A CURIOUS IRREDUCIBLE CONTINUUM

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One of Janiszewski's theorems* is to the effect that, if ab is a bounded irreducible continuum and c is a point of the first species, there is a decomposition of ab into two continua, ac and cb, such that $ac \cdot cb = c$. The fact that the condition imposed on c is not a necessary one naturally led to the question as to whether an irreducible continuum not made up of indecomposable continua can always be decomposed into two continua having only one point in common. A vain endeavor to answer this question in the affirmative resulted in the following example to the contrary, which may be of interest to workers in this field.

Although the continuum is simple, it is difficult to describe without a figure and therefore the construction should be carried out through the first two stages.

The first stage in the construction is as follows. Take a unit square in the first quadrant and mark the following points: $a=(0, 0)$, $b=(1/4, 0)$, $c=(1/2, 0)$, $d=(3/4, 0)$, $e=(1, 0)$, $f=(1/2, 1/4)$, $g=(3/4, 1/4)$, $h=(1/4, 3/4)$, $i=(1/2, 3/4)$, $j=(0, 1)$, $k=(1/4, 1)$, $l=(1/2, 1)$, $m=(3/4, 1)$, $n=(1, 1)$. Draw the straight lines bk, cf, fh, gi, il, and dm. Set $E_1=bk+cf+fh$ and $E_2=gi+il+dm$. The continuum E_1 may be regarded as the union of two continua, bk and $cf+fh+hk$, extending from ae to jn and having a common segment hk of length $1/4$. A similar statement holds for E_2. The continua E_1 and E_2 divide the square into five "strips," of which two, $bcfh$ and $ligm$, meet only one side of the square and are called *incomplete*. The other three extend from ae to jn and are called *complete*; two of them are rectangles, $R_1=abkj$ and $R_2=denm$, and the third is a polygon $P_1=$

* Z. Janiszewski, *Sur les continus irréductibles entre deux points*, JOURNAL DE L'ÉCOLE POLYTECHNIQUE, (2), vol. 16 (1926), p. 125.

cdgilkhf; all three have a uniform width of 1/4 measured parallel to the x-axis.

The first step in the second stage is to repeat this construction in the rectangle R_1 with certain modifications. The points corresponding to b, c, d, etc., are: $b' = (1/16, 0)$, $c' = (1/8, 0)$, $d' = (3/16, 0)$, $f' = (1/8, 1/8)$, $g' = (3/16, 1/8)$, $h' = (1/16, 7/8)$, $i' = (1/8, 7/8)$, $k' = (1/16, 1)$, $l' = (1/8, 1)$, $m' = (3/16, 1)$. The essential thing to notice is that the segments $c'f'$, $d'g'$, $h'k'$, and $i'l'$ each have a length equal to one-half the length of the corresponding segments cf, dg, hk, and il. We now have two new continua E_3 and E_4 which have the same properties as E_1 and E_2 except that the common segments $h'k'$ and $d'g'$ have a length of $\frac{1}{2}hk = \frac{1}{8}$. They divide R_1 into five strips, of which three are complete and have a uniform width of 1/16.

If each point (x, y) of E_3 and E_4 is moved to the position (x', y') defined by $x' = x + 1/4$ if $y \geqq 3/4$, $x' = x + 1/4 + \frac{1}{2}(\frac{3}{4} - y)$ if $1/4 \leqq y \leqq 3/4$, $x' = x + 1/2$ if $y \leqq 1/4$, and $y' = y$, the images E_5 and E_6 of E_3 and E_4 will be two continua dividing P_1 exactly as E_3 and E_4 divide R_1. Likewise the transformation $x' = x + 3/4$, $y' = y$ gives two continua E_7 and E_8 dividing R_2. Each of the continua E_5, E_6, E_7, and E_8 is the union of two lines or broken lines extending from ae to jn and having a common segment of length 1/8. We now have eight continua, $E_1, E_2, \cdots, E_8$, which define nine complete strips, each of width $1/4 \times 1/4 = 1/16$.

Now repeat the construction of the second stage in these nine strips, again making the segments corresponding to $c'f'$, $d'g'$, etc., one-half as long as before. This will give us 18 more continua, $E_9, E_{10}, \cdots, E_{26}$, each of which is the union of two lines or broken lines extending from ae to jn and having a common segment of length $1/2 \times 1/8 = 1/16$. These divide the nine complete strips of the second stage into 18 incomplete strips and 27 complete strips, each of the latter having a width of $1/4 \times 1/16 = 1/64$.

If this process is carried on indefinitely, the width of the complete strips converges to zero and we get an enumerable set of continua E_n of the type described above, whose common segments have a length converging to zero as n increases. If B is the sum of the sets E_n, $\bar{B}$ is the desired continuum C. The improper limiting points of B constitute a non-enumerable set of simple arcs joining ae and jn. Thus C is the sum of an infinity of continua joining ae and jn, which we may call the *elements* of C. We distinguish these by calling the sets E_n elements of the first class, and the simple arcs elements of the second class.

That C is a continuum and is irreducible between any point α of aj and any point β of en is easily deduced from well known theorems. The proof that C cannot be expressed as the union of two continua C_1 and C_2, which have but a single point in common, is briefly as follows. Neither C_1 nor C_2 contains points of both aj and en; let aj be a part of C_1 and let en be a part of C_2. Let x be a point of $C_1 \cdot C_2$ and lie on an element F of C. Then C_1 contains all the elements at the left of F and C_2 contains all those at the right; let C_1^* denote the sum of the first set and C_2^* that of the second. If F is one of the elements E_n, one of the two continua forming E_n lies in $\bar{C}_1^*$ and the other in $\bar{C}_2^*$. But these have a common segment of length greater than zero; hence $C_1 \cdot C_2$ contains a continuum which is not a point. If F is an element of the second class, F lies in both $\bar{C}_1^*$ and $\bar{C}_2^*$, and hence in $C_1 \cdot C_2$. Thus C cannot be expressed as the union of two continua which do not have at least one continuum which is not a point in common.

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