ON THE MAPPING OF THE SEXTUPLES OF THE SYMMETRIC SUBSTITUTION GROUP G_6 IN A PLANE UPON A QUADRIC*

BY ARNOLD EMCH

1. *Introduction.* The six permutations of three elements x_1, x_2, x_3 considered as projective coordinates in a plane determine an involution of sextuples of points which may be mapped on a rational surface.† I shall show that in case of the involution thus defined the map is a quadric whose relation with the plane, established with sufficient details, will lead to some interesting geometric applications. The map of every configuration on the quadric will be a configuration in the plane, invariant under the G_6, whose geometric properties have been investigated before.‡

2. *Mapping of the* G_6. Let ϕ_1, ϕ_2, ϕ_3 represent the elementary symmetric functions $\phi_1 = x_1 + x_2 + x_3$, $\phi_2 = x_2x_3 + x_3x_1 + x_1x_2$, $\phi_3 = x_1x_2x_3$, so that the general symmetric function of degree four has the form

$$y_i = a_i\phi_1^4 + b_i\phi_1^2\phi_2 + c_i\phi_1\phi_3 + d_i\phi_2^2\,, \tag{1}$$

depending upon three effective constants. Hence, there are four linearly independent functions y_i; $i = 1, 2, 3, 4$, which we may set proportional to the four projective coordinates of a point in a space S_3. Thus to every point in (x), and consequently to every sextuple $I_6\{(x_1x_2x_3), (x_1x_3x_2), (x_2x_1x_3), (x_2x_3x_1), (x_3x_1x_2), (x_3x_2x_1)\}$, corresponds in S_3 a point (y), and as the system of sextuples is a continuous ∞^2 manifold, the locus of such points (y) must be a surface, which will be proved to be a quadric cone Q. For the y_i's we may

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† Castelnuovo, G., *Sulla razionalità delle involuzioni piane*, Mathematische Annalen, vol. 44 (1894), pp. 125–155.

‡ Emch, A., *Some geometric applications of symmetric substitution groups*, American Journal, vol. 45 (1923), pp. 192–207.

evidently choose any four linearly independent symmetric quartics. For every choice of four such functions we obtain a certain surface Q. But all these surfaces are obviously collinearly related. The simplest choice is

$$(2)\quad \begin{cases} \rho y_1 = x_1^4 + x_2^4 + x_3^4 = \phi_1^4 - 4\phi_1^2\phi_2 + 4\phi_1\phi_3 + 2\phi_2^2, \\ \rho y_2 = x_2^2x_3^2 + x_3^2x_1^2 + x_1^2x_2^2 = \phi_2^2 - 2\phi_1\phi_3, \\ \rho y_3 = x_1^3(x_2 + x_3) + x_2^3(x_3 + x_1) + x_3^3(x_1 + x_2) \\ \qquad = \phi_1^2\phi_2 - 2\phi_2^2 - \phi_1\phi_3, \\ \rho y_4 = x_1^2x_2x_3 + x_2^2x_3x_1 + x_3^2x_1x_2 = \phi_1\phi_3. \end{cases}$$

A simple elimination process of ϕ_1, ϕ_2, ϕ_3 leads to the required relations between the y's:

$$(3)\qquad y_1(y_2 + 2y_4) + 2y_2^2 - y_3^2 - y_4^2 + 4y_2y_4 - 2y_3y_4 = 0,$$

which is a quadric cone Q with the vertex $V(4, -2, -1, -1)$, as can easily be verified.

To a plane section of Q corresponds in (x) a quartic which may be any of the reducible or irreducible types (1). Of particular importance are, of course, the exceptional elements of the (1, 6) correspondence between Q and (x). In the first place for the intersections $I(1, \omega, \omega^2)$, $J(1, \omega^2, \omega)$ of $\phi_1=0$ and $\phi_2=0$ there is $y_1=y_2=y_3=y_4=0$, so that I and J are fundamental points in (x). To the first neighborhoods of I and J, $(1+\alpha_1, \omega+\alpha_2, \omega^2+\alpha_3)$ and $(1+\alpha_1, \omega^2+\alpha_2, \omega+\alpha_3)$, correspond on Q for $\alpha_1+\alpha_2+\alpha_3\neq 0$, $\rho y_1=4(\alpha_1+\alpha_2+\alpha_3)$, $\rho y_2=-2(\alpha_1+\alpha_2+\alpha_3)$, $\rho y_3=-(\alpha_1+\alpha_2+\alpha_3)$, $\rho y_4=\alpha_1+\alpha_2+\alpha_3$, or the point $V(4, -2, -1, 1)$. To a sextuple on $\phi_1=0$, distinct from I and J, corresponds on Q the point $T(2, 1, -2, 0)$. A plane $p=y_1+\lambda y_2+\mu y_3+\nu y_4=0$ cuts Q in a conic K and the join VT in a point R on K, to which corresponds in (x) the quartic

$$(4)\quad \phi_1^4 + (\mu - 4)\phi_1^2\phi_2 + (4 - 2\lambda - \mu + \nu)\phi_1\phi_3 + (2 + \lambda - 2\mu)\phi_2^2 = 0,$$

which has $\phi_1=0$ as a double tangent. To a generic point R on VT correspond thus the first neighborhoods of I and J on $\phi_1=0$. To a plane

$$(5) \qquad y_1 + \lambda y_2 + \mu y_3 + (2\lambda + \mu - 4)y_4 = 0$$

through V, corresponds the quartic

$$(6) \qquad \phi_1^4 + (\mu - 4)\phi_1^2\phi_2 + (2 + \lambda - 2\mu)\,\phi_2^2 = 0,$$

which clearly reduces to the product of two conics of the type $\phi_1^2 + k\phi_2 = 0$. Thus a generic plane of the bundle through V cuts Q in two generatrices to which correspond in (x) two conics of the symmetric pencil $\phi_1^2 + k\phi_2 = 0$. To a tangent plane of Q corresponds a double conic $(\phi_1^2 + k\phi_2)^2 = 0$. The tangent plane at R

$$y_1 + 6y_2 + 4y_3 + 12y_4 = 0$$

touches Q along VT, and to this intersection of the tangent plane, VT counted twice, corresponds in (x) the quadruple line $\phi_1^4 = 0$.

3. *Mapping of Intersections of the Quadric Cone.* A generic surface F_n cuts Q in a space curve C_{2n} to which corresponds in (x) a symmetric curve C'_{4n} (curve in which the coordinates enter symmetrically). As F_n cuts VT in n points, $\phi_1 = 0$ is, in general, an M-fold double tangent of C'_{4n}. This also appears directly from the fact that in (1) ϕ_2^2 appears in y_1, y_2, y_3, and ϕ_1 is a factor of all other terms in which ϕ_2^2 is not contained.

Conversely to a symmetric n-ic C'_n in (x) corresponds on Q a curve, whose order can easily be determined in every case. For instance, when C'_n does not pass through I and J, which is the case when C'_n contains the term ϕ_3^m, $3m = n$, then a generic quartic C'_4 cuts C'_n in $12m$ points which form $2m$ sextuples. To these correspond on Q, $2m$ points which lie on a plane of the corresponding conic K of C_4, and which are the intersections of the curve C on Q, corresponding to C'_4. The order of C is therefore $2m$. The curve C_{2m} on Q is cut out by a surface F_m, which may possibly pass through generatrices of Q, or through the point T. For example when C'_n is a sextic, then F_m is a quadric which passes through a generatrix of Q, or is a quadric cone with its vertex at T. This is in agreement with the counting of constants. The number $N+1$ of terms of a symmetric n-ic is equal to the number of positive integral solutions of the diophantine equation $\alpha+2\beta+3\gamma=n$ resulting from the general term $\phi_1^\alpha\phi_2^\beta\phi_3^\gamma$ of the n-ic, and is equal to the nearest positive integer contained in $(n+3)^2/12$. For $n=6$, $N+1=7$. The number of constants in F_2 is 10 which is reduced to 7 by the condition to pass through a generatrix of Q. To the intersection of F_2 with Q corresponds in (x) an octavic. But to the generatrix of Q, common to F_2, corresponds a symmetric conic as a factor of the octavic, so that a sextic remains as a residual curve.

More generally F_m cuts Q in a curve to which corresponds in (x) a curve of order $4m$. In order that this reduce to $3m$ it is necessary that a factor of order m split off. These factors are of the form $\phi_1^\alpha\phi_2^\beta$, with $\alpha+\beta=m$, and F_m must pass α times through R and contains β generatrices of Q. Thus in case of $n=9$, $m=3$, F_m must be either a cubic cone with vertex at T, or a cubic surface through T and a generatrix of Q. The condition to pass through T and a generatrix of Q absorbs 5 constants of F_3, and leaves 15 (14 effective) disposable constants. But through the intersection of a quadric and a cubic surface there are ∞^3 other cubics, so that there are ∞^{12} linearly independent residual quintics on Q to which corresponds in (x) the same manifold of conics, which is in agreement with the number of solutions of the diophantine equation, in case of $n=9$.

4. *Symmetric Quartics.* To a pencil of planes through a line s cutting Q in A and B corresponds in (x) a pencil of quartics with the same double tangent $\phi_1=0$ and with two sextuples A' and B', corresponding to A and B, as base-points outside of the double tangencies at I and J. When s is tangent to Q, the quartics of the pencil all touch each other in the points of a sextuple. Now consider any two conics K' and K'' on Q. The common tangent-planes of K' and K'' envelope two cones. Through a generic

point of Q there are two tangent-planes to each of these cones. To K' and K'' correspond in (x) two quartics C_4' and C_4''. Every tangent plane of one of these cones cuts Q in a conic which touches K' and K''. To this conic corresponds a quartic which touches each C_4' and C_4'' in points of a sextuple, one for each C_4' and C_4''. Hence we may state the following theorem.

THEOREM. *Given two symmetric quartics C_4' and C_4'', then there exist two ∞^1 systems of symmetric quartics of index* 2, *such that every quartic of each system has a sextuple contact (contact in points of a sextuple) with each C_4' and C_4''.*

To the intersection of a plane through T with Q corresponds in (x) a symmetric cubic $\phi_1^3+\lambda\phi_1\phi_2+\mu\phi_3=0$. Let K' again be a conic not through I. Now I is the vertex of a quadric cone through K', whose tangent planes cut Q in conics tangent to K'. To these correspond in (x) cubics and a quartic respectively. This leads to the following theorem.

THEOREM. *For a symmetric quartic corresponding to a generic conic on Q there exists a system of cubics of index* 2 *with the property of sextuple contacts with the quartic.**

5. *A Problem in Closure.* Let K' and K'' be again two conics on Q not intersecting in real points and O a generic point in space. O as a vertex determines with K' and K'' two cones C' and C'', which intersect O in two other conics L' and L''. Assume O such that C' and C'' have no real generatrix in common, moreover so that it is possible to construct a closed pyramid of n faces inscribed to one cone, say C', and circumscribed to C''. To the four conics K', K'', L', L'' correspond in (x) four quartics C_4', C_4'', D_4', D_4''. To a conic cut out on Q by a face of P corresponds in (x) a quartic which cuts each C_4' and D_4' in a sextuple and touches each C_4'' and D_4'' in points of a sextuple. As there

* Such systems of sextuple tact cubics for the general quartic were established by A. Clebsch, Mathematische Annalen, vol. 3 (1871), pp. 45–75.

are ∞^1 such inscribed and circumscribed pyramids P the mapping upon (x) gives the following theorem.

THEOREM. *Let C_4' and C_4'' be two fixed symmetric quartics in (x). Construct a quartic $C_4^{(1)}$ with a sextuple contact with C_4'', cutting C_4' in two sextuples S_1 and S_2. Through S_2 draw another quartic $C_4^{(2)}$ with a sextuple contact with C_4'', which cuts C_4' in another sextuple S_3. Through S_3 draw similarly a third quartic $C_4^{(3)}$, cutting C_4' in a sextuple S_4, and so forth. Suppose that after drawing n such quartics, the last $C_4^{(n)}$ through S_n cuts C_4' in a sextuple S_{n+1} which coincides with S_1. If this happens once then there exists an infinite number of such series of quartics with the closure property.*

Moreover there exist two other fixed quartics D_4' and D_4'' which are related to these series in precisely the same manner as C_4' and C_4'.

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CONGRUENCES OF LINES OF SPECIAL ORIENTATION RELATIVE TO A SURFACE OF REFERENCE*

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1. *Introduction.* With each line l of a rectilinear congruence let us associate the point M in which l intersects a surface of reference S. We refer S to any orthogonal system. Let α, β, γ be the direction-cosines of l relative to the moving trihedral of S at M, the x-axis being chosen tangent to the curve $v=\text{const.}$ By congruences of special orientation relative to S, we shall mean those congruences for which the functions α, β, γ are of a special form. The present paper is concerned primarily with the case when α, β, γ are constant.

2. *Normal Congruences.* Relative to the moving trihedral the coordinates of any point P on l are

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