A NEW CHARACTERIZATION OF PLANE CONTINUOUS CURVES*

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A number of authors† have given necessary and sufficient conditions that a bounded continuum be a continuous curve. However new conditions are always of interest as no one characterization applies without difficulty to all problems. It is the purpose of this paper to give a new necessary and sufficient condition that a bounded plane continuum be a continuous curve. Also this gives a condition under which a subcontinuum of a continuous curve is itself a continuous curve. Finally we prove a new property of continuous curves.

THEOREM I. *In order that a continuum N, which is a subset of a plane continuous curve M and such that $M-N$ consists of a finite number of maximal connected subsets*‡, *be a continuous curve, it is necessary and sufficient that if $P_1, P_2, P_3, \cdots$ is any sequence of distinct points of a maximal connected subset of $M-N$ which has a sequential limit point P, then there exists an increasing sequence of positive integers $n_1, n_2, n_3, \cdots$*

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† For definitions relating to and characterizations of continuous curves, see R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, this Bulletin, vol. 29 (1923), pp. 289–302. Hereafter we shall refer to this paper as *Report*. See also R. L. Wilder, *A property which characterizes continuous curves*, Proceedings of the National Academy, vol. 11 (1925), pp. 725–728; R. L. Moore, *A characterization of a continuous curve*, Fundamenta Mathematicae, vol. 7 (1925), pp. 302–7; H. M. Gehman, *Some conditions under which a continuum is a continuous curve*, Annals of Mathematics, vol. 27 (1926), pp. 381–4; R. L. Wilder, *A characterization of continuous curves by a property of their open subsets*, this Bulletin, vol. 32 (1926), p. 217.

‡ A point set K which is a subset of a point set M is said to be a *proper subset* of M if $M-K$ is not vacuous. A connected subset K of a point set M is said to be a *maximal connected subset* of M if K is not a proper subset of any connected subset of M.

and a set of arcs of $M-N$, $P_{n_1}P_{n_2}$, $P_{n_2}P_{n_3}$, $\cdots$, *such that the set* $P+\sum_{i=1}^{\infty} P_n\, P_{n_{i+1}}$ *is closed.*

PROOF. A. The condition is necessary. Let $P_1, P_2, P_3, \cdots$ be any sequence of points of a maximal connected subset D of $M-N$ which has a sequential limit point P. There are two cases to consider.

(a). If P is a point of $M-N$, D contains P and there exists a circle C_1 with center at P which encloses no point of N. We may suppose that for every i, $P_i \neq P$, for if any P_i were P we could drop this point from the sequence and consider the remainder. Since M is connected im kleinen, there exists a circle C_2 with center at P such that $r_2 \leqq r_1/2$, where r_i denotes the radius of C_i, and such that every point of M in the interior of C_2 can be joined to P by an arc* of M which lies wholly in the interior of C_1. Let n_1 be the smallest integer so that P_{n_1} is interior to C_2. In general there exists a circle C_{i+1} with center at P such that $r_{i+1} \leqq r_i/2$ and $P_{n_{i-1}}$ lies in the exterior of C_{i+1} and such that every point of M in the interior of C_{i+1} can be joined to P by an arc of M which lies wholly in the interior of C_i. Let n_i be the smallest integer such that P_{n_i} lies in the interior of C_{i+1} and let $P_{n_i}P$ denote the arc of M (actually of $M-N$) whose existence is shown above. For every i, the set $P_{n_i}P+P_{n_{i+1}}P$ contains an arc $P_{n_i}P_{n_{i+1}}$ from P_{n_i} to $P_{n_{i+1}}$. Since every arc $P_{n_i}P_{n_{i+1}}$ lies in the interior of the circle C_i and the numbers r_i approach 0 as i increases, the set $P+\sum_{i=1}^{\infty} P_{n_i}P_{n_{i+1}}$ is closed.

(b). If P is a point of N, let C_1 be a circle with center at P and radius r so small that N and D contain points exterior to C_1. This is possible unless N is identical with M and in this case our theorem is obvious. Let $D_{11}, D_{12}, D_{13}, \cdots$ be the maximal connected subsets of $D \cdot I(C_1)$.†

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* That this can be done by an *arc*, see J. R. Kline, *Concerning the approachability of simple closed and open curves*, Transactions of this Society, vol. 21 (1920), page 453 and footnote.

† If C is a circle, $I(C)$ denotes the interior of C. If A and B are point sets, $A \cdot B$ denotes the set of points common to A and B.

We shall show that one of these sets, which we will denote by D_1, contains infinitely many of the points P_i. If this is not true, then if C_2 denotes the circle with center at P radius $r/2$, for infinitely many values of i, D_{1i} has a point within C_2 and C_1 contains a limit point of D_{1i}. Thus infinitely many of the sets D_{1i} are of diameter greater than $r/4$. But this contradicts the theorem that if $M+C_1$ and $N+C_1$ are continuous curves and $N+C_1$ is a subset of $M+C_1$ then $M+C_1-(N+C_1)$ cannot contain more than a finite number of maximal connected subsets of diameter greater than $r_1/4$.*

Let n_1 be the smallest integer such that D_1 contains P_{n_1}. Similarly one, D_2, of the maximal connected subsets of $D_1 \cdot I(C_2)$ contains infinitely many of the points P_i. Let n_2 be the smallest integer greater than n_1 such that D_2 contains P_{n_2}. In general let $C_j(j=1, 2, 3, \cdots)$ be a circle with center at P and radius r/j and let D_j be a maximal connected subset of $D_{j-1} \cdot I(C_j)$ which contains infinitely many points of the sequence $[P_i]$. Let n_j be the smallest integer greater than n_{j-1} such that P_{n_j} lies in D_j. For every j, D_j contains an arc $P_{n_j}P_{n+j_1}$.† Since for every j, the arc $P_{n_j}P_{n_{j+1}}$ lies interior to C_j we see easily that the set $P+\sum_{j=1}^{\infty} P_{n_j}P_{n_{j+1}}$ is closed.

B. The condition is sufficient. If N is not a continuous curve there exist‡ two concentric circles K_1 and K_2 and a countable infinity of continua $\overline{N}$, N_1, N_2, N_3, $\cdots$, such that (1) each of these continua belongs to N, contains a point on K_1 and a point on K_2 and is a subset of the set H which is composed of K_1+K_2+I, I denoting the annular domain between K_1 and K_2, (2) no two of these continua have a point in common and, indeed, no one of them except possibly $\overline{N}$ is a proper subset of any connected subset of $N \cdot H$,

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* See the abstract of my paper, *Concerning the arcs and domains of a continuous curve*, this Bulletin, vol. 32 (1926), p. 37.

† See R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260.

‡ See *Report*, p. 296.

(3) the set $\overline{N}$ is the sequential limiting set of the sequence of sets $N_1, N_2, N_3, \cdots$. For each i, let A_i and B_i be points of $K_1 \cdot N_i$ and $K_2 \cdot N_i$ respectively. There exist arcs X_1Y_1A and X_2Y_2B of K_1 and K_2 and an increasing sequence of integers $n_1, n_2, n_3, \cdots$, such that X_1Y_1A contains A_{n_i} for every i and in the order $X_1Y_1A_{n_1}A_{n_2} \cdots A$ and X_2Y_2B contains B_{n_i} for every i and in the order $X_2Y_2B_{n_1}B_{n_2} \cdots B$.

Let P denote a point of $\overline{N}$ which lies in I. There exists a circle C_1 with center at P such that C_1, together with its interior, lies in I. Let r_1 be the radius of C_1. Since M is connected im kleinen at P there exists in any circle C_i a concentric circle $\overline{C}_i$ such that every point of M within $\overline{C}_i$ can be joined to P by an arc of M lying wholly within C_i. Let $N_{11} \equiv N_{n_j}$, where j has the smallest value such that N_{n_j} contains a point Q_1 within $\overline{C}_1$. There exists an arc PQ_1 of M lying wholly in C_1. The arc PQ_1 from P to Q_1 contains a first point E_1 in common with N_{11} and the arc E_1P, a subset of Q_1P, has a first point F_1 in common with $\overline{N}$. The set $\{E_1F_1\}$* contains a point P_1 of $M-N$. Let C_2 be a circle with center at P and radius $r_2 \leqq r_1/2$ such that P_1 and N_{11} lie in the exterior of C_2. Let $N_{12} \equiv N_{n_j}$, where j has the smallest value such that N_{n_j} contains a point Q_2 within $\overline{C}_2$. Let us determine a point P_2 of $M-N$ as above. Continue this process indefinitely each time taking C_i with center at P and radius $r_i \leqq r_{i-1}/2$ and such that P_{i-1} and N_{1i-1} lie outside C_i. Thus we obtain an infinite sequence of points $P_1, P_2, P_3, \cdots$, and continua $N_{11}, N_{12}, N_{13}, \cdots$, such that (1) P_i belongs to $M-N$ and lies interior to C_i and thus P is the sequential limit point of the sequence $[P_i]$, (2) $\{E_iF_i\}$ contains P_i, where C_i encloses E_iF_i, and $\{E_iF_i\}$ contains no point of $N_{1i}+\overline{N}$.

Since $M-N$ consists of only a finite number of maximal connected subsets one of these must contain infinitely many of the points $[P_i]$ say $\overline{P}_1, \overline{P}_2, \overline{P}_3, \cdots$. For each i, let D_i be the maximal connected subset of $M+K_1+K_2-(\overline{N}+\overline{N}_{1i}$

* If AB is an arc from A to B then $\{AB\}$ denotes $AB-(A+B)$.

$+K_1+K_2$)* which contains $\overline{P}_i$. We see easily that there exists an integer t_2 such that $\overline{P}_1$ does not lie in D_{t_2}. Then any arc of $M-N$ from $\overline{P}_{t_1}$ ($t_1=1$) to $\overline{P}_{t_2}$† must contain a point of either K_1 or K_2. There exists an integer $t_3>t_2$ such that D_{t_3} does not contain $\overline{P}_{t_2}$. In general there exists an integer $t_i>t_{i-1}$ such that D_{t_i} does not contain $\overline{P}_{t_{i-1}}$ and thus any arc of $M-N$ from $\overline{P}_{t_{i-1}}$ to $\overline{P}_{t_i}$ must contain a point of K_1 or K_2. Let $p_i=\overline{P}_{t_i}$. Then if $k_1, k_2, \cdots$ is any increasing sequence of positive integers, the set $\overline{N}$ must contain a limit point of the set $P+\sum_{i=1}^{\infty} p_{k_i}\, p_{k_{i+1}}$ which lies on K_1 or K_2 and thus the set cannot be closed. But this set is closed by hypothesis. Thus the condition is sufficient.

THEOREM II. *In order that a bounded plane continuum M be a continuous curve, it is necessary and sufficient that* (1) *for any given positive number ϵ there are not more than a finite number of complementary domains of M of diameter greater than ϵ*; (2) *if $P_1, P_2, P_3, \cdots$ is any sequence of distinct points of a complementary domain D of M which has a sequential limit point P, then there exists an increasing sequence of positive integers, $n_1, n_2, n_3, \cdots$, and a sequence of arcs of D, $P_{n_1}P_{n_2}, P_{n_2}P_{n_3}, P_{n_3}P_{n_4}, \cdots$, such that the set $P+\sum_{i=1}^{\infty} P_{n_i}\, P_{n_{i+1}}$ is closed.*

PROOF. The necessity of condition (1) has been proved by Schoenflies.‡ The necessity of condition (2) can be proved exactly as in Theorem I since no property of the continuous curve M was used that is not also a property of the entire space. The sufficiency of the conditions is proved as in Theorem I except that the fact that some one complementary domain of M contains infinitely many of the points $P_1, P_2, P_3, \cdots$, which are chosen in the course of the argument, follows from condition (1) rather than the condition *$M-N$ consists of a finite number of maximal connected subsets.*

* If $\overline{P}_i=P_j$, then $\overline{N}_{1i}$ denotes N_{1j}.

† For the proof that such an arc exists, see R. L. Moore, *Concerning continuous curves in the plane,* loc. cit.

‡ See *Report*, pp. 290, 291.

THEOREM III. *If M is a plane continuous curve then M cannot contain, for any positive number ϵ, an infinite number of mutually exclusive continua $M_1, M_2, M_3, \cdots$, such that* (1) *the diameter of each set M_i is greater than ϵ,* (2) *$M-M_i$ is closed except for a set K_i and if η is any positive number there exists an integer n_η so that if $i>n_\eta$ then K_i can be enclosed in two circles each of radius less than η.*

PROOF. Suppose that there exists a positive number ϵ and a continuous curve M such that M contains an infinite number of continua $M_1, M_2, M_3, \cdots$, which satisfy restrictions (1) and (2) of the theorem. From condition (2) it follows that we may divide each set K_i into two subsets K_{1i} and K_{2i} such that

$$\lim_{i\to\infty} d(K_{1i}) = 0 \text{ and } \lim_{i\to\infty} d(K_{2i}) = 0.*$$

For each i and j ($i=1, 2, 3, \cdots, j=1, 2$) let A_{ji} be a point of K_{ji}, unless K_{ji} is vacuous. For no value of i can both K_{1i} and K_{2i} be vacuous. Several cases arise here according to the existence or non-existence of the various points A_{ji} but we can see easily that there exist a point A or two points A and B and an increasing sequence of integers $n_1, n_2, n_3, \cdots$, such that either (1) K_{1n_i} is vacuous for each i, and A is the sequential limit point of $[A_{2n_i}]$, (2) K_{2n_i} is vacuous for each i and A is the sequential limit point of $[A_{1n_i}]$, (3) all of the points of the sequences $[A_{1n_i}]$ and $[A_{2n_i}]$ exist and A is the sequential limit point of each sequence, or (4) all of the points of the sequences $[A_{1n_i}]$ and $[A_{2n_i}]$ exist and A and B are the sequential limit points of the sequences $[A_{1n_i}]$ and $[A_{2n_i}]$ respectively ($A \neq B$). For cases (1), (2) and (3), let $t=\epsilon$; for case (4) let $t=d(A, B)$. By condition (2) of the hypothesis of the theorem, there exists an integer k_1 so that if $i>k_1$ then

$$d(K_{1n_i})<t/12 \quad \text{and} \quad d(K_{2n_i})<t/12.$$

* If K is a set of points the notation $d(K)$ denotes the diameter of K. If A and B are two points the notation $d(A, B)$ denotes the distance from A to B.

Also there exists an integer k_2 so that if $i>k_2$ then either

Case (1) $d(A_{2n_i}, A)<t/12$,

or

Case (2) $d(A_{1n_i}, A)<t/12$,

or

Case (3) $d(A_{1n_i}, A)<t/12$ and $d(A_{2n_i}, A)<t/12$,

or

Case (4) $d(A_{1n_i}, A)<t/12$ and $d(A_{2n_i}, B)<t/12$.

In any case if $k_3=k_1+k_2$ and $i>k_3$ then the circle C_1 with center at A and radius $t/6$, or the circles C_1 and C_2 with centers at A and B and radii $t/6$, enclose every point of K_{n_i}. For every $i>k_3$, M_{n_i} contains a point p_i such that $d(A, p_i)>t/3$ and $d(B, p_i)>t/3$ (if B exists). The sequence M_{n_1}, M_{n_2}, M_{n_3}, $\cdots$, contains a subsequence $\overline{M}_1$, $\overline{M}_2$, $\overline{M}_3$, $\cdots$, such that (1) for every i, if $\overline{M}_i=M_{n_j}$ then $j>k_3$, (2) for every i, if $\overline{M}_i=M_{n_j}$ and $\overline{M}_{i+1}=M_{nm}$ then $j<m$, (3) the points $\bar{p}_1$, $\bar{p}_2$, $\bar{p}_3$, $\cdots$ * have a sequential limit point P. It follows that M contains P, that $d(P, A)\geqq t/3$ and $d(P, B)\geqq t/3$ (if B exists) and that if C_3 is a circle of radius $t/6$ with P as a center then no point of any set $\overline{K}_i$ is within C_3. As M is connected im kleinen at P the circle C_3 encloses a concentric circle C_4 such that every point of M within C_4 can be joined to P by an arc of M which lies entirely in C_3. Let $\bar{p}_s$ be the first point of the sequence $[\bar{p}_i]$ within the circle C_4. There exists an arc α from $\bar{p}_s$ to P which lies wholly in C_3. Let $\alpha_1=\overline{M}_s\cdot\alpha$ and $\alpha_2=\alpha-\alpha_1$. As α is connected one of these sets must contain a limit point of the other. The set $\overline{M}$, and consequently α_1, is closed. Then α_1 must contain a limit point q of α_2. As $\overline{M}_s$ contains α_1 and $M-\overline{M}_s$ contains α_2, by definition q must belong to $\overline{K}_s$. But no point of $\overline{K}_s$ is within C_3 while α is entirely within C_3. Thus the supposition that M contains an infinite set of this type has led to a contradiction.

The preceding theorem implies as an immediate corollary the following rather useful result.

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* If $\overline{M}_i=M_{n_j}$, then $\overline{P}_i$ denotes p_{n_j}.

THEOREM IV.* *If M is a plane continuous curve then M cannot contain, for any positive number ϵ, an infinite number of arcs of diameter greater than ϵ which are mutually exclusive except possibly for common end-points and such that if α is any one of this set of arcs then $M - \{\alpha\}$ is closed.*

That Theorem I no longer remains true when the condition that "$M-N$ consists of a finite number of maximal connected subsets" is removed, even with the addition of the condition that "for any positive number ϵ, $M-N$ contains only a finite number of maximal connected subsets of diameter greater than ϵ," is shown by the following example. The modified conditions are necessary but not sufficient.

Let N denote the set of points consisting of the intervals from (1, 0) to (0, 0) and from (0, 1) to (0, 0) together with the intervals from $(1, 1/i)$ to $(0, 1/i)$ for every positive integer i. Let M be the set of points consisting of N together with the intervals from $(j/i, 1/i)$ to $(j/i, 0)$ for every positive integer $j<i$ and for every positive integer i. The modified conditions are then satisfied, but N is not a continuous curve.

Theorem III gives a necessary condition that a bounded continuum be a continuous curve. The following example shows that this condition is not sufficient:† Let M be an indecomposable continuum of diameter $\geqq 2\epsilon$ and let $\eta < \epsilon/10$. Now suppose that $M_1, M_2, M_3, \cdots$ is any sequence of mutually exclusive subcontinua of M of diameter greater than ϵ. As each set M_i is a proper subcontinuum of an indecomposable continuum it is a continuum of condensation of M.‡ Thus for each i, $K_i \equiv M_i$. Then no matter how large i is, K_i cannot be enclosed in two circles each of radius less than η. Thus M satisfies the condition but is not a continuous curve.

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* This theorem was presented to the Society October 31, 1925. I am indebted to Dr. H. M. Gehman for the suggestion that this theorem might be generalized. The resulting study led to Theorem III of this paper.

† This example is due to Professor J. R. Kline.

‡ Cf. Z. Janiszewski and C. Kuratowski, *Sur les continus indécomposables*, Fundamenta Mathematicae, vol. 1 (1920), pp. 210–222.