4y^2z + a_5yz^2 + a_6z^3] + b_1x^3(z + my)^2 \\ + x^2(z + my)L + xM + yzN = 0, \end{aligned}$$

where L, M, and N are binary forms of degree 2, 4, and 3 respectively in y, z. Here the planes π and σ are $z+my$ and y, and yz is the double line. The trace of the surface in $z+my$ is the double line and a residual line. Two sheets of the surface are tangent to each other along yz. It is easily shown without the aid of the plane representation that there are four points where the contact is higher. They are the pinch points that separate the portions of the double line where the surface is real from those where it is imaginary.

We saw above that a new node resulted when two of the three coplanar simple lines through the triple point became coincident. Setting $y=0$ in the equation just obtained we find $x(b_1x^2+b_4xz+b_9z^2)=0$ for the three lines. The terms in L and M that do not involve y are b_4z^2 and b_9z^4. Now the four partial derivatives vanish identically at all points on xy, except the single term b_9z^4 in $\partial/\partial x$ and the terms $a_6z^3w+c_4z^4$ in $\partial/\partial y$.

The term in N that does not involve y is c_4z^3. Hence when we bring two of the above three lines into coincidence by putting $b_4=0$ we get a new conic node at $(0:0:c_4:+a_6)$. We have seen that if we also put $b_4=0$ this node is a binode B_3. But in either case it can be brought to the origin by setting $a_6=0$. This does not change the nature of the surface, apparently; but to accomplish it by the plane representation we must let the point B approach, in an arbitrary direction, the three points E coincident on ϕ.

It is known that a quintic surface having for double curve a proper conic or a pair of intersecting lines, but possessing no other singularities, is not rational. If we write the equation of a quintic having two intersecting double lines and then let the parameter which expresses the angle between them be zero, we obtain a surface of class 36 which is in general not rational, but which has four pinch points and three generally distinct triple points on the coincident double lines. It may be shown that just five relations must obtain among the coefficients of such an equation if it is to represent a rational surface of the type just discussed.

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AMOUNTS OF INVESTMENTS AT ANY NUMBER OF RATES OF INTEREST*

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The writer developed certain generalizations of the most common formulas of the mathematical theory of finance in the June-July, 1921, issue of this Bulletin. These generalizations were based upon the use of two and three rates of interest, instead of one.

It is the purpose of this short paper to extend some of these generalizations to the use of any number of rates of interest. In developing such generalizations it will be found that solutions including reasonably complete treatments of the various possible relative frequencies of conversions of the various rates of interest, while not unduly difficult, would be so complicated in form and be so monopolized by terms due only to those treatments that it would be very difficult to sift out the information which we most desire now, namely, the nature of the function itself when the number of rates

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