polar is degenerate; for $p=3$, $n=3+1$, $\epsilon=1$, we find again the 2d polar is degenerate.

If $n=\alpha p^m+\beta p^{m-1}+\cdots+\gamma p^2+\delta p$, i.e. $\epsilon=0$ in n, then all the polars of (1, 0, 0) pass through (1, 0, 0) whether or not this point lies on $f(x, y, z)=0$.

If $n<p$ we find no peculiarities like the above.

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THE CHARACTERISTIC EQUATION OF A MATRIX*

BY E. T. BROWNE

1. *Introduction.* Consider any square matrix A, real or complex, of order n. If I is the unit matrix, $A-\lambda I$ is called the *characteristic matrix* of A; the determinant of the characteristic matrix is called the *characteristic determinant* of A; the equation obtained by equating this determinant to zero is called the *characteristic equation* of A; and the roots of this equation are called the *characteristic roots* of A. If A happens to be a matrix of a particular type certain definite statements may be made as to the nature of its characteristic roots. For example, if A is Hermitian its characteristic roots are all real; if A is real and skew-symmetric, its characteristic roots are all pure imaginary or zero; if A is a real orthogonal matrix, its characteristic roots are of modulus unity. However, if A is not a matrix of some special type, no general statement can be made as to the nature of its characteristic roots. In 1900 Bendixson† proved that if $\alpha+i\beta$ is a characteristic root of a real matrix A, and if $\rho_1\geqq\rho_2\geqq\cdots\geqq\rho_n$ are the characteristic roots (all real) of the symmetric matrix $\frac{1}{2}(A+A')$, then $\rho_1\geqq\alpha\geqq\rho_n$. The extension to the case where the elements of A are com-

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† Bendixson, *Sur les racines d'une équation fondamentale*, Acta Mathematica, vol. 25 (1902), pp. 359–365.

plex was made by Hirsch* in 1902. In 1904 Bromwich† further extended the theorem as follows: If $\alpha+i\beta$ is a characteristic root of a matrix A whose elements are real or complex, and if $\rho_1, \rho_2, \cdots, \rho_n$ are the characteristic roots (all real) of $\frac{1}{2}(A+\bar{A}')$ and $i\mu_1, \cdots, i\mu_n$ are the characteristic roots of $\frac{1}{2}(A-\bar{A}')$, then α lies between the greatest and the least of $\rho_1, \cdots, \rho_n$, and $|\beta|$ does not exceed the greatest of $|\mu_1|, \cdots, |\mu_n|$.

In some cases the theorems just cited give very good limits for the characteristic roots of a matrix, while in other cases the limits are not so restricted. Thus in the case of a real orthogonal matrix these theorems may merely state that the characteristic roots lie in the square $x=\pm 1$, $y=\pm 1$. In this paper we shall give a criterion which in some cases, notably in the case of a real orthogonal matrix, give more restricted limits than the theorems above.

2. *Reduction of a Matrix to a Semi-Unitary Form.* Let A be any square matrix of order n. Then $A\bar{A}'$ is Hermitian and there exists a unitary matrix κ (that is, $\kappa\bar{\kappa}'=I$) such that

$$\kappa A\bar{A}'\bar{\kappa}' = M,$$

where M‡ is zero except in the diagonal, and the elements in the diagonal are the (real) characteristic roots $\rho_1, \rho_2, \cdots, \rho_n$ of $A\bar{A}'$. We may write

$$M = \kappa A\bar{\kappa}'\kappa\bar{A}'\bar{\kappa}' = B\bar{B}', \tag{1}$$

where

$$B = \kappa A\bar{\kappa}'. \tag{2}$$

From (1) the elements b_{ij} of B evidently satisfy the conditions

$$\sum_{t}^{1,\cdots,n} b_{it}\bar{b}_{jt} = \rho_i\delta_{ij}, \quad (i, j = 1, \cdots, n), \tag{3}$$

* Hirsch, Acta Mathematica, vol. 25 (1902), p. 367.

† Bromwich, *On the roots of the characteristic equation of a linear substitution*, Acta Mathematica, vol. 30 (1906), pp. 295–304.

‡ Hilton, *Homogeneous Linear Substitutions*, Oxford, 1914, p. 41.

where δ_{ij} is the Kronecker symbol, and equals 1 if $i=j$; 0 if $i\neq j$. In view of the conditions (3) we shall say that B is in a *semi-unitary* (*semi-orthogonal*, if B is *real*) form. If $\rho_i=1$, $(i=1,\cdots,n)$, B is unitary. We may then state the following theorem.

THEOREM I. *If A is any square matrix of order n there exists a unitary matrix κ such that $\kappa A\bar{\kappa}'=B$, where B is in a semi-unitary form.*

If M is of rank r, κ may be so chosen that $\rho_i>0$, $(i=1,\cdots,r)$; $\rho_i=0$, $(i=r+1,\cdots,n)$. Since $\rho_i=\sum_t b_{it}\bar{b}_{it}=0$, $(i=r+1,\cdots,n)$, evidently $b_{it}=0$, $(i=r+1,\cdots,n;\ t=1,\cdots,n)$; that is, the last $n-r$ rows of B consist entirely of zeros, so that B is of rank at most r. Hence, B must be of rank exactly r. Since the rank of A equals the rank of B, and the rank of $A\bar{A}'$ equals the rank of M, incidentally we have given a proof of the following well known theorem.

THEOREM. *If A is any square matrix of order n, the ranks of A and $A\bar{A}'$ are the same.**

3. *The Characteristic Roots of $A\bar{A}'$.* Referring to the matrix B defined as in (1) and (2), let us form a non-singular matrix $C=(c_{ij})$ by replacing the zeros in the last $n-r$ rows of B by elements $(x_{s1}, x_{s2},\cdots, x_{sn})\neq(0, 0,\cdots, 0)$, such that

$$(4)\qquad \sum_t^{1,\cdots,n} b_{it}\bar{x}_{st} = 0, \qquad (i = 1, \cdots, r;\ s = 1, \cdots, n-r),$$

and, moreover, such that

$$\sum_t^{1,\cdots,n} x_{it}\bar{x}_{jt} = 0, \qquad (i, j = 1, \cdots, n-r;\ i\neq j).$$

Thus, we may find $(\bar{x}_{11}, \bar{x}_{12},\cdots, \bar{x}_{1n})$ by determining a non-zero solution of the $n-r$ linear homogeneous equations (4). Having obtained $(x_{11},\cdots, x_{1n})$ we may proceed to find

* Hilton, *Homogeneous Linear Substitutions*, Exercise 4, p. 51.

$(\bar{x}_{21}, \bar{x}_{22}, \cdots, \bar{x}_{2n})$ by adjoining to the system (4) the additional linear homogeneous equation

$$\sum_{t}^{1,\cdots,n} x_{1t}\bar{x}_{2t} = 0\ ;$$

and so on. If $\sum_t^{1,\cdots,n} c_{it}\bar{c}_{it} = \rho_i$, $(i=1,\cdots, n)$, then $\rho_i > 0$ and if we write

$$\chi_{ij} = \frac{c_{ij}}{(\rho_i)^{1/2}}, \qquad (i, j = 1, \cdots, n),$$

the matrix χ thus obtained is a unitary matrix. It is evident from the manner in which χ was built up that $B\bar{\chi}'$ is zero except in the diagonal. The elements in the last $n-r$ places in the diagonal are also zero, while those in the first places are $(\rho_i)^{1/2}$, the square roots of the characteristic roots of $A\bar{A}'$. Since $B\bar{\chi}'$ is real and symmetric, the characteristic roots of

$$N = \chi\bar{B}'B\bar{\chi}' = (B\bar{\chi}')^2$$

are the squares of the characteristic roots of $B\bar{\chi}'$, and are therefore the characteristic roots of $A\bar{A}'$. But

$$N = \chi\bar{B}'B\bar{\chi}' = \chi\kappa\bar{A}'\bar{\kappa}'\kappa A\bar{\kappa}'\bar{\chi}' = \chi\kappa\bar{A}'A\bar{\kappa}'\bar{\chi}' = \psi\bar{A}'A\bar{\psi}',$$

where ψ is the unitary matrix $\chi\kappa$. Thus it follows* that the characteristic roots of $\bar{A}'A$ are the same as those of N and therefore of $A\bar{A}'$. Hence we have the following theorem.

THEOREM II. *If A is any square matrix of order n the characteristic roots of $A\bar{A}'$ are the same as the characteristic roots of $\bar{A}'A$.*

Since the unitary matrices κ, χ above are such that

$$\kappa A\bar{\kappa}' = B, \text{ and } B\bar{\chi}' = \chi\bar{B}',$$

it follows at once that

$$\kappa A\bar{\kappa}'\bar{\chi}' = B\bar{\chi}' = \chi\bar{B}' = \chi\kappa\bar{A}'\bar{\kappa}'.$$

Hence

* Hilton, *Homogeneous Linear Substitutions*, p. 20.

$$\bar{\kappa}'\bar{\chi}'\kappa A\bar{\kappa}'\bar{\chi}'\kappa = \bar{A}'.$$

Writing $\bar{\kappa}'\bar{\chi}'\kappa = \phi$, we have the following theorem.

THEOREM III. *If A is any square matrix of order n there exists a unitary matrix ϕ such that*

$$\phi A\phi = \bar{A}'. \tag{5}$$

In this connection compare Hilton, *Homogeneous Linear Substitutions*, Ex. 6, p. 124.

Since from (5)

$$A\phi = \bar{\phi}'\bar{A}' = (\overline{A\phi})',$$

$A\phi$ is Hermitian, so that we have the following theorem.

THEOREM IV. *If A is any square matrix of order n, there exists a unitary matrix ϕ such that $A\phi$ is Hermitian.*

4. *The Characteristic Roots of A.* From (2) the characteristic roots of A are evidently the same as the characteristic roots of B. Suppose then that λ is a characteristic root of B so that there exists a set $(x_1, x_2, \cdots, x_n) \neq (0, 0, \cdots, 0)$ such that

$$\sum_{t}^{1,\cdots,n} b_{ti}x_t = \lambda x_i, \qquad (i = 1, \cdots, n). \tag{6}$$

Taking the conjugates of both members of each of these equations, we have

$$\sum_{s}^{1,\cdots,n} \bar{b}_{si}\bar{x}_s = \bar{\lambda}\bar{x}_i, \qquad (i = 1, \cdots, n). \tag{7}$$

Multiplying corresponding equations in (6) and (7), member for member, and summing as to i, we find

$$\sum_{s,t}^{1,\cdots,n}\left[\sum_{i}^{1,\cdots,n} b_{ti}\bar{b}_{si}\right]x_t\bar{x}_s = \lambda\bar{\lambda}\sum_{i}^{1,\cdots,n} x_i\bar{x}_i\,;$$

that is

$$\sum_{i}^{1,\cdots,n} \rho_i x_i\bar{x}_i = \lambda\bar{\lambda}\sum_{i}^{1,\cdots,n} x_i\bar{x}_i.$$

Let G be the largest and s the smallest of the characteristic roots of $A\bar{A}'$. Then

$$\lambda\bar{\lambda} \sum x_i\bar{x}_i \leqq G \sum x_i\bar{x}_i,$$

so that $\lambda\bar{\lambda} \leqq G$. Similarly, $\lambda\bar{\lambda} \geqq s$; i. e.,

$$s \leqq \lambda\bar{\lambda} \leqq G.$$

In particular, if A is unitary so that $A\bar{A}' = I$, then $G = s = 1$, so that $1 \leqq \lambda\bar{\lambda} \leqq 1$; i.e., $\lambda\bar{\lambda} = 1$, as is well known. Hence we have the following theorem.

THEOREM V. *If λ is a characteristic root of a square matrix A and if G and s are respectively the largest and the smallest characteristic roots of $A\bar{A}'$, then*

$$s \leqq \lambda\bar{\lambda} \leqq G.$$

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