THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The two hundred sixteenth regular meeting of the Society was held at Columbia University on Saturday, April 23, 1921, extending through the usual morning and afternoon sessions. The attendance included the following sixty-seven members:

Professor R. C. Archibald, Professor R. A. Arms, Dr. Charlotte C. Barnum, Professor A. A. Bennett, Professor E. G. Bill, Professor G. D. Birkhoff, Professor H. F. Blichfeldt, Dr. R. F. Borden, Professor R. W. Burgess, Professor Abraham Cohen, Dr. G. M. Conwell, Professor Louise D. Cummings, Professor D. R. Curtiss, Professor L. E. Dickson, Dr. Jesse Douglas, Professor L. P. Eisenhart, Professor W. B. Fite, Mr. R. M. Foster, Mr. Philip Franklin, Mr. B. P. Gill, Dr. T. H. Gronwall, Dr. C. C. Grove, Dr. C. M. Hebbert, Professor E. R. Hedrick, Lieutenant R. S. Hoar, Professor L. S. Hulburt, Professor W. A. Hurwitz, Mr. S. A. Joffe, Professor Edward Kasner, Professor O. D. Kellogg, Dr. E. A. T. Kircher, Professor J. R. Kline, Mr. Harry Langman, Professor Florence P. Lewis, Mr. L. L. Locke, Mr. John McDonnell, Professor H. F. MacNeish, Professor L. C. Mathewson, Professor H. H. Mitchell, Professor G. W. Mullins, Dr. Almar Naess, Professor E. J. Oglesby, Professor W. F. Osgood, Professor F. W. Owens, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor L. W. Reid, Dr. C. N. Reynolds, Mr. L. H. Rice, Professor R. G. D. Richardson, Dr. J. F. Ritt, Dr. J. E. Rowe, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor L. L. Silverman, Professor P. F. Smith, Dr. J. M. Stetson, Miss Louise E. C. Stuerm, Professor K. D. Swartzel, Mr. H. S. Vandiver, Professor J. H. M. Wedderburn, Professor Mary E. Wells, Dr. Norbert Wiener, Miss Ella C. Williams, Professor H. S. White, Professor J. K. Whittemore, Professor Ruth G. Wood.

Professors L. E. Dickson and W. F. Osgood presided at the morning session, and Professor P. F. Smith, relieved by Professors H. S. White and Abraham Cohen, in the afternoon. The afternoon session opened with a one-hour lecture given by Professor W. A. Hurwitz at the invitation of the programme committee, on *Topics in the theory of divergent series.*

The Council announced the election of the following persons to membership in the Society:

Mr. Lewis Albert Anderson, Central Life Assurance Society, Des Moines;
Dr. Eugene Manasseh Berry, Purdue University;
Mr. Raymond Van Arsdale Carpenter, Metropolitan Life Insurance Company, New York;
Mr. James Douglas Craig, Metropolitan Life Insurance Company, New York;
Mr. Bernard Francis Dostal, Oberlin College;
Mr. George Graham, Central States Life Insurance Company, St. Louis;
Dr. Jacob Millison Kinney, Hyde Park High School, Chicago;
Mr. John Ruse Larus, Jr., Phoenix Mutual Life Insurance Company, Hartford;
Professor Daniel Acker Lehman, Goshen College;
Mr. Joseph Brotherton Maclean, Mutual Life Insurance Company, New York;
Mr. Franklin Bush Mead, Lincoln National Life Insurance Company, Fort Wayne;
Professor James Newton Michie, University of Texas;
Mr. Henry Moir, Home Life Insurance Company, New York;
Mr. William Oscar Morris, North American Life Insurance Company, Chicago;
Mr. Oliver Winfred Perrin, Penn Mutual Life Insurance Company, Philadelphia;
Professor Harris Rice, Worcester Polytechnic Institute;
Miss Jessie M. Short, Reed College;
President Wen Shion Tsu, Nanyang Railway and Mining College;
Professor Buz M. Walker, Mississippi Agricultural and Mechanical College;
Mr. Robert Montague Webb, Kansas City Life Insurance Company;
Mr. Archibald Ashley Welch, Phoenix Mutual Life Insurance Company, Hartford;
Professor Frank Edwin Wood, Michigan Agricultural College;
Mr. Joseph Hooker Woodward, Equitable Life Assurance Society, New York;
Mr. William Young, New York Life Insurance Company.

Eleven applications for membership were received.

It was voted by the Council to accept the invitations received at the preceding meeting to hold the next annual meeting of the Society at Toronto during the Christmas holidays in connection with the meeting of the American Association for the Advancement of Science, and to hold a meeting of the Society with the Chicago Section at Chicago in 1922. The time of the Chicago meeting was not determined.

Professor G. D. Birkhoff was reelected a member of the Editorial Committee of the TRANSACTIONS for a term of three years beginning October 1, 1921.

Professor P. F. Smith presented to the Council the recommendations of the committee authorized at the last meeting to consider policies for conserving the interests of the Society in foreign countries for the year 1921. These recommendations were adopted and the committee discharged with the thanks of the Council. It was voted to request the President to appoint a committee to consider and report to the Council a policy for 1922 with regard to these and related matters. A committee was also authorized to prepare nominations for officers and other members of the Council to be elected at the annual meeting in December.

It was decided that on the request of the contributor twenty-five reprints (with covers) of articles in the BULLETIN be given without charge, and that a statement be made to authors concerning the cost of reprints with a request that they forego this privilege of free copies unless they can make good use of them.

The usual luncheon and dinner were held at the Faculty Club.

The titles and abstracts of the papers read at this meeting follow below. Professor Huntington's paper was read by Professor Kellogg, and Professor Schwatt's papers by Professor Kline; the papers of Professors Poor, Lipka, Jackson, and McMahon, Dr. Walsh, Professor Fischer, Dr. Post and Professor Coble were read by title.

1. Professor W. F. Osgood: *On the gyroscope.*

When a gyroscope moves under the action of any forces, its axis describes a cone at a definite rate. The discovery of a form of the intrinsic relation between these three factors is the chief result of this paper. Since one point, O, in the axis is fixed, the applied forces (the constraint at O being omitted) can be replaced (a) by a single force, $\mathfrak{F}$, acting at a point ζ of the axis at unit distance from O and at right angles to $O\zeta$; (b) by a couple, $\mathfrak{N}$, whose plane is perpendicular to the axis. The contribution of the cone, $\mathfrak{K}$, lies in its *bending*, κ. By this is meant the rate at which the tangent plane at O turns when ζ describes its path, $\mathfrak{C}$, on the unit sphere with unit velocity. Finally, let v denote the velocity with which ζ describes its path $\mathfrak{C}$ in the case of the actual motion, and let r denote the angular velocity of the gyroscope about its axis. Then

$$Av\frac{dv}{ds} = T, \qquad A\kappa v^2 + Crv = Q, \qquad C\frac{dr}{dt} = N,$$

where s denotes the arc of $\mathfrak{C}$, T and Q the components of $\mathfrak{F}$ along the tangent to $\mathfrak{C}$ and the normal perpendicular to $O\zeta$

respectively, and C and A are the moments of inertia about the axis $O\zeta$ of the gyroscope and an axis through O perpendicular to $O\zeta$. The foregoing relations are *intrinsic*, i.e. independent of any particular choice of coordinates for the gyroscope. They apply directly to the motion of the axis. In the case most important in practice, $N = 0$, and the angular velocity, r, reduces to a constant, ν. There remain, then, merely the first two equations, in which r is replaced by ν.

2. Professor H. S. White: *Seven points in space and the eighth associated point.*

The construction given by O. Hesse in CRELLE'S JOURNAL, vol. 20, is interwoven with the relations of the eight points to three quadric surfaces on which they lie, and to one auxiliary quadric. Using the same construction by points and lines, this paper avoids all reference to quadrics. By algebraic formula in invariant form it proves the uniqueness of the derived eighth point and the symmetry of the completed set of eight, and it gives explicitly the equation of the derived point, covariant, of degree seven, in the given points.

3. Professor L. E. Dickson: *Most general composition of polynomials.*

This paper treats the problem to find three polynomials $f(x) \equiv f(x_1, \cdots, x_n)$, ϕ, F such that $f(x)\phi(\xi) = F(X)$, identically in $x_1, \cdots, x_n, \xi_1, \cdots, \xi_n$, when $X_1, \cdots, X_n$ are bilinear functions of these $2n$ independent variables. The author proves that this problem reduces to the case of a single polynomial f for which $f(x)f(\xi) = f(X)$. This case was treated by him in the COMPTES RENDUS DU CONGRÈS INTERNATIONAL DES MATHÉMATICIENS, Strasbourg, 1920, pp. 131–146. The new paper will appear in the COMPTES RENDUS.

4. Professor L. E. Dickson: *Number of real roots by Descartes's rule of signs.*

In this paper, the author gives a complete, elementary proof of Descartes's rule of signs for the number of real roots of an equation $f(x) = 0$ with real coefficients. The proof follows from the following fact, which is proved by induction. The number of variations of signs in the set $a_0, a_1, \cdots, a_n$ exceeds by a positive even integer the number of variations of signs in $b_0, \cdots, b_{n-1}$, if b_0 is any chosen number of the same sign as a_0, while the remaining b's are found as in the process of synthetic division by means of a positive multiplier.

5. Professor L. P. Eisenhart: *The Einstein solar field.*

This paper appears in full in the present number of this BULLETIN.

6. Professor J. K. Whittemore: *A special kind of ruled surface.*

In this paper are determined all ruled surfaces having the property that every pair of curved asymptotic lines cuts a constant length from a variable ruling. It is shown that these surfaces have an analogy with the right helicoid; in particular that all rulings of such a surface are parallel to a plane and that the parameter of distribution of such a surface is constant. It is proved that a surface of this type is completely determined by the choice of a plane curve, a straight line, and the constant parameter of distribution. Certain questions regarding these surfaces are considered.

7. Professor V. C. Poor: *On the theorems of Green and Gauss.*

This paper contains a definition for the divergence of a vector in terms of the point differential, namely:

$$\operatorname{div} u\, dP \times \delta P \wedge \vartheta P$$
$$= du \times \delta P \wedge \vartheta P + \delta u \times \vartheta P \wedge dP + \vartheta u \times dP \wedge \delta P,$$

with the proof for the uniqueness and the existence of the idea so defined. The statement and proof of the following very general form of Green's theorem is given:

$$\int I_1 \left(K\alpha \frac{d \operatorname{grad} \beta}{dP} - K\beta \frac{d \operatorname{grad} \alpha}{dP} \right) d\tau$$
$$= \int (K\alpha \operatorname{grad} \beta - K\beta \operatorname{grad} \alpha) \times u d\sigma.$$

Another theorem states that

$$\int \left(\frac{d\alpha}{dP} u \right) x d\tau = - \int u \times u \cdot \alpha x d\sigma - \int \alpha \frac{dx}{dP} u d\tau.$$

8. Dr. J. E. Rowe: *Pressure distribution around a breech-block.*

The breech-block of a certain gun is circular in form and is held in place by a thread on its circumference. When the gun is fired the gas presses against a circular portion of this breech-block, whose area is equal to the cross-sectional area of the powder chamber of the gun, and which is eccentrically situated with respect to the breech-block. The purpose of the paper is to derive the intensity of the pressure on any part of the thread from a given powder pressure.

9. Professor E. V. Huntington: *The mathematical theory of proportional representation. Third paper.*

The author's method of apportionment of representatives, which was presented at the December and February meetings, and which is now known as the "method of equal proportions," solves the problem of apportionment by a direct and obvious comparison between the several states, without the use of the idea of total error. A satisfactory expression for total or average error may, however, be stated, as follows: $E = \sqrt{S/N}$, where $S = \Sigma[(a - \alpha)^2/a]$. Here a is the actual and α the theoretical number of representatives assigned to a typical state A, and N is the total number of representatives. This quantity E (or S) is the quantity which is minimized by the method of equal proportions. The quantity Q, proposed at the February meeting by Professor F. W. Owens, and minimized by the Willcox method of major fractions, may be written as $Q = \Sigma[(a - \alpha)^2/\alpha]$. These quantities S and Q and other similar expressions are compared, and reasons given for preferring S.

10. Professor F. W. Owens: *On the apportionment of representatives. Second paper.*

This paper is supplementary to the author's paper on the same subject read at the February meeting of the Society and contains further developments and illustrations of the results of the different methods.

11. Professor Joseph Lipka: *On the geometry of motion in a curved space of n dimensions.*

In this paper are derived a number of geometric properties of certain systems of ∞^{2n-1} curves, termed q systems, in a curved space of n dimensions. These systems include dynamical trajectories, brachistochrones, catenaries, and velocity curves, under conservative forces or under arbitrary positional fields of force as special cases. The properties are expressed in terms of osculating and hyperosculating geodesic surfaces and hyperosculating geodesic circles and the loci of their centers of curvature.

12. Professor Dunham Jackson: *Note on an irregular expansion problem.*

This paper will appear in full in the January number of this BULLETIN.

13. Professor James McMahon: *Hyperspherical goniometry, with applications to the theory of correlation for n variables.*

The chief suggestion for this paper is contained in an article by Karl Pearson in BIOMETRIKA (vol. 11, p. 237). In Part I, the formulas of spherical trigonometry are generalized for the hypersphere in n dimensions, the underlying hypergeometry being first sketched in a form adapted to the purpose in hand. In Part II, a far-reaching connection between the geometry of the hypersphere and the theory of correlation for n variables is established by linearly transforming the 'ellipsoids' of equal frequency (on an n-dimensional correlation chart) into hyperspherical surfaces. The generalized formulas of hyperspherical goniometry are then applied to furnish easy proofs of various new and old theorems in multiple and partial correlation (including the relations referred to above), and to point the way to further developments.

14. Dr. J. L. Walsh: *On the location of the roots of polynomials.*

This paper contains a proof of the following theorem: If the coefficients of a polynomial $f(z)$ are linear in each of the parameters $\alpha_1, \alpha_2, \cdots, \alpha_k$ and are symmetric (but not necessarily homogeneous) in these parameters, and if the points α lie in a circular region C, then for any value of z we may replace the α's by k parameters *which coincide in* C and this without changing the value of $f(z)$.

This theorem has various applications; it can be easily proved that if all the k roots of a polynomial $\varphi(z)$ lie on or within a circle C whose center is α and radius r, then all the roots of

$$A_0\varphi(z) + A_1\varphi'(z) + A_2\varphi''(z) + \cdots + A_k\varphi^{(k)} = 0$$

(when the A's are arbitrary constants) lie on or within the circles of radius r whose centers are the roots of

$$A_0(z-\alpha)^k + kA_1(z-\alpha)^{k-1} + k(k-1)A_2(z-\alpha)^{k-2} + \cdots + k(k-1)\cdots 1\cdot A_k = 0.$$

15. Professor C. A. Fischer: *The kernel of the Stieltjes integral corresponding to a completely continuous transformation.*

In the 1918 ACTA MATHEMATICA, F. Riesz has proved that a linear transformation which is completely continuous, $A[f]$, can be decomposed into the sum of two orthogonal transformations, $A_1[f]$ and $A_2[f]$, such that the transformation $B_1[f] = f(x) - A_1[f]$ has an inverse, and the equations $B_2^n[\varphi] = 0$ and $B^n[\varphi] = 0$ have the same solutions. He has also proved in an earlier paper that every linear transformation can be put into the form

$$A[f] = \int_a^b f(y)d_yK(x, y).$$

In this BULLETIN, October, 1920, Professor Fischer gave the conditions which $K(x, y)$ must satisfy in order that $A[f]$ be completely continuous. In the present paper he has proved that the function $K_2(x, y)$, corresponding to Riesz' $A_2[f]$, can be put into the form

$$K_2(x, y) = \sum_{i=1}^{n} \varphi_i(x)\psi_i(y),$$

where the functions $\varphi_i(x)$ are solutions of the equation $B^\nu[\varphi] = 0$, ν being a positive integer such that every solution of $B^{\nu+1}[\varphi] = 0$ is also a solution of $B^\nu[\varphi] = 0$. The applications of this theorem to Stieltjes' integral equations of both the first and second kinds are then discussed.

16. Dr. E. L. Post: *On a simple class of deductive systems.*

In the present paper the author considers a general class of deductive systems involving primitive functions of but one argument, and solves for all such systems the following problem: to find a method for determining in a finite number of steps whether a given enunciation of the system can or cannot be asserted by means of the postulates of the system. The solution is obtained by directly analyzing the way in which assertions are generated from the primitive assertions by the rules of deduction of the system, and gives the beginning of a distinct alternative to the truth-table method which was introduced in its simplest form in a previous paper.

17. Professor W. A. Hurwitz: *Topics in the theory of divergent series.*

This paper will appear in full in the January number of this BULLETIN.

18. Dr. Norbert Wiener and Professor F. L. Hitchcock: *A new vector method in integral equations.*

Starting with the Fredholm equation

$$u(x) - \lambda \int_a^b u(y)K(x, y)dy = v(x)$$

the authors say that $f(x)$ is *conjoint* to $g(x)$ if

$$f \mid g = \int_a^b f(x)g(x)dx - \lambda \int_a^b \int_a^b f(x)g(y)K(x, y)dxdy = 0.$$

They then develop methods of rendering a closed set of functions conjoint by pairs, and they show that if $\{\varphi_n\}$ is a closed conjoint set of functions and

$$\sum_n \varphi_n(x) \frac{\int_a^b v(x)\varphi_n(x)dx}{\varphi_n \mid \varphi_n}$$

converges uniformly, it represents a solution of the Fredholm equation. Further extensions of this method give series for u that converge in the mean, and enable the determination of the characteristic numbers.

The homogeneous equation

$$u(x) - \lambda \int_a^b u(y)K(x, y)dy = 0$$

is solved by the determination of a function orthogonal to every

$$f(x) - \lambda \int_a^b f(y)K(y, x)dy.$$

A similar method is developed for the solution of the non-homogeneous equation.

All the methods of the present paper are immediate generalizations of vector methods.

19. Dr. Jesse Douglas: *On a certain type of system of* ∞^2 *curves.*

The ∞^2 straight lines of the plane form a family in which the sum of the angles of every triangle is equal to two right angles. The most general families of ∞^2 curves having the same property are the loxodromes, that is those formed of the totality of isogonal trajectories of a given simply infinite system of curves. These are characterized by differential equations of the form

$$y'' = (A + By')(1 + y'^2),$$

where A and B are any functions of x and y such that

$$A_y - B_x = 0.$$

A natural generalization is to inquire as to the systems of ∞^2 plane curves which are such that in every triangle formed of three curves of the system, the sum of the angles differs from π by an amount proportional to the area. The author bases the analysis on the equations of variation, showing that it is sufficient to restrict consideration to triangles of which two sides are formed of infinitesimally adjacent curves, while the third is finitely divergent from these two. The result is that the differential equation of the family must be of the form

$$y'' = (A + By')(1 + y'^2),$$

with $A_y - B_x = k$, where k is the constant of proportionality. A synthetic construction for curve systems of the type so defined is obtained by starting with any system of ∞^1 curves, and drawing trajectories to cut them, not under constant angle, but so that the angle increases by k times the element of area ydx under the trajectory.

The above considerations are easily extended to systems of curves on a general surface.

20. Dr. Jesse Douglas: *Concerning Laguerre's inversion.*

In a previous paper (this BULLETIN, July, 1920) the author made use of a certain representation of the (directed) lines of the plane on the cylinder $x^2 + y^2 = 1$; namely, the line of Hessian coordinates (ω, p) corresponding to the point $(x, y, z) = (\cos \omega, \sin \omega, p)$. The aspect of this representation fundamental for the present paper is that the (oriented) circles of the plane correspond to the plane sections of the cylinder. By a projective transformation of space the cylinder may be replaced by a general quadric cone. It results that the line transformations in the plane that convert circles into circles form a group, of seven parameters, of which the geometry is identical with projective geometry on a quadric cone.

The last statement is dual to the fact that the point transformations preserving cocircularity form a six-parameter group of which the geometry is identical with projective geometry on a sphere (or any ellipsoid), the identity between the two geometries being based on stereographic projection, which represents the circles of the plane as plane sections of the sphere. The representation described at the beginning thus appears as the dual of stereographic projection.

In the above G_6 of point transformations are included the ordinary inversions, which may be distinguished as corresponding to the perspectives of the sphere. On inquiring what corresponds in the G_1 of line transformations to the perspectives of the cylinder, one comes precisely upon a certain geometric construction described by Laguerre in a paper of 1882 (*Oeuvres*, vol. 2, p. 611). Laguerre studies the properties of his transformation from various points of view, bringing out especially the duality to ordinary inversion, but the aspects above developed appear to be new.

21. Professor J. R. Kline: *Closed connected point sets which are disconnected by the omission of a finite number of points.*

Suppose M is a closed connected point set such that if P is any point of M,

$$M - P = M_1 + M_2,$$

where M_1 and M_2 are two mutually exclusive sets neither of which contains a limit point of the other one. The author shows that M is a continuous curve in the sense of R. L. Moore, i.e. it is closed connected, and connected im kleinen. Suppose G is a closed connected set such that (1) if P is any point in G, $G - P$ is connected, (2) if P_1 and P_2 are any two distinct points of G, we have

$$G - P_1 - P_2 = G_1 + G_2,$$

where G_1 and G_2 are two mutually exclusive sets neither of which contains a limit point of the other one. It is shown that G is a simple closed curve.

22. Professor I. J. Schwatt: *The sum of a series as the solution of a differential equation.*

Boole (*A Treatise on Differential Equations,* 3d edition, pp. 441–450) considers the summation of series that are the solutions of differential equations of the special form

$$\sum_{k=0}^{n} f_k(D)e^{k\theta}u = 0.$$

In this paper a general method for the summation of series as the solutions of linear differential equations is developed and methods are given for solving the equations.

23. Professor I. J. Schwatt: *Method for the summation of a general case of a deranged series.*

Dirichlet, Riemann, Pringsheim and Scheibner have shown that if the terms of a convergent series are deranged, the sum of the resulting series is, in general, different from the sum of the given series. The author has treated two particular cases of deranged series (ARCHIV DER MATHEMATIK UND PHYSIK, vol. 24 (1915), p. 139 and GIORNALE DI MATHEMATICHE, vol. 54 (1916)) and considers in this paper the sum of a general case of a deranged series.

24. Professor I. J. Schwatt: *Higher derivatives of functions of functions.*

Several methods for finding the higher derivatives of functions of functions have been given but they are not in a form convenient for the purposes of application. To this seems to be due the fact that even the leading treatises on the calculus give the higher derivatives of only the simplest functions of functions, in most cases derived by special devices or by induction. Also in the expansion of functions these authors and others find as a rule by actual differentiation the first few derivatives and correspondingly obtain the first few terms of the expansion. In the course of this paper specific references are given. The author has developed methods which enable him to find the general higher derivative of functions of functions and their complete expansions.

25. Professor A. B. Coble: *A covariant of three circles.*

This paper appears in full in the present number of this BULLETIN.

R. G. RICHARDSON,
Secretary.