

## THE APRIL MEETING IN CHICAGO

The two hundred sixty-first meeting, twenty-ninth regular Western meeting of the Society was held at the University of Chicago on Friday and Saturday, April 6-7, 1928. About one hundred twenty persons attended the meeting, among whom were the following ninety-six members of the Society:

A. A. Albert, N. L. Anderson, R. W. Babcock, Bamforth, Bardell, Barnett, Bartky, Bear, Bibb, G. A. Bliss, Bussey, G. N. Carmichael, R. D. Carmichael, Carver, Chaimovitch, Conkwright, Cook, Cope, H. B. Curtis, Curtiss, H. T. Davis, Davisson, Dickson, Douglas, Earl, Everett, A. L. Foster, C. A. Garabedian, H. L. Garabedian, Garver, Georges, Gladys Gibbens, J. W. Glover, Gouwens, L. M. Graves, Griffiths, V. G. Grove, Hartung, Hassler, Hyden, Louis Ingold, Ingraham, Dunham Jackson, B. W. Jones, Kinney, E. P. Lane, Langer, LaPaz, Laun, Logsdon, C. E. Love, Lunn, J. V. McKelvey, W. D. MacMillan, W. A. Manning, March, William Marshall, Marquis, T. E. Mason, Mickelson, C. N. Moore, E. H. Moore, E. J. Moulton, A. L. Nelson, Olson, Oppenheim, Palmié, Parkinson, Pettit, H. L. Rietz, Roever, Roth, Rupp, Sanger, Schottenfels, Sherer, Shohat, W. G. Simon, Skinner, Slaught, E. R. Smith, Virgil Snyder, Stouffer, J. H. Taylor, E. L. Thompson, Thornton, Trjitzinsky, J. S. Turner, Van Vleck, Wall, L. E. Ward, Warren Weaver, Wiggin, Roscoe Woods, Wyant, Yeaton.

On Friday afternoon Professors E. B. Stouffer and E. P. Lane gave symposium addresses on *Recent Developments in Projective Differential Geometry*. On a motion by Professor Carver, a rising vote of thanks was given the speakers. These addresses appear in full in the present issue of this Bulletin

On Friday evening members and their guests attended dinner in the Del Prado Hotel. Professor Jackson acted as toastmaster. At the dinner President Snyder announced the first award of the Frank Nelson Cole prize to Professor L. E. Dickson for his book *Algebren und ihre Zahlentheorie* and other work. President Snyder presented the prize to Professor Dickson, who thanked the President. Both speakers paid tributes to the memory of Professor Cole. The toastmaster also called upon Professor Slaught, Dr.

Griffiths and Professors Manning and C. N. Moore for talks. The dinner was attended by eighty-five persons.

The Council of the Society met Friday afternoon.

The following sixteen persons were elected to ordinary membership:

Mr. Charles Trotter Bunnell, University of Rochester;  
Professor Andrew G. Clark, Colorado Agricultural College;  
Professor Howard Adams DoBell, New York State College for Teachers;  
Mr. Harold L. Dorwart, Yale University;  
Miss Allegra Eckles, Territorial Normal School, Honolulu;  
Mr. Edwin Harold Hadlock, Cornell University;  
Professor Bruce Vickroy Hill, Phillips University;  
Sister Mary Columba, Villa Maria College;  
Sister Mary St. Bernard, Villa Maria College;  
Mr. James Lewis Noecker, Thiel College;  
Mr. Oscar John Peterson, University of Michigan;  
Mr. Harry Ward Seward, Utica, N. Y.;  
Professor Clarence DeWitt Smith, Louisiana College;  
Professor Elva Elizabeth Starr, Alfred University;  
Mr. John Russell Vatnsdal, State College of Washington;  
Mr. Clement Weinstein, University of Pennsylvania.

The following were elected to membership as nominees of Members of Department of Mathematics, University of Wisconsin:

Mr. C. W. Dancer, University of Wisconsin;  
Mr. G. N. Carmichael, University of Wisconsin;  
Mr. R. O. Gilmore, University of Wisconsin.

The President announced as the Committee on Arrangements for the semi-centennial celebration of the Society: Professors T. S. Fiske (Chairman), R. C. Archibald, J. L. Coolidge, L. E. Dickson, E. R. Hedrick, Dunham Jackson, James Pierpont, M. I. Pupin, R. G. D. Richardson, and Oswald Veblen.

The following were appointed as a Committee for the nomination of officers to be elected at the next annual meeting: Professors L. E. Dickson (Chairman), E. T. Bell, D. R. Curtiss, H. M. Morse, J. K. Whittemore.

It was decided to hold a fall meeting at Cincinnati this year. It is proposed, if this meeting shows the need, to hold such a meeting each year in the Ohio Valley Region. This

meeting will probably take place on the Friday and Saturday after Thanksgiving. The President appointed as a Committee on Arrangements to act with the Associate Secretary: Professors C. N. Moore (Chairman), Henry Blumberg, P. P. Boyd, Louis Brand, and H. T. Davis.

The following dates for meetings of the Society in the Middle West were determined upon: December 31-January 1, 1928-9, University of Chicago; March 28-29, 1929, University of Chicago. As a Committee to arrange for a symposium at the latter of these meetings, the President appointed Professors E. P. Lane (Chairman), M. H. Ingraham, C. C. MacDuffee.

It was determined to hold the Western winter meeting of the Society for 1929-30 at Des Moines in connection with with the meeting of the American Association for the Advancement of Science.

The following were appointed as a Committee to study the question of how to increase the value of the summer meetings: Professors Arnold Dresden (Chairman), C. R. Adams, G. C. Evans, L. M. Graves, A. J. Kempner, and Warren Weaver.

The Council unanimously voted to thank the Carnegie Corporation for its recent gift of \$5,000 to the endowment of the Society.

At the Saturday morning meeting of the Society the following resolution expressing the Society's appreciation of the action of the Mathematical Association of America in becoming a sustaining member was unanimously passed:

The American Mathematical Society wishes to express its particular appreciation of the constant and strong support that the Mathematical Association of America has given it, and especially of its gracious action in becoming a sustaining member of the Society. It hopes that this unity may strengthen the efforts of the two organizations to further mathematics in America.

The papers whose abstracts appear below were read in three sections: (1) Geometry and Applied Mathematics—

Friday morning; (2) Algebra and Number Theory—Friday morning; (3) General, mostly Analysis—Saturday morning. Professor Roever presided at the first section, Professor C. N. Moore at the second, President Snyder and Professor Stouffer at the third. President Snyder presided at the symposium on Friday afternoon. Papers numbered 1–20 were read in the section on Geometry and Applied Mathematics, 21–30 in the section on Algebra and Number Theory, and 31–49 Saturday morning. Papers numbered 3, 8–10, 12, 14–19, 42–49 were read by title. Professor Hufford was introduced by Professor Davis, and Mr. Shook, Mr. Yank, and Mr. Morrow were introduced by Professor Dickson.

1. Professor I. A. Barnett: *The finite transformations generated by an infinitesimal projective transformation in function space.*

In a paper entitled *Projective transformations in function space*, Transactions of this Society, 1919, Professor L. L. Dines gave a method for determining the one-parameter family of projective transformations generated by a regular infinitesimal projective transformation. The method, however, made use of certain identities involving the inverse of a projective transformation in function space. In this paper, it is shown that by the introduction of homogeneous coordinates in the infinitesimal transformation, it is possible to avoid the formalities which Dines uses and obtain the result much more briefly and directly. For the particular case of the sub-group which leaves the unit sphere in function space invariant, the problem is reduced to integrating two systems of two simultaneous integro-differential equations.

2. Professor J. O. Hassler: *Plane nets whose first and minus first Laplacian transforms each degenerate into a straight line.*

This paper will appear in full in an early issue of this Bulletin.

3. Professor Arnold Emch: *Multiple systems, their geometric representation and groups.*

This paper considers triple and multiple systems with particular reference to their geometric representations and group properties. Two triple systems each invariant under a  $G_{168}$  on 7 elements without a common triple may be mapped on a torus. The configuration of triangles thus obtained is connected with the 7 hexagons on the torus invariant under a  $G_{42}$ . There exist triple systems such that all possible triples may be arranged in a complete set of triple systems, in which every triple occurs just once. This fact as shown by the author in case of 9 elements seems to be new. Two triple systems on 9 elements without a common triple may

be mapped on a torus. The group leaving a triple system on 9 elements invariant is a  $G_{432}$ . The configuration on the torus is connected with a polyhedron on the torus bounded by 6 octagons and 6 squares and is invariant in a non-cyclic  $G_6$ . A very interesting configuration is obtained by the mapping of a triple system on 6 elements in which every couple occurs twice. We obtain a polyhedron on the projective plane bounded by 6 pentagons which is invariant in an icosahedral  $G_{60}$ . The lowest non-trivial (1234) quadruple system, containing every triple one, is on 8 elements. Such a quadruple system is invariant under a  $G_{1344}$ . Two such quadruple systems without a common triple may be mapped into a hyperpolyhedron in space of four dimensions, which is invariant in a  $G_{336}$ . The quadruple system on 16 elements containing every element 6 times contains 24 quadruples and is invariant under a  $G_{192}$ . Its map is the four-dimensional cube.

#### 4. Professor Virgil Snyder: *Quadratic involutions in multiple linear complexes.*

It is shown that space involutorial birational transformations exist having the three properties: (a) Lines joining a pair  $PP'$  of conjugate points belong to a non-singular linear complex; (b) each line of the complex contains  $k$  such pairs of conjugate points,  $k$  any positive integer; (c) such transformations exist which have no surface of invariant points. Every plane contains a self-conjugate curve of order  $2k+1$ , and the correspondence  $PP'$  on it is always singular for  $k > 1$ .

#### 5. Dr. Jesse Douglas (National Research Fellow): *Reduction of the problem of Plateau to the minimization of a certain functional.*

In a previous paper (abstract in this Bulletin, March-April, 1927) the author shows how to reduce the Plateau problem to the integral equation  $(1) \int_0^1 K(t, \tau) \operatorname{ctn} \frac{1}{2}(\phi(\tau) - \phi(t)) d\tau = 0$ , or to the equivalent integral equation  $\int_0^1 K(t, \tau) d\tau / (\psi(\tau) - \psi(t)) = 0$ , where  $\psi(t) = \operatorname{ctn} \frac{1}{2}\phi(t)$ . The function  $K(t, \tau)$  is known as soon as the contour is given. The star indicates that the improper integral is to be taken in its Cauchy principal value. The function  $\phi(t)$  is required to be always increasing and continuous as  $t$  increases from 0 to 1, and  $\phi(0) = 0$ ,  $\phi(1) = 2\pi$ . The present paper points out that (1) is the Euler-Lagrange equation for the minimization of the functional  $A(\phi) = -\int_0^1 \int_0^1 K(t, \tau) \log \sin \frac{1}{2} |\phi(\tau) - \phi(t)| dt d\tau$ ; that is, (1) expresses the vanishing of the Volterra derivative  $A'(\phi, t)$ . An existence theorem is established for this minimum problem in case  $K(t, \tau)$  is positive for all  $t, \tau$  on the basis of the ideas in Fréchet's thesis (Palermo Rendiconti, vol. 22 (1906)), relative to functions on closed, compact sets.

#### 6. Dr. Jesse Douglas: *A characterization of Riemann spaces of constant non-zero curvature and $n \geq 3$ dimensions.*

Consider a projective space  $P_n$  of  $n \geq 3$  dimensions, referred to coordinates  $x^i (i = 1, 2, \dots, n)$ . A metric for angles may be imposed on  $P_n$  by means of a relative quadratic form  $A = \nu^2 g_{ij} dx^i dx^j$ , where the  $g_{ij}$  are

fixed functions and  $\nu^2$  is an arbitrarily variable function of the  $x$ 's; hence only the ratios of the  $g$ 's are important. A metric for areas may be imposed on  $P_n$  by defining the area of the infinitesimal parallelogram determined by the elements  $dx, \delta x$ , to be the square root of  $F = A_{ij,kl}(dx^i \delta x^j - dx^j \delta x^i)(dx^k \delta x^l - dx^l \delta x^k)$ , where  $A_{ij,kl}$  are arbitrarily assigned functions of the  $x$ 's. An angle-area space is identifiable with a Riemann space when and only when the  $A_{ij,kl}$  are proportional to the two-rowed determinants  $g_{ik}g_{jl} - g_{il}g_{jk}$ . The present paper shows that if  $n \geq 3$ , the only way to impose upon  $P_n$  an angle-area metric which shall be such that in every rectilinear triangle the angular excess is equal to a non-zero constant times the area, is to define  $A, F$  as in the Cayley metric based on any fixed quadric in  $P_n$ .

7. Dr. Jesse Douglas: *A new special form for the linear element of a surface.*

This paper deals with the case  $n=2$  of the problem of the preceding paper. If  $n=2$ , any  $A, F$  metric is identifiable with a Riemann metric by a proper choice of the factor  $\nu^2$ . Thus we have to find what the linear element of a surface must be in order that there may exist upon it a system of paths having the following two properties: (1) the system shall be linear (equivalent by point transformation to the straight lines of a plane); (2) the angular excess of any triangle of paths shall be proportional to its area, with a constant of proportionality  $k \neq 0$ . This problem is solved, a formula being obtained for  $ds^2$  which involves four arbitrary functions:  $U_1, U_2$  of  $u$  alone, and  $V_1, V_2$  of  $v$  alone. Only one of these functions is essential, if we assume the surface to be real. Many surfaces not of constant Gaussian curvature are included.

8. Dr. Jesse Douglas: *Necessary and sufficient conditions for the equivalence of general affine and descriptive spaces of paths.*

This paper is a generalization of the work of Christoffel on the equivalence of quadratic differential forms, and of Veblen and J. M. Thomas on the equivalence of "restricted" spaces of paths, that is, systems of paths definable by  $d^2x^i/dt^2 = \Gamma_{jk}^i(dx^j/dt)(dx^k/dt)$ , where the  $\Gamma$ 's are any functions of the  $x$ 's. For references, see Chapter V of O. Veblen, *Invariants of Quadratic Differential Forms*, Cambridge, 1927. General affine and descriptive spaces of paths have been defined by the writer in his paper *The general geometry of paths*, soon to appear in the *Annals of Mathematics*. The analytic basis here is a system of differential equations of the form  $d^2x^i/dt^2 = H_2^i(x, dx/dt)$ , where  $H_2^i$  denotes any function (generally transcendental) homogeneous of the second degree in the arguments  $dx/dt$ . The present paper develops criteria for the affine and for the descriptive equivalence of two such general spaces of paths.

9. Dr. Jesse Douglas: *Normal coordinates for a space of  $K$ -spreads.*

Normal coordinates for the "restricted" geometry of paths:  $d^2x^i/dt^2 = \Gamma_{jk}^i(x) (dx^j/dt) (dx^k/dt)$ , generalizing Riemann's normal coordinates, have been defined by O. Veblen, (Proceedings of the National Academy, vol. 8 (1922)). Normal coordinates for the general geometry of paths:  $d^2x^i/dt^2 = H_2^i(x, dx/dt)$ , have been defined by the writer in his paper cited in the preceding abstract. In the present paper, normal coordinates are defined for a space of  $K$ -spreads (abstract, this Bulletin, Jan.-Feb. 1928). They are relative to a coordinate system  $(x)$ , a fixed point  $x_0$ , and a fixed  $K$ -element at  $x_0$ . It is proved that these normal coordinates have the essential property that when the coordinate system  $(x)$  undergoes arbitrary analytic transformation, the related normal coordinates  $y$  undergo linear homogeneous transformation. This property enables us to define the process of *extension* of a tensor by means of partial differentiation with respect to the normal coordinates.

10. Dr. Jesse Douglas: *Differentiation with respect to a parameter of the Cauchy principal value of an improper integral.*

It is found that formal differentiation under the integral sign does not apply to the Cauchy principal value of an improper integral, even when the functions involved are quite regular. Thus:  $(d/dt^* (\int_0^1 K(t, \tau) (\tau-t)^{-1} dt) = \int_0^1 (\partial K/\partial t + \partial K/\partial \tau) (\tau-t)^{-1} d\tau$ . This result is generalized to the case  $\int_0^1 K(t, \tau) F(\phi(\tau) - \phi(t)) d\tau$ , where  $F$  has an infinity of order one at the zero value of its argument.

11. Professors H. T. Davis and M. E. Hufford: *Diffraction and the wave theory of light.*

The purpose of this paper is to check the energy in a diffraction pattern of 66 rings secured by passing monochromatic light through a circular orifice. The assumptions made are those of the classical wave theory of light and the mathematical development follows that given in the celebrated papers of E. Lommel. (See Gray, Mathews and MacRobert, *Bessel Functions*, 1922, Chapter 14.) The mathematical interest centers in the discovery of divergent series asymptotic to the Lommel functions  $U_1(x)$  and  $U_2(x)$ . The striking agreement of the calculated energy with that found in the diffraction pattern, particularly as shown in the broadening and darkening of the rings toward the outer part of the plate, gives renewed confidence in the classical theory of light as applied to diffraction phenomena.

12. Dr. C. D. Smith: *On generalized Tchebycheff inequalities in mathematical statistics.*

It is the main purpose of the paper to give a further development of the theory and properties of what may be appropriately called generalized Tchebycheff inequalities. The inequality is discussed as a proposition of geometry. The Markoff Lemma is generalized in such a manner as to give rise to several important forms of the inequality in a very simple way. Some closer inequalities are derived. It is shown that the closeness of the

inequality may be improved in certain cases by moving the origin. Certain types of distributions are discussed to show that rather simple types of distributions exist within which the difference between the members of the inequality attains a minimum. Types of functions are discussed for which the difference between the members of the inequality may be made arbitrarily small.

13. Professor H. W. March: *The torsion problem for prisms of non-isotropic material.*

It is shown by means of a linear transformation that the soap film method can be used to solve the torsion problem for a prism composed of non-isotropic material having three mutually perpendicular planes of elastic symmetry, two of which are parallel to the length of the prism. The torsional rigidity of the given prism is equal to that of a transformed prism of isotropic material whose modulus of rigidity is expressed in terms of the moduli of the material of the non-isotropic prism. The lines of shearing stress in the cross section of the transformed prism become, after transformation to the cross section of the original prism, the lines of shearing stress in that section.

14. Professor E. L. Dodd: *A test for periods when certain measurements are of two variates combined.*

The usual tests in periodogram analysis are decidedly untrustworthy in dealing with data where some measurements probably represent two variates erroneously united. Such an error may occur in measuring the thickness of layers not always distinctly separated. For such data, an "intensity" should be defined (1) not greatly affected by the above slipping of phase, (2) not too difficult to compute, (3) with expected value zero, and (4) with mean error unity for every trial period. The following definitions for the intensity  $I$  meets requirements (1), (3), and (4), and perhaps (2). Let the data be divided into sets of  $s$  variates, for example,  $s=60$  to test for a period of  $k$ . Suppose  $m=s/k$  is an integer. Take  $t=2\pi/k$ . Then for each of the  $m$  sub-sets of  $k$  variates  $X_r$ , compute, with  $r=0, 1, \dots, k-1$ ,  $C = \sum X_r \cos rt$ ,  $S = \sum X_r \sin rt$ ,  $U = C^2 + S^2$ . Let  $a$  be the second moment of the  $s$  variates about the mean,  $b$  the fourth moment. With  $k > 2$ , take  $A^2 = kb + (k^2 - 3k)a$ ,  $I = (\sum U - sa)/(Am^{1/2})$ . Suitable modifications are possible for  $k=2$ , and for fractional  $m$ ; and the  $I$ 's may be combined for successive sets of  $s$  variates.

15. Professor C. C. Camp: *Note on unexpected errors in Barlow's tables and others.*

One should not expect to find an error in a list of errata. M. J. Perrott in writing of Steinhauser's twenty-place table says "log 1088 doit terminer par 26 et non par 23." Actually, 226 occurs after the nineteenth decimal and Perrott must have miscounted. In the 30th edition of Schrön's table an inversion of the correct digits 42 occurs in his extended form of log 102238. Barlow's Tables as revised in 1839 were supposed to have errors of unity in the last digit of square and cube roots of some numbers beyond 1250. For



numbers under 1250 De Morgan says that "an error of a unit in the last place can . . . hardly exist . . . , though it is barely possible in numbers not much below 1250." An error is found in the cube root of 80, also it was found that the cube roots of 12, 20 were too large and those of 9, 14, 52 too small by unity in the last digit as given, the actual errors being greater than that in three cases. Moreover, errors exist in nine related cube roots. Such tables, which are widely copied, should be made entirely free of error.

16. Mr. J. H. Roberts: *Certain types of atriodic continua.*

R. L. Moore has introduced the notion *triodic continuum* and shown that if  $G$  is an uncountable set of triodic continua in the plane, then  $G$  contains an uncountable subset  $G_1$ , such that every two continua of  $G_1$  have a point in common. A continuum  $M$  is said to be triodic if it contains four distinct continua  $a, b, c$  and  $k$ , such that  $k$  is the common part of each two of the continua  $a, b$  and  $c$ . Professor Moore has raised the following question: Is it true that if  $M$  is a plane atriodic continuum, then there exists in the plane an uncountable set  $G$  of mutually exclusive continua such that each continuum of  $G$  can be thrown into  $M$  by a continuous one-to-one transformation of the plane into itself. This is shown to be true for the special case where, given any positive number  $\epsilon$ ,  $M$  can be covered by a simple chain of regions of diameter less than  $\epsilon$ . An example is given of an *unbounded* atriodic continuum  $M$  for which no such set  $G$  exists.

17. Professor G. T. Whyburn: *A theorem on plane continua.*

In this paper the following theorem is proved. Suppose  $M$  is a plane continuum having the property that each of its subcontinua contains a continuum  $N$  such that there exists a positive number  $d$  such that every subcontinuum of  $N$  contains at least one point which is accessible from at least two complementary domains of  $M$  which are of diameter  $> d$ ; then  $M$  is a continuous curve. A number of interesting corollaries of this theorem are given, among which are the following: (1) Under the same hypothesis as above, (a) every subcontinuum of  $M$  contains an arc which belongs to the common boundary of two of the complementary domains of  $M$ , and (b)  $M$  is a Menger regular curve. (2) If every subcontinuum of  $M$  contains a point which is accessible from two unbounded complementary domains of  $M$ , then every point of  $M$  is a cut point of  $M$ .

18. Professor G. T. Whyburn: *Some theorems on connected point sets.*

In this paper the following generalizations of theorems due to R. L. Wilder (*Fundamenta Mathematicae*, vol. 7, p. 371), R. L. Moore (*Proceedings of the National Academy*, vol. 9, p. 102, Theorem B+), and Kuratowski and Zarankiewicz (this *Bulletin*, vol. 33, p. 574, Remark B) are given. (1) Condition (4) in the Moore-Wilder lemma characterizing continua which are not continuous curves and proved by Wilder only for bounded continua is shown to hold for unbounded continua. (2) No connected subset  $K$  of a connected point set  $M$  contains an uncountable

collection of mutually exclusive point sets each of which cuts  $M$  but not  $K$ . (3) If  $S$  is any connected point set and  $Z$  is any collection of mutually exclusive connected subsets of  $S$  such that for each element  $X$  of  $Z$ ,  $M - X$  is neither connected nor the sum of two connected point sets, then  $Z$  is countable. (Kuratowski and Zarankiewicz imposed the unnecessary condition that each set of the collection  $Z$  is closed relative to  $S$ .) In proving (1) use was made of the following lemma: If  $N$  is a connected subset of a connected point set  $M$  and  $K$  is any component of  $M - N$ , then  $M - K$  is connected.

19. Professor G. T. Whyburn: *Concerning collections of the cuttings of continua.*

In this paper the following results are established. Let  $M$  denote any plane continuum. (1) No plane continuum  $M$  contains an uncountable collection of mutually exclusive connected point sets each of which contains a proper subset which cuts  $M$ ; hence if  $G$  is any collection of mutually exclusive connected subsets of  $M$  each of which contains a cutting of  $M$ , then all save possibly a countable number of the elements of  $G$  must be continua and must themselves be irreducible cuttings of  $M$ . (2) If  $M$  is bounded and  $G$  is any collection of mutually exclusive componentwise irreducible cuttings of  $M$  such that for each element  $g$  of  $G$ ,  $M - g$  is not the sum of two connected point sets, then  $G$  is countable. (3) If  $G$  is any collection of mutually exclusive bounded subcontinua of  $M$  each of which contains a cutting of  $M$ , then  $G$  contains a subcollection  $G^*$  which contains all but a countable number of the elements of  $G$  and which is an upper semi-continuous collection.

20. Professor C. A. Rupp: *On an extension of Pascal's theorem to a space of  $n$  dimensions.*

The edges of a simplex  $F$  meet a hyperquadric  $V$  in  $S_r$  in  $r(r+1)$  points, which may be grouped to define a second simplex  $P$ . The coordinates of the intersections of corresponding faces of the simplexes are linearly dependent. This is shown to be equivalent to saying that the pair of simplexes  $F$  and  $P$  are a Schlafli pair, or that lines in the position of Schlafli are linearly dependent. Applying the theorem to circles gives a new construction for a Stephanos pentacycle, and shows that the circles of a pentacycle are linearly dependent. In line geometry there appears the concept of linear congruences in involution, and by means of six linear complexes in involution are defined six linear congruences such that any linear congruence in involution with five of them is also in involution with the sixth.

21. Mr. A. A. Albert: *A determination of all normal division algebras in sixteen units.*

It is known that every normal division algebra in 16 units is of rank 4 and contains an element satisfying an irreducible quartic in the reference field  $F$ . In this paper the quartic is taken in the reduced form and it is shown that either all of its roots are rational functions of one of them or the algebra contains an element satisfying the equation  $\phi(\omega) = \omega^4 + \alpha\omega^2$

$\pm\beta$ ,  $\alpha$ ,  $\beta$  in  $F$ , which is irreducible in  $F$ . It is then shown that either this equation has the property that all of its roots are rational functions of one of them or its group is  $G_8$ . The algebras containing an element satisfying an equation of this type with group  $G_8$  are then determined, and hence all division algebras in sixteen units, since those algebras which contain an element satisfying an irreducible equation having the property that all of its roots are rational functions of one of them are the algebras of type  $F$  of L. E. Dickson and are known.

22. Professor W. A. Manning: *A theorem on simply transitive primitive groups.*

Let  $G_1$  be a subgroup that fixes one letter of a simply transitive primitive permutation-group. Beginning in 1917, the author published a series of theorems concerning the structure of  $G_1$  when one of its transitive constituents is doubly transitive. The most recently published theorem (1921) states that if  $G_1$  has a doubly transitive constituent of degree  $m$ , either  $G_1$  is a simple isomorphism between doubly transitive groups of degree  $m$ , or  $G_1$  has a transitive constituent whose degree is a divisor ( $>m$ ) of  $m(m-1)$ . His present theorem asserts that if  $G_1$  has a doubly transitive constituent of degree  $m$ ,  $G_1$  has also a transitive constituent whose degree is a divisor ( $>m$ ) of  $m(m-1)$ .

23. Professor L. E. Dickson: *The general Waring problem for quadratic functions.*

This paper reports on an elaborate investigation of the following difficult problem. Let the quadratic function  $q(x)$  have a positive coefficient of  $x^2$ , be an integer for every integer  $x \geq 0$ , and be negative for one or more integers  $x \geq 0$ . There is found the least integer  $E$  such that every integer  $\geq L$  is a sum of  $E$  numbers chosen from 0 to 1 and four values  $\geq 0$  of  $q(x)$  for integers  $x \geq 0$ , where  $L$  denotes the least sum of four such values. In previous papers, the writer treated the case in which  $q(x)$  is never negative when  $x$  is positive.

24. Mr. R. C. Shook: *Two extended Waring problems.*

The first problem considers the conditions that a form  $f = a_1x_1^6 + \dots + a_nx_n^6$ ,  $x_i \geq 0$ , shall satisfy in order that it may represent all integers. A minimum order  $n = 10$  is established and twenty-two forms of order 10 are shown to represent all integers up to 4,096. The second problem discusses the maximum gaps in a table of sums by four of (i) values of quadratic functions  $f(x)$ , having integral values  $\geq 0$  for  $x \geq 0$ , and (ii) the positive values of quadratic functions  $q(x)$  having integral values not all equal to or greater than zero for  $x \geq 0$ .

25. Professor L. E. Dickson: *Positive binary quadratic forms  $B$  such that every positive integer is a sum of  $s$  values of  $B$ .*

Since  $B$  must represent unity, we may transform  $B$  into  $f = x^2 + gxy + hy^2$  where  $g = 0$  or 1, and  $h > 0$ . Hence the case  $s > 3$  is trivial. For  $s = 2$  or 3, it

is shown that every positive integer is a sum of  $s$  values of  $f$  if and only if  $h=1, 2$ , or  $3$  if  $s=2$ , and  $h=1, \dots, 7$  if  $s=3$ . The paper will appear in *Journal de Mathématiques*.

26. Professor L. E. Dickson: *Generalization of Waring's theorem on cubes*.

This paper contains a further consideration of the theorems announced in the *American Mathematical Monthly*, April, 1927.

27. Mr. K. C. Yang: *All positive integers are sums of 9 pyramidal numbers  $(x^3 - x)/6$ ,  $x=0, 1, 2, \dots$*

E. Maillet, in the *Bulletin de la Société de France*, vol. 23, (1895), pp. 40-49, proved that every integer  $\geq 19272$  is a sum of at most 12 pyramidal numbers. By an elementary proof, it is here shown that nine suffice.

28. Dr. Lois W. Griffiths: *Representations of integers in the form  $x^2 + 2y^2 + 3z^2 + 6w^2$* .

The number of representations  $2T(N)$  of a positive odd integer  $N$  in the form  $x^2 + 2y^2 + 3z^2 + 6w^2$  is not known exactly. In this paper new conditions on, and new limits for,  $T(N)$  are obtained in terms of the divisors of  $N$ . The determination of  $T(N)$  for  $N$  odd and composite is reduced to that of  $T(P)$ , where  $P$  is in turn the distinct prime factors of  $N$ . The paper is to appear in the *American Journal of Mathematics*.

29. Mr. B. W. Jones: *Representation of integers by positive ternary quadratic forms*.

L. E. Dickson (*Annals of Mathematics*, (2), vol. 28 (1927), p. 333) made the following definition: "All the integers not represented by a regular form  $f [= ax^2 + by^2 + cz^2$ , where  $a, b$  and  $c$  are positive integers] coincide with all the positive integers given by certain arithmetical progressions," and proved that not more than seventeen forms  $f$  are regular, where  $a=1$ ,  $b$  and  $c$  relatively prime and less than certain large integers. Applying his methods and extension of them, this paper proves first that not more than 103 forms  $f$  are regular when 1 is the greatest common divisor of  $a, b$  and  $c$ . Then, using methods of Dirichlet and Dickson, and certain extensions of these, together with elementary transformations and previously known results for certain ternaries, all but six of these forms are proved regular. (Partial results are known for these six.) Many quadratic ternary forms with cross products are similarly proved regular. Also numerous semi-regular forms are considered.

30. Mr. D. C. Morrow: *All quaternary quadratic forms which represent every positive integer*.

It is proved that all positive quaternary quadratic forms are equivalent to certain reduced forms. For forms which represent all positive integers, it is shown that the number of reduced forms is finite. The conditions on the coefficients, which are necessary in order that the

duced form shall represent every positive integer, are derived, and certain equivalent cases are eliminated. The paper then proceeds to determine all those reduced forms which represent every positive integer.

31. Dr. F. R. Bamforth: *A classification of boundary value problems for a system of ordinary differential equations of the second order.*

Boundary value problems of the form  $y'(x) = (A(x) + \lambda B(x))y(x)$ ,  $My(a) + Ny(b) = 0$ , are considered where  $\lambda$  is a parameter, capital letters denote square matrices of order two, and small letters vectors. It is shown that every boundary value problem of this form, where the matrix  $B(x)$  satisfies certain hypotheses, can be reduced to one and only one of a set of four normal forms. This reduction can be made by means of a non-singular transformation of the form  $y = U_n$ . Necessary and sufficient conditions are found for a boundary value problem in one of these normal forms to be self-adjoint according to the definition of G. A. Bliss. It is pointed out which of the normal forms have been studied and a boundary value problem in one of these forms, hitherto not studied, is solved completely. Finally, a general boundary value problem is deduced from an isoperimetric problem in the calculus of variations.

32. Dr. L. E. Ward: *On third order boundary value and expansion problems.*

A classification of boundary value problems of the system consisting of the differential equation  $u''' + \rho^2 u = 0$  and three boundary conditions linear and homogeneous in the values of  $u$  and its first two derivatives at two real points is given. The forms of the boundary conditions in one of the irregular cases are considered, and some of the properties of an infinite series of the characteristic functions arising from this case are derived.

33. Dr. L. E. Ward: *On the uniqueness of the coefficients in a certain expansion problem.*

In a paper published in the Transactions of this Society, vol. 29 (1927), the author showed that a certain irregular boundary value problem leads to characteristic functions in an infinite series in which any analytic function can be expanded, and that the formal series for the function converges uniformly to the function in the interior of a certain triangle. In this paper it is shown that zero can be expanded in such a series, uniformly convergent in the interior of the same triangle, and that therefore the coefficients in this type of expansion are not unique.

34. Mr. T. F. Cope: *An analogue of the Jacobi condition for the problem of Mayer with variable end points.*

In this paper the generalized Mayer problem formulated by G. A. Bliss (Transactions of this Society, vol. 19 (1918), p. 305) is considered. The second variation for this problem is computed and reduced to a particularly simple form. It is then shown that there is a boundary value

problem associated with the second variation, from which a necessary condition for a solution of the original problem is deduced. This condition is, essentially, that for a minimizing arc for the original problem, the boundary value problem can have no solution for negative values of its parameter. Finally, the boundary value problem associated with the second variation is transformed into an equivalent problem, which is shown to be definitely self-adjoint according to the definition of G. A. Bliss (Transactions of this Society, vol. 28 (1926), p. 570).

35. Dr. W. J. Trjitzinsky: *A class of quasi-analytic functions.*

In the field of quasi-analytic functions (they are non-analytic, but determined by their initial values at a point) some of the unsolved questions are: (1) construction of such functions, (2) determination of conditions under which an assigned set of initial values defines such a function. In this paper it is shown that functions of the form  $\sum a_n/(x-c_n)$  are quasi-analytic in the real interval  $(-1, +1)$ , if  $\sum_n |a_n| \log_k^{n+1} n$  ( $k \geq 2$ ) converges and  $c_n$  is an enumerable set of points whose limiting points are everywhere dense on  $(-1, +1)$  and whose distances from the real axis approach zero not faster than  $1/\log_k n$ . Using this result, sufficient conditions are found for the existence of quasi-analytic functions when a set of initial values is assigned.

36. Professor C. N. Moore: *On Gibbs' phenomenon for the developments in Bessel's functions.*

In a paper of 1911 (Transactions of this Society, vol. 12, pp. 181-206) the author developed an asymptotic formula for the coefficients of the developments in Bessel's functions which made it possible to infer facts concerning the convergence of these developments in the neighborhood of the origin which have not as yet been obtained in any other manner. In the present paper it is shown that the nature of the Gibbs' phenomenon in the neighborhood of the origin for these developments can be readily deduced from the same asymptotic formula.

37. Professor Dunham Jackson: *A note on closest approximation.*

This paper is concerned with a variation of the problem of closest approximation, which may be formulated as follows: Let  $f(x)$  be a given continuous function of period  $2\pi$ . Let  $h$  be a given positive number—less than or greater than  $2\pi$ . (The value  $h=2\pi$  is not excluded of necessity, but because the results in this case are already known.) For a given trigonometric sum  $t_n(x)$ , of the  $n$ th order, let  $I(t_n)$  be the maximum of the integral of  $[f(x)-t_n(x)]^2$  over an interval of length  $h$ , as the initial point of the interval is allowed to vary. Among all trigonometric sums of the  $n$ th order, let  $T_n(x)$  be the one for which  $I$  has the smallest possible value. The minimum problem has a unique solution, and the minimizing sum  $T_n(x)$  converges uniformly toward  $f(x)$  as  $n$  becomes infinite, if  $f(x)$  satisfies suitable hypotheses (for example, if  $f(x)$  satisfies a Lipschitz

condition). The problem can be greatly generalized without increasing the difficulty of the treatment.

38. Professor J. A. Shohat: *On the convergence of mechanical quadratures on an infinite interval.*

A function  $\Psi(x)$  defined on an interval  $(a, b)$ , finite or infinite non-decreasing, with infinitely many points of increase, gives rise to a system of orthogonal Tchebycheff polynomials  $\{\phi_n(x)\}$  of degree  $n$  ( $=0, 1, 2, \dots$ ) having each roots  $(x_i)$  real, distinct and between  $a, b$ . We then construct the formula of "Mechanical Quadratures"

$$(1) \quad \int_a^b f(x) d\Psi(x) = \sum_{i=1}^n H_i f(x_i) + R_n(f),$$

where  $R_n(f) \equiv 0$ , if  $f(x)$  is a polynomial of degree  $< 2n$ . In the present paper the author investigates the convergence of formula (1), that is, for what classes of functions  $f(x)$  is  $\lim_{n \rightarrow \infty} R_n(f) = 0$ . A complete solution is given in the most important case:  $\Psi(x)$  is continuous in  $(a, b)$ , which includes that of  $\int_a^b f(x) p(x) dx$ , as a particular case. The case  $(a, b)$  finite is herein included, thus generalizing a result due to Stieltjes. Applications are given to Tchebycheff polynomials, also to the theory of probability. Part of these results appeared in a note in the *Comptes Rendus*, Feb., 1928.

39. Professor J. A. Shohat: *On the convergence of the expansion of continuous functions in series of Sturm-Liouville functions.*

Consider the expansion  $f(x) = \sum_{n=1}^{\infty} A_n U_n(x)$ , ( $A_n = \int_a^b f(x) U_n(x) dx$ ), where  $U_n(x)$  denotes the solution of the reduced Sturm-Liouville differential equation with the known boundary conditions. The author shows that the convergence of the expansion (1) can be settled immediately, by means of the most elementary propositions of integral calculus, in case  $f(x)$  satisfies a Lipschitz condition. In many cases we thus readily derive the order (with respect to  $n$ ) of  $A_n$  and of the remainder in (1). The method is applicable to differential equations of higher order, also to many other expansions in series of orthogonal functions.

40. Mr. W. C. Risselman: *On the approximate representation of a given function by means of polynomials in another given function.*

The problem is that of approximating to a given function  $f(x)$  by means of a polynomial  $P_n$  of given degree in a function  $\phi(x)$  so as to minimize  $\int |f(x) - P_n[\phi(x)]|^m dx$ , the integral being taken between finite limits  $(a, b)$ . On the assumption that the inverse of  $\phi(x)$  is single-valued and under certain further hypotheses, which are less restrictive in case  $m=2$  than for other values of  $m$ , uniform convergence of the sequence  $\{P_n[\phi(x)]\}$  to  $f(x)$  is proved. When the inverse of  $\phi(x)$  is multiple-valued, there is convergence to a suitably defined mean of the values of  $f(x)$  corresponding to a single value of  $\phi(x)$ . The method of proof is the same in principle

as one which has been applied recently in discussing the particular case  $\phi(x) \equiv x$ .

41. Mr. J. M. Earl: *On the convergence of polynomials of approximation on an infinite interval.*

This paper deals with the approximation to a given function  $f(x)$  by means of the polynomial  $P_n(x)$  of degree  $n$  or less determined so as to minimize  $\gamma(n)$ , the integral from  $-\infty$  to  $+\infty$  of the  $m$ th power of the absolute value of the error multiplied by a non-negative weight function  $\rho(x)$ . It represents therefore a broad generalization of the problem of Hermite series, corresponding to the special case in which  $\rho(x) = e^{-x^2}$  and  $m = 2$ . Under suitable restrictions on  $f(x)$  and  $\rho(x)$ , it is shown that  $P_n(x)$  converges uniformly to  $f(x)$  on any finite interval on which  $\rho(x)$  has a positive lower bound. The proof is deduced from theorems on polynomial approximation leading to the calculation of an upper bound for the integral  $\gamma(n)$ . A similar discussion for the interval 0 to  $\infty$  can be made to depend on the above.

42. Dr. Elizabeth Carlson: *A simplified proof for the extension of Bernstein's theorem to Sturm-Liouville sums.*

By using a method similar to one used by de la Vallée Poussin in proving the corresponding theorem for trigonometric sums, the author has proved (Transactions of this Society, vol. 26) the following theorem: The maximum of the absolute value of the derivative of a Sturm-Liouville sum of order  $n$  ( $n \geq 1$ ) can not exceed  $npM$ , where  $M$  is the maximum of the absolute value of the sum itself, and  $p$  is independent of  $n$  and of the coefficients in the sum. In this paper, a simpler proof is given by using a well known asymptotic expression for the characteristic solution  $v_k(x)$  of a Sturm-Liouville system, in the form  $v_k(x) = \cos kx + (1/k)\beta(x)\sin kx + (1/k^2)\alpha(xk)$ .

43. Mr. T. S. Peterson: *A class of invariant functionals of quadratic functional forms.*

In his paper *Invariant functionals of functional forms* (abstract in this Bulletin, vol. 34 (1928), pp. 8-9), A. D. Michal considered the law of transformation of a few important invariant functionals of a quadratic functional form. In particular, there were considered the Fredholm determinant and the resolvent kernel of the functional coefficients of the form. The present paper makes a detailed study of the functional invariants arising out of the Fredholm minors of the Fredholm determinant. Use is made of a theorem of Platrier to obtain the law of transformation of these invariants.

44. Professor R. L. Moore: *Concerning upper semi-continuous collections.*

A bounded continuous curve  $M$  will be called a *cactoid* (*opuntoid*) if every non-degenerate maximal cyclic subset (see. G. T. Whyburn, Pro-



ceedings of the National Academy, vol. 13, pp. 31-38) of  $M$  is a simple closed surface and no point of  $M$  lies in a bounded complementary domain of any subset of  $M$ . A cactoid  $M$  is said to be *aspiculate* if it does not contain any arc  $t$  such that no point of  $t$  except its end points is a limit point of  $M-t$ . An aspiculate cactoid  $M$  is *simple* if no two cut points of  $M$  are separated in  $M$  by infinitely many different points. If  $G$  is an upper semi-continuous collection of mutually exclusive continua filling up a spherical surface  $S$  then the space whose elements are the continua of  $G$  is, in the sense indicated in the author's paper *Concerning upper semi-continuous collections of continua* (Transactions of this Society, vol. 27, pp. 416-428), topologically equivalent to a cactoid; and, conversely, every cactoid bears this relationship to some  $G$  and  $S$ . This theorem remains true if "cactoid" is replaced by "simple aspiculate cactoid" and the condition is imposed that the non-degenerate continua of  $G$  form a contracting sequence.

45. Professor K. P. Williams: *The symbolic development of the disturbing function.*

The paper gives a modification of Newcomb's symbolic development of the disturbing function. A set of fundamental operators are isolated and lead directly to Newcomb's operators. The paper will appear in the American Journal of Mathematics.

46. Professor K. P. Williams: *A comment on certain equations in the theory of radiative equilibrium.*

In Eddington's discussion of the internal constitution of a star a set of equations occurs, which determine the coefficients in an expansion in terms of Legendre polynomials. In this paper it is shown that the equations do not warrant the conclusion that Eddington draws from a hasty examination. An integrable case is assumed, and the results confirm the conclusion of the general discussion. The paper appeared in the May number of the Astrophysical Journal.

47. Professor L. M. Graves: *The second variation for discontinuous solutions in the calculus of variations.*

In a paper presented to the Society (Sept., 1927) the author treated the extension of the Jacobi-Carathéodory conditions for discontinuous solutions to space problems, and gave a new formulation and new proof of the Carathéodory conditions which must hold at the corners. The present paper treats the same condition by means of the second variation. A new definition of conjugate point is formulated to include all cases, and the Jacobi-Carathéodory condition is that no minimizing arc can contain a pair of conjugate points. This method takes care of the exceptional cases of the previous methods, and proves also that no minimizing arc can contain a cusp. The equivalence of this formulation of the condition with the earlier one is shown.

48. Mr. E. R. C. Miles: *Fredholm solution of a generalized Neumann problem in the plane.*

Consider the class of functions expressible as potentials of a single layer,  $V(M) = \int \log(1/MP) d\mu(s_p)$  where  $\mu(s_p)$  is of limited variation on the curve  $C$ . The normal derivative of  $u(m)$  is discontinuous across  $C$ , but wherever  $\mu'$  exists, formulas analogous to the usual ones, but in terms of Stieltjes integrals, are obtained, if  $C$  has a curvature at the point. The generalized Neumann problem states that  $(1+\lambda)/(2\lambda)$  times the total flux across a curve which approaches an arc  $AB$  of  $C$  uniformly from inside minus  $(1-\lambda)2\lambda$  times the corresponding outside limit shall be a given function  $g(B) - g(A)$ , of limited variation and with regular discontinuities on  $C$ . This problem has a unique solution provided  $\lambda$  is not one of a set of characteristic values. The value  $\lambda = -1$  yields a Neumann problem for the exterior region, and is not a characteristic value;  $\lambda = +1$  yields a Neumann problem for the interior region, and is a characteristic value. In the latter case a necessary and sufficient condition for a solution is that  $\int_C dg(s) = g(C) = 0$ , and the solution is determined except for an arbitrary constant.

49. Mr. E. R. C. Miles: *Fredholm solution of a generalized Dirichlet problem in the plane.*

Analogous results to those of the previous paper, with reference to the generalized Dirichlet problem, are obtained for the class of functions expressible as potentials of a double layer,  $u(m) = \int [(\cos nr)/r] dv(s_p)$ . Both problems are handled by solving Stieltjes integral equations, and include the usual treatments as particular cases. One interest in these problems is the fact that the methods used are extensible to the three-dimensional situation, as will be shown in a paper which the author is writing in conjunction with Professor Evans.

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