REMARK ON A THEOREM OF OSGOOD CONCERNING CONVERGENT SERIES OF ANALYTIC FUNCTIONS

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1. *Introduction.* One of the most fundamental properties of analytic functions, that a sequence of such functions which converges in a region R must converge uniformly in at least one subregion R_1 of R, was discovered by Osgood* as early as 1901. It seems, however, that it has not yet been remarked that the theorem of Osgood holds also for meromorphic functions if, as is usual now, one extends the definition of uniform convergence in the neighborhood of a point so as to include functions having poles.

2. *Regular Convergence.* Although the principal object of the present note is to prove the theorem we have just stated, it is perhaps worth while to point out that all the theorems used in connection with the uniform convergence of analytic functions can be proved more easily if one replaces the concept of uniform convergence by another differing only slightly from it. We shall say that a sequence of functions $f_n(z)$, meromorphic on a closed region A, is *regularly convergent* at a point z_0 of A, if for every sequence of points $z_1, z_2, \cdots,$ belonging to A, and converging towards z_0, the (finite or infinite) limit

$$\lim_{n\to\infty} f_n(z_n)$$

exists†. We note that if this limit (which is obviously

* W. F. Osgood, *Note on the functions defined by infinite series whose terms are analytic functions,* Annals of Mathematics, (2), vol. 3 (1901), pp. 25–34.

† Although this very concept has been already used by Hans Hahn for the general theory of real functions under the very pregnant name of "*Stetige Konvergenz*" [H. Hahn, *Theorie der reellen Funktionen*, Berlin,

independent of the chosen sequence) is finite at every point of a closed and bounded subset B of A, and if the functions $f_n(z)$ converge regularly at every point of B, they must also converge uniformly on this set.

Moreover, it is readily seen that all the extensions of the concept of uniform convergence that have been made are included in the above definition.*

3. *Extension of Osgood's Theorem.* We consider now a sequence of meromorphic functions $f_n(z)$ which converges towards a function $f(z)$ at every (interior) point of the region R. It is well known that if there exists a subregion S of R and two positive numbers M and N such that for $n>N$, one has $|f_n(z)|<M$ at every point of S, the sequence $f_n(z)$ converges *regularly* in S.

Accordingly, it follows that the theorem of Osgood holds also for meromorphic functions, provided that the limiting function $f(z)$ is *finite* at every point of R. For if this were not the case one could find regions $S_1, S_2, \cdots$, with the property that R contains S_1 and S_k contains S_{k+1}, and an increasing sequence of indices $n_1<n_2<\cdots$, such that $|f_{n_k}(z)|>k$ for all z in S_k. The regions S_k contain at least one point z_0 common to all of them. At this point the limit function cannot be finite, and we thus read a contradiction to our hypothesis.

4. *Singularities Everywhere Dense.* We conclude from this that if the theorem of Osgood is not true, the points at which the limit function $f(z)$ is infinite must form a set everywhere dense in R.

Moreover, since we can apply the same reasoning to the sequence of meromorphic functions $1/\{f_n(z)-a\}$, which

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J. Springer, 1921, p. 238], I take the liberty to speak here of "regular", instead of "continuous", convergence, because in doing this the points which for a long time have been called "irregular" coincide with those at which the convergence is not regular in the above sense.

* See my paper, *Stetige Konvergenz und normale Familien*, which is to appear soon in the Mathematische Annalen.

converges in R towards the function

$$\frac{1}{f(z) - a},$$

we see that for every finite number a the roots of the equation $f(z) - a = 0$ must form a set that is everywhere dense in R.

5. *Reductio ad Absurdum.* We consider now the continuous and bounded set of real functions $\phi_n(x,y)$, where $z = x + iy$, and

$$\phi_n(x,y) = \begin{cases} 1 \text{ at the points of } R \\ \qquad \text{for which} \qquad R[f_n(z)] > 1, \\ -1 \text{ at the points of } R \\ \qquad \text{for which} \qquad R[f_n(z)] < 1, \\ R[f_n(z)] \text{ at the points of } R \\ \qquad \text{for which} \quad -1 \leqq R[f_n(z)] \leqq 1, \end{cases}$$

$R[f_n(z)]$ designating the real part of $f_n(z)$.

If the theorem of Osgood does not hold for our functions $f_n(z)$, the functions $\phi_n(x, y)$ have the following properties:

(a) the limit,

$$\lim_{n \to \infty} \phi_n(x,y) = \phi(x,y)$$

exists everywhere in R;

(b) for every real number α such that $|\alpha| < 1$, the roots of the equation $\phi(x, y) - \alpha = 0$ are everywhere dense in R.

As these properties are in contradiction with the well known theorem of Baire,* that the limit of a sequence of bounded continuous functions cannot be totally discontinuous, the result we have stated must be true.

6. *Questions of Connectivity.* If a sequence of functions $f_n(z)$, meromorphic in R, converges everywhere in that region, the points E of R at which the sequence converges

* See Carathéodory, *Vorlesungen über reelle Funktionen*, Leipzig, Teubner, 1927, §176.

regularly must accordingly be everywhere dense in R. Moreover, E contains a denumerable set of subregions $R_1, R_2, \cdots$, in which the limit function $f(z)$ is meromorphic or equal to the infinite constant, and the interior points of R_k form also a set that is dense in R.

If the functions $f_n(z)$ have no poles in R, the regions R_k must be simply connected and, as F. Hartogs and A. Rosenthal* have proved in a very remarkable paper, must fulfill certain other conditions less easy to state. *Nothing of this kind happens in our case.*

Consider for instance a finite or denumerable set of regions $R_1, R_2, \cdots$, whose connectivity may be infinite. Suppose that no two of these regions overlap, and that every point of the z plane is either an interior point or a boundary point of some R_k or that it is a limiting point of such points. Assume, moreover, for simplicity that the boundary points of all the R_k form with their limit points a bounded and closed set S.

Let $\zeta_1, \zeta_2, \cdots$ be a denumerable sequence of points on S such that every point of S is a limiting point of this sequence; the ζ_k thus must not only be everywhere dense on S, but an infinite number of them must cover every isolated point of S.

Designate by γ_n the set of points which are at a distance $1/n$ from S. Let z_n be a point of γ_n such that there is no point of γ_n nearer to ζ_n than z_n. Then the distance $|z_n - z_j| > 1/[n(n+1)]$, $j \neq n$, and the distance from z_n to S is likewise $> 1/(n(n+1))$. Furthermore, every point ζ of S is a limiting point of the set $z_1, z_2, \cdots$. It follows that the various circles κ_n

$$\kappa_n : \qquad |z - z_n| < \frac{1}{2n(n+1)}, \qquad (n = 1, 2, \cdots),$$

have no points in common, and contain no points of S.

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* F. Hartogs and A. Rosenthal, *Über Folgen analytischer Funktionen*, Mathematische Annalen, vol. 100 (1928), pp. 212-263.

7. *Approach by Rational Functions to Zero.* We consider now the functions

$$g_n(z) = \frac{1}{n^2}\cdot\frac{1}{4n(n+1)(z-z_n)-1}, \qquad (n = 1,2,\cdots).$$

We note that for all z exterior to the circle k_n,

$$|g_n(z)| < \frac{1}{n^2}\cdot$$

With every number n we can associate an integer p_n such that no point ζ of S is at a distance from the set of points $z_n, z_{n+1}, \cdots, z_{p_n}$ greater than $1/n$.

The sequence of rational functions

$$f_n(z) = \sum_{k=n}^{k=p_n} g_k(z)$$

then converges identically towards zero in the whole plane. This convergence is regular at every interior point of each R_k, but the convergence is irregular at every point ζ of S, because every such point is a limiting point of poles of the $f_n(z)$.

8. *Approach by Rational Functions to Meromorphic Functions.* It is possible to choose the rational functions $f_n(z)$ such that they converge on R_k to an arbitrary function $\phi_k(z)$, meromorphic in R_k, and converge on S to any function $\phi(z)$ which may be represented as the limit of a sequence of rational functions. The proof of this fact can be based upon the proposition that any function which is meromorphic in a region D, bounded by regular curves and of finite connectivity, can be represented by a definite integral taken over the boundary of D, the integral vanishing identically outside of D. By approximating these integrals with finite sums of rational functions, and applying a method analogous to that of §7, we can prove the theorem stated above without great difficulty.

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