NOTE ON THE PROJECTIVE GEOMETRY OF PATHS

BY T. Y. THOMAS*

1. *Projective Geometry of Paths.* It was first shown by Weyl† that the functions $\Gamma^i_{\alpha\beta}$ and the functions

$$(1) \qquad \Lambda^i_{\alpha\beta} = \Gamma^i_{\alpha\beta} + \delta^i_\alpha \psi_\beta + \delta^i_\beta \psi_\alpha,$$

where ψ_α is an arbitrary covariant vector, and

$$\delta^i_\alpha = 0, \quad \text{for } i \neq \alpha; \qquad = 1, \quad \text{for } i = \alpha,$$

define the same geometry of paths. This leads to the consideration of properties of the paths which are independent of the particular set of functions $\Gamma^i_{\alpha\beta}$ by means of which the paths are defined. Theorems expressing such properties constitute the projective geometry of paths. In the following note we give a few theorems belonging to the projective geometry of paths.

2. *Projective Tensors.* Theorems of the projective geometry of paths appear to have their statement in terms of what may be called *projective tensors,* i. e. tensors which are independent of the particular set of functions $\Gamma^i_{\alpha\beta}$ defining the paths. We shall show how a set of projective tensors may be derived by covariant differentiation from an n-uple of mutually independent vectors.

Let $h_{(\alpha)i}$ denote an n-uple of independent covariant vectors. Then the determinant

$$(2) \qquad h = |h_{(\alpha)i}|$$

does not vanish identically. We may therefore define an n-uple of contravariant vectors $h^{(\alpha)i}$ as the cofactors of the

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† H. Weyl, Göttinger Nachrichten, 1921, p. 99. See also O. Veblen, Proceedings of the National Academy, vol. 8 (1922), p. 347; and O. Veblen and T. Y. Thomas, Transactions of this Society, vol. 25, p. 557.

corresponding $h_{(\alpha)i}$ in the determinant h divided by h. Hence

$$(3)\qquad h_{(\alpha)i}\, h^{(\alpha)j} = \delta_i^j,$$

and

$$(4)\qquad h_{(i)\alpha}\, h^{(j)\alpha} = \delta_i^j,$$

where the left members represent a summation in α from $\alpha = 1$ to $\alpha = n$. It will be understood in the following that each index which appears twice in a term, once as a subscript and once as a superscript, is to be summed over the values 1 to n.

The change $[h_{(\alpha)i,j}]$ in the covariant derivative $h_{(\alpha)i,j}$ of the n-uple $h_{(\alpha)i}$, based on the functions $\Gamma^i_{\alpha\beta}$, when the Γ's are replaced by the above functions $\varDelta$, is

$$(5)\qquad [h_{(\alpha)i,j}] = -h_{(\alpha)i}\, \psi_j - h_{(\alpha)j}\, \psi_i .$$

Hence

$$(6)\qquad (n+1)\, \psi_i = -h^{(\alpha)\beta}\, [h_{(\alpha)\beta,\, i}] .$$

The vector γ_i defined by

$$(7)\qquad (n+1)\gamma_i = -h^{(\alpha)\beta}\, h_{(\alpha)\beta,\, i}$$

will therefore change by ψ_i when $\Gamma^i_{\alpha\beta}$ is replaced by $\varDelta^i_{\alpha\beta}$, i.e.

$$(8)\qquad [\gamma_i] = \psi_i .$$

Hence

$$(9)\qquad h_{(\alpha)ij} = h_{(\alpha)i,j} + h_{(\alpha)i}\gamma_j + h_{(\alpha)j}\gamma_i$$

represents a set of n projective covariant tensors of the second order.

In a similar way we may treat the case of an n-uple of independent contravariant vectors $h^{(\alpha)i}$. This leads to a set of n projective mixed tensors of the second order given by

$$(10)\qquad h_j^{(\alpha)i} = h^{(\alpha)i}{}_{,j} - h^{(\alpha)i}\lambda_j - \delta^i_j h^{(\alpha)\beta}\lambda_\beta$$

where the vector λ_i is defined as

$$(11)\qquad (n+1)\lambda_i = h_{(\alpha)\beta} h^{(\alpha)\beta}{}_{,\, i},$$

in which the covariant n-uple $h_{(\alpha)i}$ is obtained from the contravariant n-uple $h^{(\alpha)i}$ by dividing the cofactors of the elements of the determinant

$$h = |h^{(\alpha)i}|$$

by the determinant h. Other projective tensors will appear in the following paragraphs.

3. *The n-uple of Parallel Vectors.** If $h_{(\alpha)i}$ represents an n-uple of independent parallel covariant vectors for some set of functions $\Gamma^i_{\alpha\beta}$ defining the paths, then

$$(12) \qquad h_{(\alpha)ij} = 0$$

identically. The condition (12) is also sufficient for the n-uple of independent covariant vectors $h_{(\alpha)i}$ to be parallel. Hence we have the following theorem.

THEOREM I. *A necessary and sufficient condition for the n-uple of independent covariant vectors $h_{(\alpha)i}$ to be parallel in the projective geometry of paths is that* (12) *be satisfied.*

Forming the equations

$$(13) \qquad h^{(\alpha)i}_{j} = 0,$$

we may state the corresponding theorem for the case of a contravariant n-uple.

THEOREM II. *A necessary and sufficient condition for the n-uple of independent contravariant vectors $h^{(\alpha)i}$ to be parallel in the projective geometry of paths is that* (13) *be satisfied.*

4. *Reduction to the Euclidean Geometry.* The affine geometry of paths becomes a euclidean geometry if there exists an n-uple of independent parallel covariant vectors $h_{(\alpha)i}$. This is obviously a necessary condition and it is seen to be sufficient since the existence of an n-uple of parallel vectors $h_{(\alpha)i}$ leads to the equations

$$(14) \qquad h_{(i)\sigma} B^{\sigma}_{\alpha\beta\gamma} = 0,$$

where $B^{\sigma}_{\alpha\beta\gamma}$ is the affine (ordinary) curvature tensor, and hence

* L. P. Eisenhart, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 8 (1922), p. 207. See also O. Veblen and T. Y. Thomas, loc. cit., p. 589.

$$(15) \qquad B^i_{\alpha\beta\gamma} = 0,$$

owing to the independence of the vectors $h_{(\alpha)i}$. Hence we have the following theorem.

THEOREM III. *A necessary and sufficient condition for the projective geometry of paths to be a euclidean geometry is that the Γ's be such that there exists an n-uple of independent covariant vectors $h_{(\alpha)i}$ satisfying* (12).

A similar theorem may be stated for the n-uple of contravariant vectors.

5. *Reduction to the Riemann Geometry.* For the projective geometry of paths to be a Riemann geometry it is necessary and sufficient that the Γ's be such that there exists a symmetric tensor $g_{\alpha\beta}$ whose determinant

$$(16) \qquad g = |g_{\alpha\beta}|$$

does not vanish identically, and a vector ψ_i such that

$$(17) \qquad \tilde{g}_{\alpha\beta,\gamma} = 0,$$

where $\tilde{g}_{\alpha\beta,\gamma}$ is the covariant derivative of $g_{\alpha\beta}$ based on the functions $\Lambda^i_{\alpha\beta}$ given by (1). Writing (17) in the form

$$(18) \qquad g_{\alpha\beta,\gamma} - 2g_{\alpha\beta}\psi_\gamma - g_{\gamma\beta}\psi_\alpha - g_{\alpha\gamma}\psi_\beta = 0,$$

where $g_{\alpha\beta,\gamma}$ is the covariant derivative of $g_{\alpha\beta}$ based on the functions $\Gamma^i_{\alpha\beta}$, and multiplying by the contravariant tensor $g^{\alpha\beta}$, formed in the ordinary manner from the tensor $g_{\alpha\beta}$, we have

$$(19) \qquad 2(n+1)\psi_\gamma = g^{\alpha\beta} g_{\alpha\beta,\gamma}.$$

Substituting this value of ψ_γ in the left member of (18) and denoting the resulting expression by $g_{\alpha\beta\gamma}$, we find

$$(20) \; g_{\alpha\beta\gamma} = g_{\alpha\beta,\gamma} - \frac{g^{\mu\nu}}{n+1}\left(g_{\alpha\beta} g_{\mu\nu,\gamma} + \frac{1}{2} g_{\gamma\beta} g_{\mu\nu,\alpha} + \frac{1}{2} g_{\alpha\gamma} g_{\mu\nu,\beta}\right).$$

Hence (18) takes the form

$$(21) \qquad g_{\alpha\beta\gamma} = 0.$$

The tensor $g_{\alpha\beta\gamma}$ defined by (20) may be shown to be projective.

The equations (21) constitute a necessary condition for the projective geometry of paths to be a Riemann geometry based on the tensor $g_{\alpha\beta}$ as the fundamental metric tensor. This condition is easily shown to be sufficient. Hence we have the following theorem.

THEOREM IV. *A necessary and sufficient condition for the projective geometry of paths to be a Riemann geometry is that the Γ's be such that there exists a tensor $g_{\alpha\beta}$ which satisfies the equations* (21).

6. *Reduction to the Weyl Geometry.** Let us denote by $g_{\alpha\beta}$ a covariant symmetric tensor whose determinant g does not vanish identically, as in the preceding paragraph, and also by φ_α a covariant vector. Let us then form the equations

$$g_{\alpha\beta\gamma} = g_{\alpha\beta}\varphi_\gamma, \tag{22}$$

where $g_{\alpha\beta\gamma}$ is now defined by

$$\begin{aligned} g_{\alpha\beta\gamma} = g_{\alpha\beta,\gamma} - \frac{g^{\mu\nu}}{n+1}\left(g_{\alpha\beta}g_{\mu\nu,\gamma} + \frac{1}{2}g_{\gamma\beta}g_{\mu\nu,\alpha} + \frac{1}{2}g_{\alpha\gamma}g_{\mu\nu,\beta}\right) \\ + \frac{n}{n+1}\left(g_{\alpha\beta}\varphi_\gamma + \frac{1}{2}g_{\gamma\beta}\varphi_\alpha + \frac{1}{2}g_{\alpha\gamma}\varphi_\beta\right), \end{aligned} \tag{23}$$

in which $g_{\alpha\beta,\gamma}$ is the covariant derivative of $g_{\alpha\beta}$ based on the functions $\Gamma^i_{\alpha\beta}$. The tensor $g_{\alpha\beta\gamma}$ defined by (23) is a projective tensor. We may prove the following theorem.

THEOREM V. *A necessary and sufficient condition for the projective geometry of paths to be a Weyl geometry is that the Γ's be such that there exists a tensor $g_{\alpha\beta}$ and vector φ_α which satisfy the equations* (22).

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* By the Weyl geometry is meant the geometry used by Weyl as the basis of his combined theory of gravitation and electricity. See H. Weyl, *Raum, Zeit, Materie*, 4th ed., p. 113.