OSGOOD'S ADVANCED CALCULUS

Advanced Calculus. By William F. Osgood. New York, The Macmillan Company, 1925. xvi+530 pages.

Almost twenty years ago, in a presidential address delivered before the American Mathematical Society*, Professor Osgood outlined his conception of the aims and methods that should underly the teaching of the calculus. As between the formalists and the reformists of the Perry school, he pointed out that though the former were right in insisting on the necessity of rigorous training in formal work in order to acquire the ideas of the calculus, still this drill must appear to the student "as having for its direct object the power to solve some of the real problems of pure and applied mathematics, and these problems must always be kept before his eye." This idea was emphasized again with the words "That which is most central in the calculus is its *quantitative* character, through which it measures and estimates the things of the world of our senses. And instruction in the calculus that does not point out—not merely at the beginning or at the end, but all through the course—this close contact with nature, has not done its duty by the student." He felt, however, that too often those who had attempted to interpret physical phenomena mathematically had started from incomplete and vaguely stated hypotheses and had used methods so slipshod as to be beneath the respect of the undergraduate student of the calculus.

Later in the same year his *First Course in the Differential and Integral Calculus* was published. Here the program of his presidential address was carried out. There was a large amount of material for drill in technique; statements of hypotheses and conclusions were clear and accurate; proofs were carried through with all the rigor suitable for a first course in the calculus, and a large part of the book was devoted to applications, with the aim not merely to impart knowledge, but to foster appreciation of the spirit of the calculus and to give power to use it as a tool in the interpretation of nature. The book had a wide sale, and many of the younger generation of American mathematicians there acquired their first appreciation of what mathematics means. Still, however great the service thus rendered, it was surpassed by the influence for the better that this text exerted upon all succeeding works on the calculus published in this country, and on many published abroad. Few could successfully imitate the style of Osgood, many saw only imperfectly what he was aiming at, yet if they could not draw the bow of Ulysses they at least modeled their armory after his and shot as nearly as they could in the same direction.

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Some instructors found the *First Course* too "hard"; others, with more reason, felt that the material presented was far too copious for a three-hour year course, but considerably short of enough for a second course. The author himself felt that rearrangement, revision, and extension were advisable. This plan was carried out by the publication in 1922 of his *Introduction to the Calculus* (in the previous year the first half had been brought out under the title *Elementary Calculus*), followed three years later by the volume which is the subject of this review.

The author does not here attempt to duplicate any part of his *Introduction*, hence no general restatement or summary of a first year course in the calculus is included. Instead there are numerous references to the *Introduction*. The instructor who uses this text must therefore see to it that the *Introduction* is accessible to his class, or he must be prepared to substitute other explanations and references. Even the syllabus of elementary solid analytic geometry that found a place in the *First Course* has been dropped, and the reader is referred to Osgood and Graustein's *Plane and Solid Analytic Geometry*. It may be surprising to many that an Advanced Calculus should contain almost nothing on the convergence or divergence of infinite series, and little on Maclaurin's and Taylor's developments. The author, has, however, preferred to devote considerable space in the *Introduction* to these topics and thus can dispense with their treatment here.

On the other hand he has included two chapters on general methods of integration and on reduction formulas, which ordinarily are considered a part of elementary rather than of advanced calculus. However, it is a satisfaction to find here a good proof of the theorems concerning partial fractions, a systematic treatment of rationalizing substitutions, and a proper appreciation of the waste of ingenuity involved in proofs of reduction formulas except as the inverse of differential relations. Otherwise the book is distinguished by its wealth of applications to physical problems and only here presents material beyond that covered by the normal class in advanced calculus.

The author explains that the order in which he has written is not a necessary order for the reader. The latter may begin with the chapter on Partial Differentiation, or with Double Integrals, or Differential Equations. Again, the omission of certain sections on first reading of a chapter is suggested. This very flexibility indicates clearly the author's perception of the important part the teacher must play in determining the success or failure of a course in the calculus. The instructor who, without looking ahead, assigns "the next four pages with all the odd-numbered problems", will have poor sailing and a mutinous crew before long. If, for instance, the lesson includes, without previous explanation, the proof of pages 73–74 that a density function in two dimensions must be continuous although this does not hold in one dimension, there may be trouble. Along with the easier and more formal problems will be found others that will tax the best in the class, others that anticipate later chapters. The instructor must be prepared to pick and choose according to the strength and needs of his class.

Although it is true that the author does not hesitate to include both proofs and problems of some difficulty, he does not follow the too prevalent plan of plunging through a stiff demonstration as quickly as possible and emerging to simpler matters calculated to reassure the bewildered neophyte. He is quite willing, in fact, to take a hard journey in several stages. Frequently there is a "heuristic proof", which is shown in its true light as a discussion pointing the way to the truth, followed by a demonstration of more satisfying rigor. For example, there are three proofs of the formula which reduces a double integral to iterated integrals in rectangular coordinates. The first is based on geometric intuition, the second is arithmetic but incomplete, the third, some distance on, indicates how the second is to be made satisfactory. Everywhere there is the greatest care to say precisely what is meant. The author does not suffer from the delusion that accuracy of statement makes a subject more difficult or less attractive.

There are enough exercises. Many are formal and relatively easy, but others will require more than ordinary ability for their solution. Some test the student's capacity to anticipate matters taken up later in the text. In fact there is no gentleman's agreement to make only backwards references. Both in text and exercises there is not infrequently a look forward, and sometimes results are used whose truth is for the time being assumed, but is proved later. There is a good supply of examples worked out. These often emphasize critical points as well as matters of routine. In fact exercises, solved or unsolved, in a few cases supply the proofs of formulas that follow in later pages of the text, thus affording the class an excellent opportunity to find whether the instructor knows what sort of a lesson he has assigned.

Up to the end of Chapter X much material has been taken from the *First Course*, but besides Chapters I and II, Chapter IX is new, as is practically all that follows the tenth chapter. There has been, however, much revision of the treatment of topics from the *First Course*, and nearly all the changes are, to the reviewer's mind, improvements. As one leaves behind this first part of the book, one cannot but sense that the author feels a certain relief in being through with this too familiar, often-worked-over field. There is an atmosphere of added enthusiasm, an interest more tense, as we pass on to chapters where applications are of major importance and technique is less stressed. There are passages also where the author unbends to make an apt remark or to indulge in a picturesque or even a homely phrase. To the reader we leave the pleasure of digging these out and chuckling over them. They are not too frequent, and are well balanced by the more serious exhortations against such transgressions as the uncritical use of infinitesimals or juggling proofs.

We have already indicated the subject matter of Chapters I and II, on general methods of integration and reduction formulas. Chapters III and IV consider double and triple integrals. Here much has been taken from the *First Course*, but there are additions and changes. The proofs depend largely on geometric intuition, as the author is careful to point out; more rigorous demonstrations are deferred to Chapter XII. Thus the student obtains some insight into the methods and uses of integration before he encounters existence proofs of an arithmetic character. Besides the applications generally to be found in other texts, there are to be noted the sections on attractions, on density, pressure at a point and specific force, and on potential in three dimensions.

The next four chapters, from V to VIII, are concerned with partial differentiation and its applications, these latter being chiefly geometric. Part of the corresponding material of the *First Course* was used in the last nine pages of the *Introduction*; most of the rest is to be found here in revised form, with some important additions. Thus the former admirable treatment of the total differential is here somewhat expanded, and the approach to it has been made easier. In section 12 of Chapter V, existence theorems are carefully stated for implicit functions defined by one equation or by several simultaneous equations, but the reader is referred to other books for proofs. On pages 180–185, Chapter VII, there is a discussion of Lagrange's method of multipliers in the theory of maxima and minima. Chapter VIII, on envelopes, develops a sufficient condition that a family of curves have an envelope, whereas most texts stop with necessary conditions.

Elliptic integrals are treated in Chapter IX. Most of the thirteen pages are devoted to the standard forms for integrals of the first and second kind and the reduction of integrals to these forms. This is followed by a brief but skillful explanation of Landen's transformation, and by a half page defining the elliptic functions and giving references.

Chapter X, on indeterminate forms, has, in the main, been transferred from the *First Course*. It is probably more the fashion to take this subject up in elementary calculus, but here, at least, we have satisfactory proofs and a good sense of relative values.

In Chapter XI the notion of work done on a particle by a force leads up to definitions of line integrals. Green's Theorem in two dimensions is then discussed, and is used in treating integrals independent of path. In this connection the author pauses, for the space of three pages, to consider definitions and elementary properties of simply and multiply connected regions. He then passes on to line integrals and Green's Theorem in three dimensions (called Green's Lemma in a later chapter, page 287). Stokes' Theorem is next proved more carefully than is usual in other texts. Sections 11–17 develop the equations of the flow of heat and of electricity in conductors. A valuable feature is the clear distinction between mathematical proof and the inductive processes by which a physical law is inferred and expressed in a mathematical formula. It would be hard to improve on these sections, which are adaptations of material from the author's *Allgemeine Funktionentheorie*. In sections 18 and 19 there is a glance forward to related topics treated in later chapters.

Most readers will find Chapter XII, on the transformation of multiple integrals and on the equation of continuity, more difficult than any other in the book. There is no help for this; it could not be otherwise in a treatment at all adequate. The first section begins with a satisfactory arithmetized existence proof for the ordinary definite integral, with some novelty in its form, and so stated as to help in proving the existence theorem for double integrals and the formulas for their reduction to iterated integrals considered in the following sections. The property of uniform continuity is explicitly assumed, and the property of zero area for the boundary is tacitly supposed to hold for all double integrals considered. This latter matter may trouble the reader of inquiring mind when he tries to verify the assertion "the further development of the proof" (for double integrals) "follows precisely the lines of the earlier case" (for the ordinary definite integral). The reader referred to may also be puzzled to find why the author assumes, on page 254, that $f(x)$ is positive. The transformation of double and triple integrals is next considered, a "heuristic proof" employing infinitesimals and Duhamel's Theorem being in each case followed by a logically more satisfactory one using, respectively, line and surface integrals. The last four sections of the chapter are devoted to obtaining in various forms the equation of continuity in hydrodynamics. Section 11 will be difficult reading for the average student, but the effort to master it will be well worth while. We may note again the author's insistence on a distinction between what is proof and what is not (see especially pages 278–279).

An elementary treatment of vector analysis occupies Chapter XIII. Green's and Stokes' Theorems are put in vector form, and there is a section on curvature and torsion of twisted curves. In section 9 there are very sensible remarks on limitations to the usefulness of vector notations.

The next chapter is a long one of 67 pages on differential equations, with emphasis on applications. Thus of the first third, on equations of the first order, seven pages are devoted to solutions of standard cases, while sixteen give applications. The other subdivisions are on linear equations, geometrical interpretation and singular solutions, solutions by series and by integrating factors, and partial differential equations. Section 14 will hardly be appreciated unless the reader is familiar with the pages of Appell's *Mécanique Rationnelle* given as a reference, and the top of page 348 may also prove troublesome owing, apparently, to an unexplained change of notation. The brief discussion of singular solutions is excellent. In sections 24 and 25 there is a treatment of partial differential equations of the first order by the method of characteristics. Naturally, this is not one of the simplest parts of the book.

Chapters XV and XVI are especially valuable, though many classes will omit one or both in a one-year course. The matters here presented can hardly be found elsewhere in such concise and elegant form. The former chapter begins with a treatment of forced vibrations and then derives in exact form the partial differential equations for both transverse and longitudinal motion of a vibrating string. The familiar simpler forms are obtained as approximations. The vibrating membrane is also briefly considered. Chapter XVI, on Fourier's series and orthogonal functions, considers formal developments in Fourier's series, power series as proceeding from problems of approximate representation, and series of orthogonal functions in general, especially as suggested by the principle of least squares. Five pages are given to zonal harmonics and Bessel's functions.

Chapter XVII, on the calculus of variations and Hamilton's principle, carries the first subject only as far as the derivation and solution of Euler's equation, i.e., the problem of finding extremals, with applications. Most of these applications are familiar enough, though the derivation of Laplace's equation in curvilinear coordinates is not so well known. Isoperimetric problems and variable end points are considered, also integrals in parametric form. The author's insistence on accurately stated definitions of variations is to be commended. Hamilton's Principle and the Principle of Least Action occupy the last fourteen pages. Their application to a number of relatively simple problems is a most valuable feature.

Chapter XVIII, on thermodynamics and entropy, will help clear up some fundamental notions in this field.

With Chapters XIX and XX we return to matters of a more formal sort. The first takes up definite integrals with parameters, improper integrals and tests for their convergence or divergence, the Gamma and Beta Functions, improper double integrals, and special methods for the evaluation of improper integrals. The last chapter gives a brief introduction to the elementary theory of functions of a complex variable.

There are some misprints, though not an undue number. As part or all may have been corrected in the plates since the first printing, a list of them would be useless here.

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HEATH'S EUCLID

The Thirteen Books of Euclid's Elements translated from the text of Heiberg with introduction and commentary. By Sir Thomas L. Heath, K.C.B., K.C.V.O., F.R.S. Cambridge, University Press, 1926. 8 vo. 3 volumes, pp. xii+432; 436; iv+546. £3, 10 s.

It is about eighteen years ago that this writer published in this periodical (this BULLETIN, (2), vol. 15, pp. 386-391) a review of the first edition of this noteworthy example of English scholarship and power of exposition. The titlepage then gave the author as T. L. Heath, C. B., Sc. D.; it now records honors which this and other works upon the history of mathematics have brought to the author, and with the approval of the whole scientific world. As regards the general treatment of the subject, the significance of such a publication, and its influence upon modern education, little need be added to the review above mentioned. The only matter demanding special consideration at this time relates to the changes which characterize the new edition.

In general, the work is a reprint of the first impression, with such corrections and minor changes as are naturally desirable after such a lapse of years. It is a gratifying tribute to the scholarship of the author that the changes in the original text are so few, as also that the demand for the work has made this edition necessary. It should not be thought, however, that the text has not been thoroughly revised or that it fails to include the latest information relating to discoveries in the field considered. The care shown in the revision is seen in numerous changes in the foot-