relations are

$$h_1^{d_1} = 1, \quad h_2^{d_2} = 1, \quad \ldots, \quad h_r^{d_r} = 1 \tag{3}$$

where those of the exponents $d_1, d_2, \ldots, d_r$ which are not 1 are the invariant factors of the matrix $|\gamma_{it}|$.

An operation of $\tilde{G}$ is of finite period if and only if it is in the group H generated by the operations $h_1, h_2, \ldots, h_r$ and the relations (3). Hence H contains all operations of $\tilde{G}$ of finite period. Moreover H is a finite group because it is commutative and generated by a finite number of operations each of finite period. The invariant factors of $|\gamma_{it}|$ are invariants of H (Kronecker, Berliner Monatsberichte, 1870, p. 885).

These invariant factors are what Tietze calls the Poincaré numbers of the group G. They are invariants of G because G determines the commutative group $\tilde{G}$ uniquely and $\tilde{G}$ determines the finite group H uniquely and H determines the invariant factors uniquely.

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ANALYTIC FUNCTIONS AND PERIODICITY*

BY J. F. RITT

This paper presents two theorems which show that the condition that a function be periodic can be analyzed, in different ways, into a set of requirements, from the satisfaction of only one of which, if the function is analytic, the periodicity of the function can be inferred. The theorems are

Theorem A. *If $f(z)$ is a uniform analytic function, and if an $a > 0$ exists such that every z_1 at which $f(z)$ is analytic is the center of a circle of radius a on which a z_2 lies at which $f(z)$ is analytic and assumes the same value as at z_1, then $f(z)$ is periodic, and has a period of modulus a.*

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THEOREM B. *If $f(z)$ is a uniform analytic function, and if every z_1 at which $f(z)$ is analytic belongs to a sequence of non-coincident points in arithmetic progression, at each of which $f(z)$ is analytic and assumes the same value as at z_1, then $f(z)$ is periodic.*

In A, the condition that $z_1 - z_2$ have a constant modulus can be replaced by the condition that it have a constant argument, or, for that matter, a constant real or pure imaginary part. Also, in both theorems, the condition that $f(z)$ be uniform is not essential, and is made only to keep cumbersome statements out of the proof.

Theorems similar to the above can be proved for other peculiarities than periodicity. Many types of functional equations can be studied from the point of view of this paper.

We shall use the following lemma.

LEMMA. *Given a non-constant uniform analytic function $f(z)$, and a transformation which carries every z_1 at which $f(z)$ is analytic into a z_2 at which $f(z)$ is analytic and assumes the same value as at z_1, there exists an analytic function $\varphi(z)$ such that $z_2 = \varphi(z_1)$ for a set of points z_1 dense in some area. For every z in this area, $f[\varphi(z)] = f(z)$.*

If $f(z)$ is analytic at z and if $f'(z) \neq 0$, there are circles with z as center inside of which $f(z)$ is analytic and assumes no value more than once. For any z, one of these circles —call it c—has a maximum (perhaps infinite) radius. Those maximum circles c for the center of which, $z = x + iy$, both x and y are rational, are countable. Let them be arranged in a sequence

$$c_1, c_2, \ldots, c_n, \ldots.$$

It is easy to see that every z for which $f(z)$ is analytic and $f'(z) \neq 0$ is interior to some c_n.

Take now any circle γ inside of which $f(z)$ is analytic and assumes no value more than once. For every z_1 in γ, the hypothesis of the lemma provides a z_2. Those points z_2 at which $f'(z_2) = 0$ are countable, for the zeros of an analytic function which is not identically zero are isolated. Every other z_2 lies within some c_n; associate with its z_1 the c_n of smallest n in which z_2 lies. In this way those points z_1 of γ for which $f'(z_2) \neq 0$ are separated into a countable number of sets. Not all of these sets can be nowhere dense, for their sum differs from all of γ by a countable set.

Hence there is a c_n and a set of points—call it M—dense in some area within γ, such that for any z_1 of M, z_2 lies within c_n.

Take, and hold fast momentarily, any z_1 of M which is the center of some circle—call it Γ—in which M is dense. As $f(z_1) = f(z_2)$, we can take Γ so small that $f(z)$ assumes no value in Γ which it does not assume within c_n. To associate with every point of Γ that point of c_n at which $f(z)$ assumes the same value is to define a $\varphi(z)$ which is analytic in Γ. Furthermore, for a set of points z_1 dense in Γ, $z_2 = \varphi(z_1)$. This proves the lemma.

It is now easy to prove Theorem A. Because $|z_2 - z_1|$ is constant, the modulus of $\varphi(z)$ is constant for a set of points dense in Γ. As the modulus of $\varphi(z)$ is continuous, it is constant throughout Γ. An analytic function with a constant modulus is a constant. Let h be the constant value of $\varphi(z)$; of course $h \neq 0$. Then $f(z+h) = f(z)$ for every z in Γ, and hence throughout the domain of existence of $f(z)$.

Considering Theorem B, suppose that, for every z_1 at which $f(z)$ is analytic, a number $h(z_1) \neq 0$ exists such that $f(z)$ is analytic, and equal to $f(z_1)$, at every $z_1 + mh(z_1)$ $(m = 1, 2, \ldots)$.

According to the lemma, there exists a circle Γ, a set M dense in Γ, and an analytic $\varphi(z)$, such that $\varphi(z_1) = z_1 + h(z_1)$ for every z_1 in M.

Suppose that $\varphi(z) - z$ is not constant. Barring out the trivial case in which $f(z)$ is constant, we may assume that neither $f(z)$ nor $\varphi(z) - z$ assumes any value more than once in Γ; this can always be brought about by replacing Γ by some circle within Γ. Let Γ' and Γ'' be two circles concentric with Γ, and of radii respectively one-half and one-quarter that of Γ.

Consider the function

$$\frac{z}{m} + [\varphi(z) - z],$$

where m is any positive integer. As m increases this function approaches $\varphi(z) - z$ uniformly in Γ, so that, for m large enough, it assumes no value more than once in Γ', but assumes in Γ' every value which $\varphi(z) - z$ assumes in Γ''. The values assumed in Γ'' certainly cover some circle Δ.

Consequently, for m large enough, the function

$$\psi_m(z) = z + m[\varphi(z) - z]$$

maps Γ' conformally, in a one-to-one manner, upon a region Σ_m which covers a circle Δ_m, of radius m times that of Δ.

Let z be in Γ'. Associate with $\psi_m(z)$ the number $f(z)$. This defines a $g_m(z)$, analytic in Σ_m. Now $\psi_m(z)$ maps those points of M which lie in Γ' into a set of points dense in Σ_m. Since $\psi_m(z_1) = z_1 + mh(z_1)$ for z_1 in M, $f(z)$ and $g_m(z)$ are equal for a set of points dense in Σ_m, and therefore throughout Σ_m.

As $f(z)$ assumes no value more than once in Γ', $f(z)$ assumes no value more than once in Σ_m. Consider any z_1 of M such that $\psi_m(z_1)$ is very close to the center of Δ_m. Since $h(z) = \varphi(z) - z$ for z in M, $h(z)$ is bounded for z at once in M and in Γ'. Hence if m is large, so that Δ_m is large, $\psi_m(z_1) + h(z_1)$, at which $f(z)$ takes the same value as at $\psi_m(z_1)$, will lie in Δ_m, and so in Σ_m. This contradiction with the statement at the head of the paragraph proves that $\varphi(z) - z$ is constant; it is not zero, for no $h(z)$ is zero. As $f[\varphi(z)] = f(z)$ for z in Γ, $f(z)$ is periodic.

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