THREE REPRINTS FROM THE ENCYKLOPÄDIE

Neuere Untersuchungen über Funktionen reeller Veränderlichen, nach den Referaten von L. Zoretti, P. Montel, und M. Fréchet. By A. Rosenthal. Leipzig, Teubner, 1924. 351 pp.

Neuere Untersuchungen über trigonometrische Reihen. By E. Hilb and M. Riesz. Leipzig, Teubner, 1924. 40 pp.

Neuere Entwicklung der Theorie partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus. By L. Lichtenstein. Leipzig, Teubner, 1924. 58 pp.

The first article is a revision of the corresponding article in the French Encyclopedia, prepared under the direction of Professor Borel by Professors Zoretti, Montel and Fréchet. The original plan was to translate the French article as it stood, but a desire for greater completeness and accuracy led to an arrangement for its revision instead. As a result of the interruption in the work caused by the war, a ten year interval elapsed between the conclusion of this arrangement and the publication of the article. Hence the new material added in the revision took on the proportions of an entirely new article, the present version being about two and a half times the size of the original one. The additions are all indicated by the symbols $_{*} \cdots {}^{*}$, in accordance with the agreement made with Professor Borel and the usage in the French revision of articles appearing first in the German Encyclopedia.

The first part of the article, which forms the revision of the portion of the original article due to Zoretti, deals with the theory of point sets, and is based both on the literature prior to 1912 (the date of the French edition) and subsequent to that date. In the chapter entitled "Verallgemeinerungen" the author gives a brief account of work in General Analysis, to which he refers as "die weitgehenden Untersuchungen von E. H. Moore und seinen Schülern." He stresses particularly the very great generality of the results attained. A detailed account of these results is not given, since such an account would belong more properly in the article by Pincherle on functional operations and equations (ENCYCLOPÄDIE, II A 11), which was completed before the publication of any of the work in General Analysis.

The second part of the article forms the revision of the portion of the original version contributed by Montel, and deals with integration and differentiation. Among the noteworthy additions to be found in this part may be mentioned work on integration due to Pierpont, F. Riesz, Denjoy, Hellinger, and Perron, and the notion of the approximate derivative due to Khintchine and Denjoy.

The third part of the present article deals with sequences of functions and forms the revision of the first half of that portion of the original article which is due to Fréchet. The second half of Fréchet's contribution, dealing with trigonometric series, is omitted here, since that topic is completely covered by the article of Hilb and Riesz, reviewed below, and the previous article by Burkhardt (ENCYCLOPÄDIE, II A 12) on trigonometric series and integrals prior to 1850.

Among the important additions in this part to the topics treated in the original version may be mentioned the following: Osgood's fundamental results regarding the distribution of points in the neighborhood of which the convergence of a series is respectively uniform or non-uniform, together with generalizations of these results, the conception of relative uniform convergence due to E. H. Moore, the relation between the classification of Borel assemblages and Baire's classification of functions, the relations between measurable functions and Baire's classes, and the recent results of Blumberg relative to properties of entirely arbitrary functions.

The second article gives an excellent bird's-eye view of the extensive domain of mathematical theory with which it deals. It includes the more important results of all the modern researches in this field, beginning with Riemann's fundamental memoir, and gives occasional indications of methods of proof that have been particularly fruitful. In addition the numerous references to the literature found in the footnotes will serve to orient the reader in a more detailed study of any particular topic which he may select.

In one of the early sections the high lights in the development of the theory from the historical standpoint are indicated by the authors. These include Riemann's noteworthy contribution, in which the introduction of a general theory of integration greatly enlarged the scope of Fourier series, while at the same time the foundations were laid for a study of trigonometric series in general. Closely related to Riemann's work is Cantor's study of the uniqueness of the development, and Du Bois-Reymond's theorem that a convergent trigonometric series, representing a function with a Riemann integral, is necessarily a Fourier series, as well as the latter's examples of continuous functions with divergent Fourier series. Fejér's investigation of the summation of Fourier's series by Cesàro's means of the first order revealed the significance of the methods of divergent series in the field of trigonometric series. Finally Lebesgue's fundamental definition of the integral paved the way for developments that have added much to the unity and completeness of the whole theory.

After this brief sketch of the development of the theory, the logical rather than the historical order is used by the writers in their detailed accounts of methods and results. In dealing with Fourier series these are grouped with respect to certain central topics, such as Fourier coefficients, general convergence theory, conjugate series, uniform and absolute convergence, special features of convergence and divergence, summation processes, Parseval's theorem, and the Riesz-Fischer theorem. In the presentation of the general theory of trigonometric series a summary of Riemann's work is followed by an account of the developments and generalizations of his ideas, due to various later writers. An appendix is devoted to the topics of multiple Fourier and trigonometric series and the approximate representation of continuous functions by means of trigonometric polynomials. The discussion of multiple series is extremely brief, it being pointed out by the authors that a general view of the present status of the theory may be gained by an examination of two papers dealing with this field, one by W. H. Young (PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY, vol. 11 (1912)), and the other by H. Geiringer (MONATSHEFTE FÜR MATHEMATIK UND PHYSIK, vol. 29 (1918)). The discussion of approximation theory is considerably more complete and includes the salient features of recent work in that field.

The third article connects up with the article by A. Sommerfeld (ENCYKLOPÄDIE, II A 7c) on boundary value problems in the theory of partial differential equations. The latter article was dated 1900 and gave an account of the developments in the field under consideration up to that date. Professor Lichtenstein's article deals with the additional developments since that time in the more restricted field indicated in his title.

As the author points out, the introduction of the methods of integral equations into the study of boundary value problems gave a very considerable impulse to the growth of the theory, and that portion of it dealing with linear differential equations of elliptic type may now be regarded as having reached a certain state of finality. A brief but comprehensive account of this portion of the theory, with complete references to the literature, is given in the present article. Even a casual examination of these references will readily reveal what an extensive portion of the recent developments in this field is due to Professor Lichtenstein himself.

The last two sections of the article are concerned with boundary value problems in the case of non-linear differential equations of elliptic type. Here, of course, the theory is much less developed, but many important additions have been made to it in the past quarter of a century. A systematic account of these researches, with references to the literature, is given in the sections mentioned.

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