ON TRANSFORMABLE SYSTEMS AND COVARIANTS OF ALGEBRAIC FORMS*

BY C. C. MACDUFFEE

1. *Introduction.*—The purpose of this article is to give a rigorous demonstration of an important theorem in the theory of covariants of algebraic forms in p variables; namely, that if $(G_1, \ldots, G_h) = (0, \ldots, 0)$ is an invariantive property, the G_i being polynomials in the coefficients of the forms, there exists a set $V_1, \ldots, V_\nu$ of relative covariants in $p-1$ sets of cogredient variables, such that $(V_1, \ldots, V_\nu) = (0, \ldots, 0)$ when and only when $(G_1, \cdots, G_h) = (0, \cdots, 0)$. The corresponding theorem for invariants, i.e., for $h = 1$, is given by Bôcher, *Introduction to Higher Algebra* (p. 232). Bôcher there states that "a projective relation expressed by the identical vanishing of a covariant or contravariant is typical of what we shall usually have when a single equation is not sufficient to express the condition." This paper shows that such a projective relation can in general be characterized by the simultaneous vanishing of a number of covariants.

The special case of this theorem for binary forms is mentioned without proof by Clebsch, *Binäre Formen* (p. 91). J. P. Gram in the MATHEMATISCHE ANNALEN (vol. 7), and J. Deruyts in a book entitled *Essai d'une Théorie Générale des Formes Algébriques* (Brussels, 1891) consider the characterization by covariants of particular forms defined by the holding of identical relations among their coefficients. Both proofs are incomplete, however, and Gram's method actually leads to a false result in case the given conditions are non-homogeneous.

2. *Transformable Systems.* Consider a system of l algebraic forms in p variables

$$(1) \qquad f_i(a_{i1}, a_{i2}, \cdots a_{iq_i}; x_1, x_2, \cdots, x_p) \equiv f_i, \quad (i = 1, \cdots, l)$$

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of degree r_i in the variables. The linear homogeneous transformation

$$x_i = \sum_{j=1}^{j=p} \alpha_{ij} x_j', \qquad (i = 1, \cdots, p) \tag{2}$$

of determinant $|\alpha| \neq 0$ induces upon the coefficients of the forms (1) a system of transformations

$$a_{ij}' = \sum_{k=1}^{k=q_i} \beta_{ijk} a_{ik}, \qquad (j = 1, \cdots, q_i) \tag{3}$$

of determinants $|\beta_i| = |\alpha|^{k_i}$ where * k_i is an integer.

According to the usage of J. Deruyts,† a *transformable system* is defined as a system of linearly independent polynomials $F_1, \cdots, F_s$ in the coefficients a_{ij} which are transformed by (3) according to the law

$$F_i' = \sum_{j=1}^{j=s} \gamma_{ij} F_j, \qquad (i = 1, \cdots, s), \tag{4}$$

where the F_i' are the same functions of the primed coefficients a_{ij}' as the F_i are of the a_{ij}, and where the γ_{ij} are polynomials in the β_{ijk} and hence in the α_{ij}. In particular, (3) is a transformable system for every i.

It follows that the determinant $|\gamma|$ of (4) is a power of $|\alpha|$.‡ For if possible choose a set of numbers α_{ij} such that $|\alpha| \neq 0$ and $|\gamma| = 0$. It would then follow that the F_i' were linearly dependent, contrary to definition. Hence $|\gamma|$ is different from zero for all values of the α_{ij} for which $|\alpha| \neq 0$, and hence $|\gamma| = c|\alpha|^n$ where n is a positive integer or zero. Now choose $\alpha_{ij} = \delta_{ij}$ and (2) becomes the identity transformation. Then (3) and (4) likewise become identity transformations, so $c = 1$. The following theorem may now be proved.

THEOREM. *Let $G_1, \cdots, G_h$ be a system of polynomials in the a_{ij}, and $G_1', \cdots, G_h'$ the same functions of the a_{ij}' having the properties that*

(*a*) *It is possible to effect a transformation* (2) *making* $(G_1', \cdots, G_h') = (0, \cdots, 0)$ *when and only when* $(G_1, \cdots, G_h) = (0, \cdots, 0)$,

* Hurwitz, *Zur Invarianttheorie*, MATHEMATISCHE ANNALEN, vol. 45, pp. 381–404.

† BULLETINS DE L'ACADÉMIE DES SCIENCES DE BELGIQUE, (3), vol. 32 (1896), p. 82. Deruyts requires that the F_i be homogeneous, and does not require that they be linearly independent.

‡ Proved more at length by Deruyts, loc. cit., p. 434

(*b*) *If* $(G_1, \cdots, G_h) = (0, \cdots, 0)$, *then* $(G_1', \cdots, G_h') = (0, \cdots, 0)$ *is true for all transformations* (2).

Then there exists a transformable system $(F_1, \cdots, F_s)$ *such that* $(F_1, \cdots, F_s) = (0, \cdots, 0)$ *when and only when* $(G_1, \cdots, G_h) = (0, \cdots, 0)$.

The case where $(G_1, \cdots, G_h) = (0, \cdots, 0)$ for no set of values of the a_{ij} may be disposed of first. In this case the transformable system may consist of the integer 1, which is never zero and is undisturbed by the transformation (2).

It is then permissible to assume that in the remaining case there actually exists at least one set of coefficients a_{ij} for which $(G_1, \cdots, G_h) = (0, \cdots, 0)$. If there are linear relations between the polynomials $G_1, \cdots, G_h$, we shall henceforth consider a subset $G_1, \cdots, G_g$ which are linearly independent and have the property that all the G_i which have been discarded are linear combinations with constant coefficients of them. Evidently $(G_1, \cdots, G_g) = (0, \cdots, 0)$ when and only when $(G_1, \cdots, G_h) = (0, \cdots, 0)$.

From relations (3) we can express $G_1', \cdots, G_g'$ in the form

$$G_i' = \sum_{j=1}^{j=t} \zeta_{ij} H_j, \qquad (i = 1, \cdots, g), \tag{5}$$

where the ζ_{ij} are polynomials in the β_{ijk} and therefore in the α_{ij}. If m represent the maximum degree of the G_i' in the a_{ij}', then the degree of every H_j in the a_{ij} will be $\leqq m$.

Choose the α_{ij} so that (2) becomes the identity transformation. Then (3) likewise become identity transformations and (5) becomes

$$G_i' = G_i = \sum_{j=1}^{j=t} c_{ij} H_j, \qquad (i = 1, \cdots, g),$$

where the c_{ij} are constants. Since the G_i are by hypothesis linearly independent, the rank of the matrix $(c_{ij}) = c$ is g. We assume that our notation is so chosen that the first g columns of the matrix form a determinant which is different from zero, and solve for $H_1, \cdots, H_g$ in terms of $G_1, \cdots, G_g, H_{g+1}, \cdots, H_t$. The relations (5) may, by the substitution of these values, be made to assume the form

$$G_i' = \sum_{j=1}^{j=g} \xi_{ij} G_j + \sum_{j=g+1}^{j=t} \eta_{ij} H_j, \quad (i = 1, \cdots, g).$$

Moreover we may assume that there exists no linear relation with constant coefficients between the polynomials G_j and H_j, or between the columns of the matrix η; for in either case a condensation would be possible reducing the number of functions H_j.

It will now be shown that every H_j vanishes for such values of the a_{ij} as make $(G_1, \cdots, G_g) = (0, \cdots, 0)$. For if c_{ij} is any particular set of values of the a_{ij} which reduce $(G_1, \cdots, G_g)$ to $(0, \cdots, 0)$ and the H_j to constants $\bar{H}_j$, we have by hypothesis $(G_1', \cdots, G_g') = (0, \cdots, 0)$, and therefore

$$\sum_{j=g+1}^{j=t} \eta_{ij}\bar{H}_j = 0, \qquad (i = 1, \cdots, g).$$

Since the columns of this matrix are linearly independent, we have $\bar{H}_j = 0$ for $j = g + 1, \cdots, t$.

Since the set of polynomials $(G_1, \cdots, G_g, H_{g+1}, \cdots, H_t) = (0, \cdots, 0)$ when and only when $(G_1, \cdots, G_g) = (0, \cdots, 0)$, we denote the augmented set by $(G_1, \cdots, G_t)$ and obtain, just as before, the relation

$$G_i' = \sum_{j=1}^{j=t} \xi_{ij}' G_j + \sum_{j=t+1}^{j=u} \eta_{ij}' H_j, \quad (i = 1, \cdots, t). \tag{6}$$

We consider this system of equations to be so reduced that no H_j is expressible linearly with constant coefficients in terms of the G_j and the remaining H_j, and that the columns of the matrix η' are linearly independent. Moreover, the degree of every H_j is $\leqq m$, the maximum degree of the original polynomials $G_1, \cdots, G_g$. Proceeding in this way, after a finite number of steps we reach a relation of the form (6) in which every H_j is identically zero; for there are but a finite number of linearly independent polynomials of degree $\leqq m$ in a finite number of variables a_{ij}. This final set of G's form our transformable system $F_1, \cdots, F_s$. For $(F_1, \cdots, F_s) = (0, \cdots, 0)$ when and only when $(G_1, \cdots, G_g) = (0, \cdots, 0)$, which is true when and only when $(G_1', \cdots, G_g') = (0, \cdots, 0)$, which in turn is true when and only when $(F_1', \cdots, F_s') = (0, \cdots, 0)$.

3. *Absolute Covariants.* The elements γ_{ij} of the matrix γ of the transformation (4) are polynomials in the elements α_{ij} of the transformation (2). Hence γ may be considered as a function $T(\alpha)$ of the matrix α. As was noted by Deruyts,*

* Loc. cit., p. 437.

the matrix $\gamma = T(\alpha)$ of a transformable system of linearly independent polynomials obeys the functional equation

$$T(\alpha\beta) = T(\beta) \cdot T(\alpha).$$

Such matrices have been called *invariant matrices*, and the operation T an *invariant operation* by Schur,* who derived many of their properties.

If we denote by F the matrix

$$\begin{bmatrix} F_1 \\ \cdot \\ \cdot \\ \cdot \\ F_s \end{bmatrix},$$

then (4) may be written $F' = \gamma F$. If $p = (p_{ij})$ denote a non-singular constant matrix, and if we set $\bar{F} = pF$, then the functions $\bar{F}_i$ are linear combinations of the F_j having the property that $(\bar{F}_1, \cdots, \bar{F}_s) = (0, \cdots, 0)$ when and only when $(F_1, \cdots, F_s) = (0, \cdots, 0)$. From composition of transformations we have $\bar{F}' = pF' = p\gamma F = p\gamma p^{-1}\bar{F}$. Hence the matrix $p\gamma p^{-1}$ is *equivalent* to the matrix γ in the sense that there exists a set of functions $\bar{F}_i$ which are transformed by $p\gamma p^{-1}$ and which all vanish when and only when all the F_i vanish. Schur calls a matrix $T(\alpha)$ reducible if there exists a matrix equivalent to it which reduces into distinct blocks, i.e., if

$$pT(\alpha)p^{-1} = \begin{pmatrix} M_1 & \\ & M_2 \end{pmatrix}.$$

The blocks M_1 and M_2 are themselves invariant matrices.† An invariant matrix is reducible into irreducible invariant matrices in essentially but one way,‡ and the elements of an irreducible matrix are linearly independent homogeneous polynomials in the coefficients α_{ij} of the original transformation

* *Ueber eine Klasse von Matrizen die sich einer gegebenen Matrix zuordnen lassen*, Berlin thesis, 1901, p. 5. In this paper the functional equation is taken as $T(xy) = T(x) \cdot T(y)$. The transposes of these matrices satisfy the equation given above.

† Loc. cit., p. 6.

‡ Loc. cit., p. 39.

(2).* An irreducible matrix cannot transform a set of linearly dependent functions except they all be zero.†

We now consider the matrix γ to be replaced by an equivalent matrix consisting of irreducible blocks. Suppose $\varphi = T(\alpha)$ is one of these blocks and K_i for $i = 1, \cdots, t$, linear combinations of the F_i which are transformed by this invariant matrix. By showing that the K_i are coefficients of a covariant V_1, we show that the condition $(F_1, \cdots, F_s) = (0, \cdots, 0)$ can be replaced by the equivalent condition $(V_1, \cdots, V_v) = (0, \cdots, 0)$.

It is evident from (2) that if $x_i^{(j)}$ are p systems of variables cogredient with the x_i, then

$$(7) \quad \begin{bmatrix} x_1^{(1)} & x_1^{(2)} & \cdots & x_1^{(p)} \\ x_2^{(1)} & x_2^{(2)} & \cdots & x_2^{(p)} \\ \cdot & \cdot & \cdot & \cdot \\ x_p^{(1)} & x_p^{(2)} & \cdots & x_p^{(p)} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1p} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2p} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{p1} & \alpha_{p2} & \cdots & \alpha_{pp} \end{bmatrix} \begin{bmatrix} x_1^{(1)\prime} & x_1^{(2)\prime} & \cdots & x_1^{(p)\prime} \\ x_2^{(1)\prime} & x_2^{(2)\prime} & \cdots & x_2^{(p)\prime} \\ \cdot & \cdot & \cdot & \cdot \\ x_p^{(1)\prime} & x_p^{(2)\prime} & \cdots & x_p^{(p)\prime} \end{bmatrix}.$$

Denoting these matrices by X and X' respectively, we have $X = \alpha X'$. Now since T is an invariant operation, where $T(\alpha) = \varphi$, we have $T(X) = T(X')\cdot\varphi$, or, denoting the elements of $T(X)$, $T(X')$ by X_{ij}, X_{ij}', respectively, we have ‡

$$X_{ij} = \sum_{k=1}^{k=t} \varphi_{kj} X_{ik}', \qquad (i, j = 1, \cdots, t).$$

Since φ is irreducible, the functions X_{ij} are all linearly independent, for X_{ij} is the result of substituting $x_i^{(j)}$ for α_{ij} in the element of φ lying in the ith row and jth column. Now

$$\begin{aligned} \sum_{j=1}^{j=t} K_j X_{ij} &= \sum_{j=1}^{j=t} K_j \sum_{k=1}^{k=t} \varphi_{kj} X_{ik}' \\ &= \sum_{k=1}^{k=t} X_{ik}' \sum_{j=1}^{j=t} \varphi_{kj} K_j = \sum_{k=1}^{k=t} K_k' X_{ik}'. \end{aligned}$$

Therefore $\sum_{j=1}^{j=t} K_j X_{ij}$ is an absolute covariant of the system of forms (1) for every value of $i = 1, \cdots, t$.

4. *Relative Covariants.* We have shown that $\sum_{j=1}^{j=t} K_j X_{ij} \equiv V_i$ for $i = 1, \cdots, t$ is transformed into itself by every transformation of type (2). Moreover the X_{ij} are linearly independent functions of p sets of variables $x_i^{(j)}$ cogredient with x_i. We shall now show that *there exists a relative covariant with the same coefficients as V_i in $p - 1$ sets of cogredient variables.*

* Loc. cit., p. 56.

† Loc. cit., p. 70.

‡ Compare Deruyts, loc. cit., p. 438.

Let $|\alpha|^\mu$ be the highest power of $|\alpha|$ that is contained in every element of the matrix φ. We denote the quotient $\phi_{ij}/|\alpha|^\mu$ by ψ_{ij}, and the result of substituting $x_i^{(j)}$ for α_{ij} in $|\alpha|$ by A, and the result of substituting $x_i^{(j)}$ for α_{ij} in ψ_{ij} by Y_{ij}. Then $X = A^\mu Y$, and $X' = A'^\mu Y'$, where A' and Y' are the same functions of the $x_i^{(j)\prime}$ that A and Y are of the $x_i^{(j)}$. From (7) we have $A = |\alpha| A'$. It follows that

$$\begin{aligned}\textstyle\sum_{k=1}^{k=t} K_k' Y_{ik}' &= A'^{-\mu} \textstyle\sum_{k=1}^{k=t} K_k' X_{ik}' = A'^{-\mu} \textstyle\sum_{j=1}^{j=t} K_j X_{ij} \\ &= A^\mu A'^{-\mu} \textstyle\sum_{j=1}^{j=t} K_j Y_{ij} = |\alpha|^\mu \textstyle\sum_{j=1}^{j=t} K_j Y_{ij}.\end{aligned}$$

Hence in this case $\sum_{j=1}^{j=t} K_j Y_{ij}$ is a relative covariant of weight μ for $i = 1, \cdots, t$. It will now be shown that for some i, $\sum_{k=1}^{k=t} K_k Y_{ik}$ can be expressed in $p-1$ sets of cogredient variables. Since at least one function

$$Y_{ik}(x_1^{(1)}, \cdots, x_1^{(p)}, x_2^{(1)}, \cdots, x_2^{(p)}, \cdots, x_p^{(1)}, \cdots, x_p^{(p)})$$

does not contain A as a factor, and since A is irreducible, there is at least one set of constants c_{ij} such that $C = |c_{ij}| = 0$, and

$$Y_{ik}(c_{11}, \cdots, c_{1p}, c_{21}, \cdots, c_{2p}, \cdots, c_{p1}, \cdots, c_{pp}) \neq 0.$$

There is a linear relation with constant coefficients between the columns of C, say

$$\kappa_1 c_{i1} + \kappa_2 c_{i2} + \cdots + \kappa_p c_{ip} = 0, \quad (i = 1, \cdots, p),$$

where at least one κ, say κ_1, is different from zero. Hence

$$c_{i1} = \lambda_2 c_{i2} + \lambda_3 c_{i3} + \cdots + \lambda_p c_{ip}, \quad (i = 1, \cdots, p). \tag{8}$$

Therefore

$$Z_{ik} \equiv Z_{ik}(c_{12}, \cdots, c_{1p}, c_{22}, \cdots, c_{2p}, \cdots, c_{p2}, \cdots, c_{pp}) \neq 0,$$

where Z_{ik} is obtained from $Y_{ik}(c_{kl})$ by substitution (8).

Hence if we make the substitution

$$x_i^{(1)} = \lambda_2 x_i^{(2)} + \lambda_3 x_i^{(3)} + \cdots + \lambda_p x_i^{(p)}, \quad (i = 1, \cdots, p) \tag{9}$$

upon Y_{ik}, we obtain

$$Z_{ik}(x_1^{(2)}, \cdots, x_1^{(p)}, x_2^{(2)}, \cdots, x_2^{(p)}, \cdots, x_p^{(2)}, \cdots, x_p^{(p)}) \not\equiv 0,$$

for in particular it is not zero when $x_i^{(j)} = c_{ij}$, $(i, j = 1, \cdots, p)$. Transformation (9) does not destroy the cogrediency of the variables. Since $Z_{ik} \not\equiv 0$, it follows that Z_{ik} for $k = 1, \cdots, t$ are linearly independent. For we have

$$\textstyle\sum_{k=1}^{k=t} K_k Z_{ik} A^\mu = \sum_{j=1}^{j=t} K_j' Z_{ij}' A'^\mu.$$

The polynomials $Z_{ik}A^{\mu}$ are transformed by the adjoint of φ, and according to the theorem of Schur mentioned above, a matrix which transforms a system of linearly dependent polynomials which are not all zero is reducible. Hence if the $Z_{ik}A^{\mu}$ were linearly dependent, the matrix φ would be reducible, contrary to our assumption.

5. *Conclusion.* We have proved the following theorem:

THEOREM. *If $G_1, \cdots, G_h$ are a system of polynomials in the a_{ij}, and $G_1', \cdots, G_h'$ the same functions of the a_{ij}' such that*

$$(G_1, \cdots, G_h) = (0, \cdots, 0)$$

is an invariantive property, then there exists a set of rational integral relative covariants $V_1, \cdots, V_v$ in $p-1$ sets of cogredient variables such that $(V_1, \cdots, V_v) = (0, \cdots, 0)$ when and only when $(G_1, \cdots, G_h) = (0, \cdots, 0)$.

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A CORRECTION

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In my paper in the November number of this BULLETIN (vol. 28, No. 8), the word *integers* should be replaced by the word *rationals* in line 16 of page 398 and in the table on page 399.