THE TRANSFORMATION OF A REGULAR GROUP INTO ITS CONJOINT

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The main purposes of the present article are to correct an assumption which appeared in a former article with the same title,* and to exhibit in a definite manner a method of finding a particular substitution of order 2 which transforms a given regular substitution group into its conjoint. It is well known that a regular group R_1 which is simply isomorphic with an arbitrary finite group G whose operators are 1, s_2, s_3, $\cdots$, s_g can be obtained by letting $s_\alpha(\alpha=2, 3, \cdots, g)$ correspond to the permutation of the operators of G when all of these operators are multiplied on the right by s_α. Another simply isomorphic regular group R_2 may be obtained by letting s_α correspond to the permutation of these operators when all of them are multiplied on the left of s_α^{-1}. From the associative law it results that every substitution of R_1 is commutative with every substitution of R_2, and hence R_1 and R_2 are the conjoint of each other.†

From the preceding paragraph it results that any regular group and its conjoint can be made simply isomorphic as follows: Make all the cycles involving a particular letter of the regular group correspond to their inverses in the conjoint group, and adjoin to each such inverse its conjugates under the given regular group. A necessary and sufficient condition that a substitution corresponds to its inverse in this simple isomorphism is that it is invariant under the given regular group. In particular, all the substitutions of this group will thus correspond to their inverses when the

* This BULLETIN, vol. 25 (1919), p. 326.

† W. Burnside, *The Theory of Groups*, 1911, p. 88.

group is abelian, and only then. It is important to observe that a substitution t of order 2 which transforms the given regular group into its conjoint can be obtained by letting the common letter of the said inverse cycles correspond to itself and noting the permutation of the remaining letters involved in these corresponding cycles. The number of letters omitted by t is just one larger than the number of substitutions of order 2 found in the regular group.

In the article to which reference was made in the first paragraph it is assumed, page 328, that in some simple isomorphism between every regular group and its conjoint every two corresponding substitutions involve a common cycle. From the results of the preceding paragraph it is easy to see that it is sometimes impossible to establish such a simple isomorphism between these groups. For instance, when R_1 is the non-cyclic group of order 21 it is obvious that the cycles of order 3 found in a co-set with respect to the subgroup of order 7 include all their conjugates under the group, and that one of these co-sets cannot be transformed into the other by an operator of the holomorph of the group. Hence it results that no two corresponding substitutions of order 3 in a simple isomorphism between this group and its conjoint can involve a common cycle. On the other hand it has been proved that *a simple isomorphism between a regular group and its conjoint can always be established in such a manner that at least two cycles in every pair of corresponding substitutions are inverses of each other.*

The substitution t noted above which transforms R_1 and R_2 into each other is commutative with every substitution of the subgroup composed of all the substitutions of the double holomorph of R_1 (when R_1 is non-abelian) which omit the letter common to the cycles of R_1 which were made to correspond to their inverses in the given simple isomorphism between R_1 and R_2. This results from the facts that t transforms each of these cycles into its inverse and the totality of these cycles is transformed into itself by all the substitutions of the subgroup in question. In particular, we have established the theorem that *every regular group can be transformed into its conjoint by an operator of order 2 which is commutative with every operator of the group of isomorphisms of this regular group.* When R_1 is non-abelian its double holomorph is composed of all the substitutions on the letters of R_1 which transform R_1 either into itself or into R_2. The subgroup composed of all the substitutions of this double holomorph which omit a given letter thereof is therefore the direct product of the group of isomorphisms of R_1 and a substitution of order 2.

The number of letters in the cycles which correspond to their inverses in a particular pair of corresponding substitutions in the simple isomorphism between R_1 and R_2 obtained by transforming R_1 by t is equal to the number of the substitutions of R_1 which are commutative with the substitution of R_1 involving these cycles. In particular, all of these cycles appear in the same substitutions of R_1. In the special case when R_1 is a dihedral group whose order is twice an odd number only one cycle of order 2 corresponds to its inverse in such an isomorphism, while two such cycles correspond to their inverses in the non-invariant substitutions of order 2 when the order of the dihedral group is divisible by 4 and exceeds 4. In each of these cases t is of a smaller degree than any substitution besides the identity found in the holomorph of the dihedral group in question. A necessary and sufficient condition that t omits only one letter of R_1 is that the order of R_1 is odd.

When the group G contains no invariant operator besides the identity the substitution t may be exhibited as follows: Represent G as a substitution group and write it similarly on two distinct sets of letters. Extend the direct product of the conjugate groups thus obtained by a substitution s of order 2 which merely interchanges their corresponding letters. The subgroup of index g formed by extending by means of s the simple isomorphism between these two groups, in which all substitutions are commutative with s, involves no invariant subgroup of the entire group besides the identity and is not invariant. Hence the entire group can be represented as a transitive group of degree g with respect to this subgroup. When it is thus represented the factors of the said direct product will be a regular group and its conjoint respectively, and s will obviously be the substitution t as regards these groups. This process exhibits also clearly some of the properties of a group and its conjoint when the group involves no invariant operator besides the identity.

When G is a complete group this process clearly gives the double holomorph of G. Hence it throws light on the difficult problem of constructing a complete group of odd order. The subgroup composed of all the substitutions which omit one letter of the holomorph of such a group must be of degree $g-1$ and its transitive constituents can be combined into pairs which are transformed into each other by t. Moreover, *every substitution which transforms a regular complete group of odd order g into its conjoint is of degree $g-1$.* In particular, such a group cannot be made simply isomorphic with its conjoint in such a way that some two corresponding substitutions involve a common cycle.

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