

THE OCTOBER MEETING IN NEW YORK

The two hundred thirty-seventh regular meeting of the Society was held at Columbia University on Saturday, October 25, 1924, extending through the usual morning and afternoon sessions. The attendance included the following fifty members of the Society:

Alexander, Archibald, Babb, C. R. Ballantine, J. P. Ballantine, Barnum, W. M. Bond, R. W. Burgess, B. H. Camp, Cole, L. D. Cummings, Dantzig, Doak, Eisenhart, Fite, Fort, Philip Franklin, Gafafer, Garretson, Gehman, Gill, Glenn, Gronwall, Hille, Himwich, Dunham Jackson, Joffe, Kline, Langman, Lefschetz, Longley, MacColl, Meder, C. L. E. Moore, Northcott, Pell, R. G. Putnam, Rainich, Reddick, R. G. D. Richardson, Ritt, Ruger, Seely, C. E. Smith, J. H. Taylor, J. M. Thomas, Veblen, M. E. Wells, Wiener, R. G. Wood.

At the May Meeting of the Society a new type of membership was created and designated Sustaining Membership. This action was taken on the conviction that there are individuals and institutions who realize the fundamental part that mathematical research plays in our modern civilization and who are willing to render financial assistance to its publication. At the present meeting the Secretary took great pleasure in announcing the following list of Sustaining Members who had been solicited by the Committee on Endowment:

Allyn and Bacon, Boston;
 Babcock and Wilcox Company, New York;
 Dartmouth College, Hanover;
 General Electric Company, Schenectady; Patron;
 Ginn and Company, Boston;
 Insull Interests, Chicago (comprising the Commonwealth Edison Company, the People's Gas Light and Coke Company, the Public Service Company, the Middle West Utilities Company, and the Chicago, North Shore and Milwaukee Railroad); Patron;
 Mr. E. W. Rice, Jr., General Electric Company, Schenectady;
 Union Central Life Insurance Company, Cincinnati;
 University of Washington, Seattle;
 Western Electric Company, New York; Patron;
 Westinghouse Electric and Manufacturing Company, New York; Patron.

The Secretary announced also that the following persons had been elected to ordinary membership in the Society by

the Council since April, 1924, either by mail vote* or at a meeting:

Mr. Anantanarayana Aiyar, Judson College, Rangoon;
 Mr. William Leake Ayres, University of Texas;
 Miss Wealthy Babcock, University of Kansas;
 Mr. Frank Swan Beale, University of Maine;
 Professor Ella Edna Bernstorff, Friends University;
 Professor Harold Blair, Western State Normal College, Kalamazoo;
 Mr. Leonard Marcot Blumenthal, University of Michigan;
 Mr. Robert Bogue, Omaha;
 Professor Zechariah Chafee, Harvard University Law School;
 Professor Tsaihsin H. Chen, Yen-Ching University, Peking;
 Mr. Starling W. Childs, Jr., Yale University;
 Professor Jung Lu Chin, Nankai University, Tientsin;
 Superintendent William Horner Cocke, Virginia Military Institute;
 Mr. Redmond S. Colnon, St. Louis;
 Miss Mae Elizabeth Conn, University of Southern California;
 Sister Mary Cordia, Notre Dame College;
 Professor Moses Eugene Cox, Agricultural and Mechanical College of Texas;
 Professor Marian Elizabeth Daniells, Iowa State College;
 Professor Robert Dodds Daugherty, Iowa State Teachers College;
 Professor Raymond Mark Deming, Upper Iowa University;
 Mr. Howard Theodore Engstrom, University of Maine;
 Professor Claire Beatrice Fisher, Potomac State College;
 Mr. Donald Alexander Flanders, University of Pennsylvania;
 Mr. Roger Lee Flanders, Norwich University;
 Mr. Alfred Douglas Flinn, Engineering Foundation, New York;
 Miss Neda Belle Freeman, Irving College;
 Professor Benjamin Curtis Glover, Otterbein College;
 Professor George Wollam Gorrell, University of Denver;
 Professor Vernon Guy Grove, Michigan Agricultural College;
 Mr. Ira James Gwinn, Morningside College;
 Professor John Alexander Hardin, Centenary College of Louisiana;
 Mr. Floyd S. Harper, University of Nebraska;
 Professor Nathan W. Harter, Thiel College;
 Professor Frederick Charles Hartwick, University of Dayton;
 Mr. Hubert Banks Huntley, University of New Hampshire;
 Miss Mabel Hutchins, Blue Mountain College;
 Mr. Saburo Isayama, Niigata High School for Girls;
 Professor Ralph Lent Jeffery, Acadia University;
 Professor Elijah Newton Johnson, Butler College;
 Professor Jesse Brelande Johnson, Baylor University;
 Miss Dora Evelyn Kearney, University of Minnesota;
 Mr. Ernest George Keller, University of Wisconsin;
 Mr. Arthur J. Koeppe, Union Central Life Insurance Company, Cincinnati;
 Mr. Abraham Boris Kouperman, University of Chicago;
 Mr. William H. Kurzin, Crane Junior College;
 Mr. Benjamin George Lehenbauer, Union Central Life Insurance Company, Cincinnati;
 Mr. John J. Lichter, St. Louis;
 Mr. Hugh Gray Lieber, Columbia University;

* See this BULLETIN, vol. 30, p. 481.

Professor William Thomas MacCreadie, Norwich University;
 Professor Anna E. Many, Newcomb College;
 Mr. Thomas Midgley, General Motors Chemical Company, Dayton;
 Professor Kevin Nowlan, S. J., Loyola University;
 Professor Yoshitomo Okada, Tôhoku Imperial University;
 Professor Earle Kerr Paxton, Washington and Lee University;
 Mr. Walter Otis Pennell, Southwestern Bell Telephone Company,
 St. Louis;
 Professor Sue Scott Perkins, Missouri State Teachers College, Springfield;
 Mr. Howard Manchester Phillips, DeKalb Junction, N. Y.;
 Dr. Robert Garfield Putnam, New York University;
 Miss Margaret Ramsey, Linfield College;
 Professor Caroline M. Reaves, Coker College;
 Dr. Bernard Paul Reinsch, University of Illinois;
 Dr. Norman Hurd Ricker, Humble Oil and Refining Company, Houston;
 Professor Joseph Alphonsus Rooney, Mount St. Charles College;
 Mr. Norman Eby Rutt, University of Pennsylvania;
 Professor Samuel Thomas Sanders, Louisiana State University;
 Dr. Paul Schnurmann, Pittsburgh;
 President John Stephen Sewell, Alabama Marble Company, Birmingham;
 Professor Charles E. Shull, Bridgewater College;
 Professor Elmer McClellan Stahl, Midland College;
 Professor Will B. Stokes, Southern Louisiana Institute;
 Miss Lucretia Mae Switser, University of Kansas;
 Dr. Brandreth Symonds, Mutual Life Insurance Company;
 Mr. Keizo Takahashi, Shonan Middle School, Banagawa-Ken;
 Professor William Duane Tallman, Montana State College;
 Mr. Ward Hastings Taylor, University of Arkansas;
 Mr. Benjamin Franklin Tillson, New Jersey Zinc Company;
 Mr. Paul S. Wagner, Johns Hopkins University;
 Miss Esther May Weaver, Northwestern University;
 Dr. David Vernon Widder, Bryn Mawr College;
 Mr. Frank Harry Mead Williams, Drexel Institute.

Ordinary Members nominated by Sustaining Members:

Mr. W. L. Abbott, Commonwealth Edison Company, Chicago;
 Mr. R. J. Andrus, Twin State Gas and Electric Company, Boston;
 Mr. C. C. Argabrite, Interstate Public Service Company, Indianapolis;
 Mr. H. E. Bates, Peoples Gas Light and Coke Company, Chicago;
 Mr. L. V. Bewley, General Electric Company, Schenectady;
 Mr. J. E. Black, Babcock and Wilcox Company, New York;
 Mr. E. J. Blair, Chicago Rapid Transit Company;
 Mr. L. F. Blume, General Electric Company, Pittsfield;
 Mr. G. T. Bogard, Kentucky Utilities Company, Louisville;
 Mr. Aram Boyajian, General Electric Company, Pittsfield;
 Mr. B. O. Buckland, General Electric Company, Schenectady;
 Mr. B. I. Budd, Chicago, North Shore and Milwaukee Railroad Company;
 Mr. F. H. Buller, General Electric Company, Pittsfield;
 Mr. C. M. Burrill, General Electric Company, Schenectady;
 Mr. J. R. L. Carrington, Union Central Life Insurance Company, Cin-
 cinnati;
 Mr. P. R. Cassidy, Babcock and Wilcox Company, New York;
 Miss E. Clarke, General Electric Company, Schenectady;
 Mr. S. B. Cushing, Public Service Company of North Illinois, Chicago;
 Mr. G. F. Davis, General Electric Company, Schenectady;

Dr. Saul Dushman, General Electric Company, Schenectady;
 Mr. J. W. Evers, Commonwealth Edison Company, Chicago;
 Mr. C. L. Fortescue, Westinghouse Company, East Pittsburgh;
 Mr. E. J. Fowler, Commonwealth Edison Company, Chicago;
 Mr. R. F. Franklin, General Electric Company, Schenectady;
 Mr. G. W. Hamilton, Middle West Utilities Company, Chicago;
 Mr. E. E. Hardecastle, Union Central Life Insurance Company, Cincinnati;
 Mr. R. B. Harper, Peoples Gas Light and Coke Company, Chicago;
 Mr. Isaac Harter, Babcock and Wilcox Company, New York;
 Mr. J. L. Hecht, Public Service Company of North Illinois, Chicago;
 Mr. G. J. Heimberger, Babcock and Wilcox Company, Bayonne;
 Mr. Walter Helmer, Chicago Rapid Transit Company;
 Mr. R. E. Hellmund, Westinghouse Company, East Pittsburgh;
 Mr. Jesse Hyatt, Chicago, North Shore and Milwaukee Railroad Company;
 Mr. E. E. Johnson, General Electric Company, Schenectady;
 Mr. H. A. Johnson, Chicago Rapid Transit Company;
 Mr. Charles Jones, Chicago Rapid Transit Company;
 Mr. W. A. Jones, Babcock and Wilcox Company, Bayonne;
 Mr. E. W. Kellogg, General Electric Company, Schenectady;
 Mr. H. J. Kerr, Babcock and Wilcox Company, New York;
 Mr. A. L. Kimball, General Electric Company, Schenectady;
 Mr. C. H. Koch, General Electric Company, Schenectady;
 Mr. A. J. Koeppe, Union Central Life Insurance Company, Cincinnati;
 Mr. L. L. Langvand, Babcock and Wilcox Company, Barberton;
 Mr. E. S. Lee, General Electric Company, Schenectady;
 Mr. N. E. Lewis, Babcock and Wilcox Company, New York;
 Mr. D. W. McLenegan, General Electric Company, Schenectady;
 Mr. Campbell MacMillan, General Electric Company, Schenectady;
 Mr. H. B. Marvin, General Electric Company, Schenectady;
 Mr. S. T. Maunder, General Electric Company, Schenectady;
 Mr. M. S. Mead, General Electric Company, Schenectady;
 Mr. F. W. Merrill, General Electric Company, Fort Wayne;
 Mr. C. W. Middletown, Babcock and Wilcox Company, New York;
 Mr. Harold Mott-Smith, General Electric Company, Schenectady;
 Mr. Harold Otis, Chicago Rapid Transit Company;
 Mr. W. E. Paul, General Electric Company, Schenectady;
 Mr. C. W. PenDell, Public Service Company of North Illinois, Chicago;
 Mr. J. F. Peters, Westinghouse Company, East Pittsburgh;
 Mr. A. D. Pratt, Babcock and Wilcox Company, New York;
 President A. G. Pratt, Babcock and Wilcox Company;
 Mr. C. W. Rice, Schenectady;
 Mr. J. D. Roberts, Central Illinois Public Service Company, Springfield;
 Mr. E. L. Robinson, General Electric Company, Schenectady;
 Mr. R. F. Schuchardt, Commonwealth Edison Company, Chicago;
 Mr. E. O. Schweitzer, Commonwealth Edison Company, Chicago;
 Mr. E. B. Shand, Westinghouse Company, East Pittsburgh;
 Mr. I. H. Summers, General Electric Company, West Lynn;
 Mr. H. D. Taylor, General Electric Company, Schenectady;
 Mr. C. E. Thompson, Chicago, North Shore and Milwaukee Railroad
 Company;
 Mr. Stephan Timoshenko, Westinghouse Company, East Pittsburgh;
 Mr. L. A. Umansky, General Electric Company, Schenectady;
 Mr. G. B. Warren, General Electric Company, Schenectady;
 Mr. Hosea Webster, Babcock and Wilcox Company, New York;
 Mr. R. W. Wieseman, General Electric Company, Schenectady.

During the summer, the Council voted thanks for the following gifts to the Endowment Fund: from Stone and Webster, \$1000; from the Cutler-Hammer Company, \$250; from Mr. C. H. Coffin, ex-president and founder of the General Electric Company, \$2000. It was voted that sustaining members may be elected by the Council at the time of the first presentation of their names, and that not more than five persons may be nominated to ordinary membership by a sustaining member for each \$100 of dues paid.

The President made the following appointments to represent the Society: at the Pan-American Scientific Congress at Lima, Professors E. V. Huntington and D. N. Lehmer; at the inauguration of President Vinson of Western Reserve University, Professor W. G. Simon; at the celebration of the one hundredth anniversary of the Rensselaer Polytechnic Institute, Dr. G. A. Campbell; at the International Mathematical Congress at Toronto, Professor G. A. Bliss and the Secretary; at the inauguration of President Irving Maurer, of Beloit College, Professor H. H. Conwell.

The President appointed as Committee to Administer the Eliakim Hastings Moore Fund, Professors Arnold Dresden (chairman), J. W. Alexander, and H. E. Slaughter. Mr. Robert Henderson, of the Equitable Life Assurance Society, was designated to deliver the second Josiah Willard Gibbs Lecture at the coming Annual Meeting in Washington; his subject will be *Life insurance as a social service and as a mathematical problem*.

A committee consisting of Mr. S. A. Joffe and Professor Tomlinson Fort was appointed to audit the accounts of the Treasurer for the current year. A list of nominations of officers, members of the Council and members of the Board of Trustees was adopted and ordered printed on the official ballot for the annual election. An invitation was received from the University of Pennsylvania to hold the summer meeting at that institution in 1926.

The report of the Committee on the Cole Prize (see this BULLETIN, vol. 29, p. 14) recommending that the prize be offered for the most notable advance in the theory of Linear Algebras was adopted. Further details will be published shortly by the Committee of which Professor H. S. White is chairman.

It has been decided to continue the printing of the BULLETIN and TRANSACTIONS in Hamburg for the year 1925.

Titles and abstracts of the papers read at this meeting follow below. President Veblen presided at both sessions. Mr. Huber was introduced by Professor G. E. Wahlin. The papers of Alexander, Cole, Franklin (second paper), Garabedian, Hille (first paper), Huber, Hutchinson, Michal, Pennell, Walsh, and Williams were read by title.

1. Professor B. H. Camp: *Note on a transformation of the hypergeometric series.*

The sum of the first t terms of the hypergeometric series which occurs in the theory of probability is* $(1/n^{(r)}) \sum_{t=0}^{t=t} (r^{(t)}/t!) y^{(s)} (n-y)^{(t)}$, where $s = r-t$, and $r^{(t)} = r(r-1) \dots (r-t+1)$. This may be transformed to the form: $(r^{(t+1)}/n^{(r)} t!) \sum_{y=s-1}^{y=y-1} y^{(s-1)} (n-y-1)^{(t)}$, the summation now being with respect to y . The process by which this is obtained may be of value in the transformation of other sums. The details will appear in a paper on *Probability integrals for a hypergeometric series*, in BIOMETRIKA.

2. Professor Louise D. Cummings: *A new type of double sextette closed under a binary (3,3) correspondence.*

This paper shows that cyclic double sextettes closed under a binary (3,3) correspondence are reducible to two non-congruent types. Type I was determined in a previous paper, and type II is here exhibited.

3. Dr. J. H. Taylor:† *Reduction of Euler's equations to a canonical form.*

This paper will appear in full in an early issue of this BULLETIN.

* Cf. Elderton, *Frequency Curves and Correlation*, p. 37.

† National Research Fellow.

4. Mr. H. M. Gehman: *Necessary and sufficient conditions that every closed and connected subset of a continuous curve be a continuous curve. Second paper.*

The conditions previously announced to the Society (Chicago Meeting, April 19, 1924) are contained in this more general theorem: *A necessary and sufficient condition that every closed and connected subset of a continuous curve M be a continuous curve is that M contain no closed and connected point set N , containing a continuum of condensation W , such that $N-W$ is uniformly connected im kleinen "relative to W ."* Another necessary and sufficient condition is that, given any positive number ϵ , M contains at most a finite number of mutually exclusive closed and connected sets of diameter greater than ϵ . A condition which is sufficient but not necessary is that, given any positive number ϵ , M can be expressed as the sum of a finite number of closed and connected sets each of diameter less than ϵ , each pair having at most a finite number of points in common.

5. Dr. Philip Franklin: *The rotating disc.*

A homogeneous incompressible disc of radius r , when caused to rotate with constant angular velocity ω , suffers a contraction in radius, according to the theory of relativity. Eddington, assuming that the geometry on the disc remains euclidean, finds the new radius to be, approximately, $r' = r(1 - \omega^2 r^2 / 8)$. The author discusses the results when this assumption is dropped, certain others being retained, and finds for the new radius approximately $r' = r(1 - \omega^2 r^2 / 14)$. Some properties of the geometry on the disc are also given.

6. Dr. Philip Franklin: *The electric currents in a network.*

If constant E. M. F.'s are inserted in a network of conductors, the resulting currents are uniquely determined. Their explicit expression in terms of the resistances of the conductors and the topological properties of the network was given by Kirchhoff in 1847. In this paper the result is obtained by a briefer method, based on the matrices for a linear graph.

7. Professor F. N. Cole: *On simple groups of low order.*

This paper appeared in full in the November-December, 1924, number of this BULLETIN.

8. Professor J. I. Hutchinson: *On the roots of the Riemann zeta function.*

The object of this paper is to simplify the methods and formulas in use in connection with numerical investigations on the zeta function. The limits of an empirical law noticed by Gram, regarding the separation of the roots, are determined and the results used in locating 59 additional roots, thus completing the list of those smaller than 300. The values of 14 roots are calculated, thus including the remaining unknown roots smaller than 100. The number of positive roots less than 500 is calculated and found to be 269.

9. Professor Einar Hille: *A note on regular singular points.*

This paper contains a discussion of the differential equation $w'' + G(z)w = 0$ in the neighborhood of the origin, which is supposed to be a regular singular point. The discussion is based upon the transformation $Z(z; z_0) = \int_{z_0}^z G(z) dz; W(Z) = [G(z)]^{-1/4} w(z)$, and involves among other things a characterization of the nature of the singular point by means of the geometrical singularity at the origin of the curve net $X = \text{const.}, Y = \text{const.}$ The orientation of the zeros and the extrema of the solutions upon a Riemann manifold in the neighborhood of the origin is studied, and regions are determined in which these points are regularly distributed. The results are incomplete in the case in which one of the roots of the indicial equation is greater than $+1$. The investigation is completed with a discussion based on the method of zero-free regions.

10. Professor C. A. Garabedian: *Four methods for solving the problem of the rectangular beam.*

Upon applying to beams of rectangular section the series already employed by the author in connection with plates and rods*, one is led to four distinct methods. Method A deals with the broad shallow beam, while method B is concerned with the beam that is narrow and deep. Methods AA and BB (announced in the closing paragraph of the author's note on *Thick rectangular plates*†) are

* See abstract in this BULLETIN, vol. 30, p. 295, with accompanying references.

† COMPTES RENDUS, Feb. 11, 1924, and this BULLETIN, loc. cit.

analogous to methods A and B respectively. All four methods treat the beam rigorously and in three dimensions, despite the two-dimensional aspect of the problems. Methods A and B excel in rigor, yet AA and BB yield interesting solutions expressible in polynomial form. It is possible to vary the depth in A and the width in B, and in methods AA and BB both width and depth may be varied. A note concerning these methods appeared in the *COMPTES RENDUS* of August 18, 1924; the author hopes later to publish the further developments in extenso.

11. Professor J. L. Walsh: *Some two-dimensional loci.*

This paper solves the following problem: If two points of the plane, A and B , have as their respective loci the interiors of two circles, determine the locus of the point P such that (I) the angle APB is a prescribed constant, (II) the ratio $PA:PB$ is a prescribed constant. In each case, the locus of P is a region bounded in whole or in part by a cartesian oval or an arc thereof.

12. Professor K. P. Williams: *A uniqueness theorem for the Legendre and the Hermite polynomials.*

Let

$$\varphi(y) = a_0 + a_1 y + (a_2/2!)y^2 + (a_3/3!)y^3 + \dots,$$

and put

$$\varphi(2xz + z^2) = P_0 + P_1(x)z + P_2(x)z^2 + \dots$$

In this paper it is shown that a simple recursion relation between three successive polynomials of the set P_0, P_1, P_2, \dots will exist only in case we have to do essentially with the generalized polynomials of Legendre or the polynomials of Hermite.

13. Mr. C. M. Huber: *On the prime divisors of the cyclotomic functions.*

In this paper the author deals with a generalization of the test of Sylvester as to the character of the primes as divisors or non-divisors of the cyclotomic functions. As preliminary, a theorem is given for determining the prime ideal factorization of the rational primes in cyclotomic sub-fields. A necessary and sufficient condition is then developed for a rational prime to be a divisor or non-divisor of the cyclotomic functions. Particular examples are then discussed in application of the test found.

14. Mr. A. D. Michal: *Functional invariants, with a continuity of order p , of one-parameter Fredholm and Volterra transformation groups.*

This paper considers functionals of a function $y(x)$ and its first p derivatives $y'(x), \dots, y^{(p)}(x)$ admitting a given arbitrary one-parameter *Fredholm* group of linear functional transformations. Sufficient conditions for invariance are given in the form of completely integrable functional equations with partial functional derivatives when the kernel of the transformation is of a certain type. In the case of the linear functional necessary and sufficient conditions for invariance are given. The paper concludes with a theorem showing the unique rôle played by a linear functional of $y(x)$ and its derivative $y'(x)$ in the invariant theory of linear functionals, of continuity order p , which admit a given arbitrary one-parameter *Volterra* group of linear functional transformations.

15. Professor J. W. Alexander: *On certain new topological invariants.*

The author derives certain topological invariants analogous to but in general different from the Betti numbers and coefficients of torsion.

16. Mr. W. O. Pennell: *The interpretation of non-integral exponents with notes on the theory of subponents.*

In this paper the author makes use of the following vector notation: a_α represents a vector in the XY plane of length a making an angle α with the X axis. He calls α the subponent of the vector X , and points out that in multiplication subponents follow laws similar to those of exponents.

17. Professor Norbert Wiener: *The representation of bounded functions by trigonometric integrals.*

The author investigates the condition under which a bounded function may have a trigonometric expansion analogous to a Fourier integral. He studies the convergence in the mean of this expansion both over a finite range and (in the sense of Bohr) over an infinite range.

18. Professor Einar Hille: *A general type of singular points.*

The present paper gives a unified treatment of a fairly general type of singular points of linear differential equations of the second order, based upon the transformation of Liouville. It is assumed that there exists a region abutting upon the singular point which is carried over into a right half-plane by the transformation and, further, that in this half-plane the transformed equation is asymptotic (in an easily characterized sense) to a sine-equation. Such an equation can be integrated by the method of successive approximations and the form of the integral gives directly the asymptotic properties of solutions of the transformed equation in the half-plane. In particular, the distribution of the zeros can be read off easily. Any irregular singular point can be handled by this method; also a wide class of regular singular points, though not all. Further, any singular point which is transformable into one of the previously mentioned types by a change of independent variable can be attacked directly by this method, the detour over the intermediate transformation being unnecessary.

19. Professor Dunham Jackson: *The geometry of frequency functions.*

This paper has appeared in the January-February issue of this BULLETIN.

20. Professor J. F. Ritt: *Elementary functions and their inverses.*

The chief item of this paper is the determination of all elementary functions whose inverses are elementary. The elementary functions are understood to be those obtained by performing algebraic operations and taking exponentials and logarithms. For instance, the function $\tan [e^{z^2} - \log z (1 + \sqrt{z})] + [z^z + \log \arcsin z]^{1/2}$ is elementary. It is proved that if $F(z)$ and its inverse are both elementary, there exist n functions $\varphi_1(z), \varphi_2(z), \dots, \varphi_n(z)$, where each $\varphi(z)$ with an odd index is algebraic, and each $\varphi(z)$ with an even index is either e^z or $\log z$, such that $F(z) = \varphi_n \varphi_{n-1} \dots \varphi_2 \varphi_1(z)$, each $\varphi_i(z)$ ($i < n$) being substituted for z in $\varphi_{i+1}(z)$.

This paper is a continuation of Liouville's work of almost a century ago on the classification of the elementary functions, on the possibility of effecting integrations in finite terms, and on the impossibility of solving certain differential equations, and certain transcendental equations, in finite terms.

21. Professor J. F. Ritt: *On the impossibility of solving certain differential equations in finite terms.*

Liouville, in 1841, proved that the solutions of Bessel's equation $z^2w'' + zw' + (z^2 - \nu^2)w = 0$ cannot be expressed in terms of z by performing elementary operations* and quadratures. But it is customary to regard a differential equation as solved if one finds a finite equation in w and z ; one does not generally try to express w itself in finite terms. In the present paper it is shown that the solution of Bessel's equation does not satisfy any equation in w and z found by performing elementary operations upon those variables and integrating exact differentials.

22. Mr. G. Y. Rainich: *On singularities in analytical physics.*

The introduction, in place of the electromagnetic tensor and its dual, of their sum w , which is a complex tensor, and of complex vectors simplifies calculations. The energy tensor is the norm of w , i. e., w multiplied by its conjugate. Residues of an electromagnetic field are introduced as values of surface integrals surrounding singular lines (electron paths); they are complex numbers only the moduli of which are determined by space-time. Residues can be used to complete the determination of the electromagnetic field, which, in a region free from singularities, is given, by the curvature of space-time, only to a constant of integration (see the author's paper read before the Society in May, 1924). To determine this constant we can agree to make all the residues real, which can be done because the arguments of the residues of every electromagnetic field differ only by multiples of π , proved by the fact that magnetic charges do not exist. The analogy of the electromagnetic field with the analytic function on a minimal surface (see abstract of the author's second paper, below) is discussed.

* See preceding abstract.

23. Mr. G. Y. Rainich: *The analytic function on a minimal surface.*

The Codazzi equations, to which the tensor usually represented by the second differential form is subjected, have in the case of a minimal surface the form of the Cauchy-Riemann equations which are satisfied by an analytic function. To make the analogy complete, we have to consider an analytic function as defining in each point of the plane a linear vector transformation. The Cauchy integral theorem is immediately applicable to the function on a minimal surface, and this leads to the definition of a residue. The Gaussian curvature at each point is equal to minus the square of the modulus of the function at that point. If a surface is given by its first form and the coefficients satisfy a certain differential equation, the analytic function is defined to a constant of integration, the values of which correspond to associated surfaces. Certain of the results are extended to surfaces of constant mean curvature.

24. Professor Dunham Jackson: *On the gradient of a functional.* Preliminary communication.

This paper is concerned with a definition of the directional derivative of a functional, the determination of the direction in function space for which the value of this directional derivative is a maximum, and the evaluation of the derivative in any other direction as the product of the above-mentioned maximum value by the cosine of the angle between the two directions in question.

25. Dr. T. H. Gronwall: *Extension of a theorem due to Lerch.*

Lerch has shown that when $f(x)$ is continuous for $0 < x < 1$ and $\int_0^1 x^n f(x) dx = 0$ for $n = 0, 1, 2, \dots$, then $f(x)$ vanishes identically. In the present note it is shown that when $f(x)$ is absolutely integrable and the above equation holds, $f(x)$ vanishes identically on any sub-interval where it is continuous.

26. Dr. T. H. Gronwall: *On the Cesàro summability of Fourier's and Laplace's series.*

Fejér has shown that any Cesàro sum of order one of the Fourier series of a continuous function takes only

values between the extreme values of the function, and also that the same property holds for the sums of order two of the Laplace series. Considering also fractional values of the order k of the sums, the present paper proves that $k=1$ and $k=2$, respectively, are the smallest values of k with the above property.

27. Professor J. R. Kline: *A note concerning closed non-dense linear sets which are enumerable.*

In both the first and the second editions of Hobson's *Theory of the Functions of a Real Variable*, the author attempts to prove that a non-dense closed set is enumerable if its complementary intervals are such that every one abuts on another at each of its ends. Professor R. L. Moore gave an example of a set satisfying the conditions of the hypothesis of Hobson's theorem, which is not enumerable. In the present paper it is shown that in case it is stipulated that the non-dense closed set and all its derived sets have the property of having their complementary intervals each abut on another at each of its ends, then the set is enumerable.

R. G. D. RICHARDSON,
Secretary.

THE OCTOBER MEETING OF THE SAN FRANCISCO SECTION

The forty-fourth regular meeting of the San Francisco Section was held at the University of California on October 25, 1924. In the absence of the Chairman of the Section, Professor Carpenter, the meeting was called to order by the Secretary of the Section. Professor Hedrick was elected temporary chairman. The total attendance was thirty, including the following twenty-five members of the Society:

Alderton, Allardice, Andrews, Bell, Bernstein, Blichfeldt, Growe, Haskell, E. R. Hedrick, Hotelling, Irwin, Lehmer, Sophia Levy, McCarty, McFarland, F. R. Morris, Moreno, Noble, Pehrson, T. M. Putnam, Robertson, Schmiedel, Pauline Sperry, A. R. Williams, Wong.