THE SECOND EDITION OF THE HURWITZ-COURANT FUNKTIONENTHEORIE

Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen von Adolf Hurwitz, herausgegeben und ergänzt durch einen Abschnitt über geometrische Funktionentheorie von R. Courant. Zweite vollständig umgearbeitete und vermehrte Auflage. Berlin, Julius Springer, 1925. xii+496 pp. Vol. III of the series Die Grundlagen der Mathematischen Wissenschaften in Einzeldarstellungen.

The first edition* of this book consisting of 1500 copies, was exhausted in two years. In spite of general economic conditions which greatly inflated book sales in Germany, this fact is evidence of a very favorable reception.

As is indicated in the title, the book consists of three parts, Part I dealing with the general theory of analytic functions, mainly from the point of view of Weierstrass, Part II, with elliptic functions, and Part III, with the theory of functions from the geometric point of view. The first two parts are based upon a manuscript and upon lecture notes of Hurwitz, and from the start have left little to be desired in arrangement and clarity. Accordingly, it was found that in the new edition but little change was desirable. Such alterations as have been made consist of improvements of typography, in the figures, in a reduction of a somewhat superfluous use of italics and numbers for formulas, and in occasional changes in the form of presentation. All of these bear witness to a detailed revision, and, in general, render the work still more readable. Two additions of several pages each have been made, namely, a treatment of the Gamma function, and some examples of the Lagrange series for a function of an implicitly given function. Both because of their inherent interest, and their helpfulness in the comprehension of the theories they illustrate, these additions will be welcome.

Part III, the geometric theory of functions, is consistently faithful to the Riemann point of view. It carries the reader from the discussion of conformality and the elementary functions to such questions as abelian integrals and the existence of algebraic functions corresponding to a given algebraic Riemann surface, the existence of automorphic functions with given fundamental region, the mapping of multiply connected regions, and the problems of uniformization. The keynote of the whole treatment is the use of the Dirichlet integral which appears in the basic existence theorems.

---

* The following reviews appeared: W. Wirtinger, MONATSHEFTE, vol. 33 (1923), p. 13; Rolin Wavre, ENSEIGNEMENT, vol. 23 (1923), p. 118; O. D. Kellogg, this BULLETIN, vol. 29 (1923), pp. 415-417; L. Bieberbach, JAHRESBERICHT DER VEREINIGUNG, vol. 32 (1924), pp. 54-55; H. Kneser, GÖTTINGISCHE ANZEIGEN, vol. 187 (1925), pp. 87-88.

The task of making an exposition of these subjects modern, readable, and fairly compact, was by no means an easy one. Much of the material was to be found only in original memoirs, and its reduction to a form adapted to the student's needs was here attempted for the first time. It is therefore not surprising that in the interim between the two editions, numerous ideas for improving the treatment came to the author As a result, Part III has been completely recast, for the most part rewritten, and considerably extended.

Specific difficulties which were encountered in the first edition have not, as a rule, been patched up, but eliminated by completely new modes of attack. A great improvement has been made in the general mapping theorem of chapter eight. In the new edition, all difficulties connected with the boundary have been separated out by the expedient of first treating only regions bounded by a finite number of circular arcs, and then generalizing by considering any region as the limit of an expanding sequence of regions of the simpler type. This procedure is, in fact, typical of the skill with which the author resolves his more complicated theorems into simpler unitary parts, in such a way that the reader may follow the argument with more ease and without loss of perspective.

The material which follows the introductory portion is grouped about two fundamental mapping theorems. The first is the theorem of Riemann* to the effect that the general simply connected open region with more than one boundary point can be mapped on the interior of the unit circle. To this is attached a study of the functions which map straight line and circular arc polygons on the upper half-plane. The second theorem, although present in embryo in analytic form in Riemann's theory of abelian functions,† was first enunciated in its full generality as a mapping theorem by Hilbert,‡ with all the essential elements of the proof, and states that the general region of planar character can be mapped on a slit-region (Schlitzbereich), i.e. the whole plane except for boundary points, with the property that each connected subset consists of a straight line segment parallel to the axis of reals. To the results and methods of this theorem are attached the further topics mentioned above, including a deeper penetration into the automorphic functions.

In the first edition, the mapping of polygons was discussed as illustrating the Riemann theorem, the establishment of the theorem being incidental to a more general theorem coming considerably later (p 378). In giving it more prominence, the second edition makes a distinct improvement, for both didactic and historical reasons. Two proofs are

---

* WERKE, 2d ed., Leipzig, Teubner, 1892, p. 40.

† Loc. cit., p. 100.

‡ *Zur Theorie der konformen Abbildung*, GÖTTINGER NACHRICHTEN, 1909, pp. 314-323. The last three references are supplied by the reviewer. No writings of Hilbert are cited in the book. The notes on the literature, although sufficiently copious, often have a tendency to vagueness, and would be more satisfactory if they assigned to their authors more systematically credit for specific results.

supplied. The first, based on methods due to Koebe and Carathéodory, gives a particular satisfaction, since it proves a theorem in the geometry of analytic functions purely by the use of geometry and the immediate properties of such functions, and without the usual appeal to real harmonic functions. It is doubtful whether an essentially more elegant presentation could be given. The second proof, while based upon the classic foundation of Schwarz' alternating process, is by no means lacking in more modern elements. These are due to the expedient above referred to of treating the general region as the limit of a set of simply bounded regions, and come to light in the convergence proof.

The skill in the arrangement of the proof of the Hilbert theorem has already been alluded to. As a very general existence theorem, it is inevitable that its establishment involves a considerable mass of detail. The author has given it much thought, extending over a period of thirteen years or more (see the footnote on page 491), and it is difficult to suggest any essential improvement in the admirable treatment now before us.

Space is lacking for a characterization of the work in detail. The field of geometric function theory is so broad that it is not surprising that a reviewer of the first edition regretted omissions. The thing which is striking is the number of topics which do find a place in the comparatively brief space of 233 pages. This is made possible, in part, by the willingness of the author to forego, in certain places, particularly in the last chapter, complete generality or complete detail in proof. The effect of this policy is certainly to enhance the interest by reducing tedium and allowing greater breadth of treatment. Its danger lies in a possible obscuring of the importance or difficulty of the details.

But for the most part, and in all the basic work, sufficiently complete proofs for the thoughtful reader are given. While the reasoning is close, and the results valid, it need not be surprising that in places there are difficulties which will trouble the student, or even throw him off the track. Some of these he should be on guard against, and instances follow.

One occurs in Part I. The theorem on page 101, "A function which is regular in the neighborhood of a point a, and whose modulus is bounded there, is regular at this point itself," ignores the possibility of a removable singularity. The proof that to a harmonic function there always corresponds a conjugate function is omitted in the new edition, although the fact is needed repeatedly.

There is trouble on page 310. Here $\beta_1(s)$, $\beta_2(s)$, ... are continuous functions of s, and it is stated that $\int_{-\epsilon}^{+\epsilon}\beta_4(s)ds$ vanishes with ϵ. This looks innocent, and therein lies the danger, for $\beta_4(s)$ is also a function of y, which in turn, depends on ϵ. It is true that for sufficiently small ϵ, $|\beta_4(s)|$ has a bound independent of y, so that the result is correct. But real care is required to verify the fact, and a difficulty of this sort should not be masked.

The high general level of clarity of the book is only occasionally impaired; when it is, the obscurity is correspondingly great. An example is the treatment of the source-free flow, on page 312. The author here attempts the difficult task of explaining such a flow without contrasting examples or discussion of a flow with sources. He bases his concept of freedom from sources on the time rate of change of quantity of fluid within any subregion of the region G in which the flow is defined, although this rate must *always* vanish in the stationary flow of fluid with constant density under discussion, whether the flow be source-free or not. He takes over twice the space to say what could better be said somewhat as follows: in a source-free flow, the time rate of flow across the boundary of any subregion of G vanishes; this rate is measured by the integral $\int v_n ds$ where C is the contour of the subregion and v_n the component of the velocity in the direction of the normal; hence the integral vanishes for every such contour.

The student not previously informed is likely to have trouble with the matter of singularities of analytic functions. Except for the absence of mention of removable singularities, Part I (p. 57ff.) would give satisfactory ideas of the singularities of one-valued functions. But in Part III occur a number of statements apt to cause confusion. Thus, on page 316, the function of $f(z)=\sqrt[n]{z}$ is said to have a singular point at $z=0$, a statement apparently at variance with the definition at the end of page 358. The footnote on the latter page would justify the reader in inferring that $\log z$ had a pole at the origin.

The compromise attempted for the concept of Riemann surface between a critical one and an elementary one, cannot be described as successful. According to the definition (page 352, with later extensions), the Riemann surface belonging to $f(z)$ is made up of "points," which consist each of a value of z, and an element of the function $f(z)$. The Riemann surface is thus inseparable from the function to which it belongs, so that the theorems on the existence of functions with preassigned Riemann surfaces lose their meaning. To be sure, as the author says (p. 359), Riemann surfaces can be defined "purely geometrically"; so the proofs of the theorems do have a meaning, even if based on a concept whose definition has not been finally given. If, for didactic reasons, it seems advisable to employ a concept which is not entirely adequate, why does not sound pedagogy dictate using a simple one, like the old notion of the plane of the variable z with various duplicates, passing one into another in specified ways between branch points, where they coalesce?

This edition, like its predecessor, attains a high standard of typographical excellence and accuracy. Misprints are few. On page 308, in the last three lines, there are two places where s_0 should appear in place of s. The formula for the linear transformation corresponding to a rotation of the sphere which is stereographically projected onto the plane (p. 330) needs attention; so does the expression for S_2 in figure 119, page 461.

It will be observed that such objections as have been mentioned above will not be serious for the reader who is aware of their existence, and it is questionable whether a student should attempt the reading of Part III without a fair degree of maturity. Doubtless the purpose of combining this material with the two earlier parts was largely to give access to the necessary training. If the reader comes to his task sufficiently equipped, there is little doubt that he will find this presentation of the geometric function theory most stimulating, and that he will be eager to pursue the subject farther. And this is one of the highest ends a text book can attain.

O. D. KELLOGG

BRILL'S LECTURES ON ALGEBRAIC CURVES

Vorlesungen über ebene algebraische Kurven und algebraische Funktionen. By Alexander Brill. Braunschweig, Vieweg, 1925. x+340 pp. R. M. 17.50.

The book under review embodies in final form the course of lectures which, for many years, Brill has given at the University of Tübingen. As is well known Brill together with Noether are the twin-stars of German geometricians who did the important pioneer work concerning the geometry on algebraic curves.

One may therefore expect that a treatise on the subject by such a man should contain much that is of value and of fundamental importance for the student of geometry and, in a wider sense, for the mathematician in general.

It is true that the lectures are intended for the beginner, i.e., in American terminology, for the first and possibly second year graduate student. In other words, the student will have a fairly good start in algebraic geometry after he has mastered Brill's lectures.

It is obvious that Brill's purpose does not aim at the comprehensiveness and extensive vistas of Enriques' beautiful *Lezioni* or Severi's penetrating *Vorlesungen.* Brill is anxious to stress the function theoretic aspect of the subject more or less in the Weierstrassian spirit, and expects much from such a systematic treatment of the fundamental ideas of algebraic geometry. Thus, referring to the results of the Italian school and in particular to appendices F and G of Severi's *Vorlesungen,* for which Brill has written such a sympathetic introductory preface, Brill expresses the opinion that an attempt to put Severi's method of proof in algebraic form might turn out to be worth while.

Probably the majority of academic teachers who in the course of years repeatedly lecture on the same topic find it advisable to make progressive changes in the choice of the subject matter and its method of presentation. This is of course as it should be if the teacher keeps track of recent advances, in short, if he is up to date.

Brill states that in former years he put more stress upon the projective point of view, while in later years he returned more and more to the standpoint of the first discoverers in this field, of Descartes, Newton, Cramer, Euler, in so far as the graphical or geometric form relations (gestaltlichen Verhältnisse) are concerned. The reviewer may be prejudiced, but he nevertheless regrets that the projective point of view should have been relegated to second or third place.