ROTATING CYLINDERS AND RECTILINEAR VORTICES.

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§ 1. *Rectilinear Vortex and Rotating Cylinder in a Stream of Incompressible Fluid.*

We shall assume that the rotation of the cylinder produces a circulation around the cylinder which may be approximately represented by placing a rectilinear vortex along the axis of the cylinder, a device which was adopted by Lord Rayleigh* in his paper "On the irregular flight of a tennis ball." Let V be the velocity of the stream, $2\pi k$ the circulation around the cylinder and $2\pi c$ the strength of the rectilinear vortex at the side of the cylinder. Assuming that the axis of this vortex is parallel to the axis of the cylinder, the motion is two-dimensional and we may represent the velocity potential ϕ and stream line function ψ as follows:

$$\phi + i\psi = U\left(z + \frac{a^2}{z} \right) + ik \log z + ic \log \frac{z - z_0}{z - z_1},$$

where a is the radius of the cylinder, $z = x + iy$ is a complex variable specifying the position of a point relative to the axis of the cylinder, z_0 specifies the position of the vortex, and z_1 that of its image.

Let (u, v) be the component velocities of the fluid at (x, y), (u_0, v_0) the component velocities of the vortex; then differentiating the above equation with regard to z we find that

$$u - iv = U\left(1 - \frac{a^2}{z^2} \right) + \frac{ik}{z} + ic\left[\frac{1}{z - z_0} - \frac{1}{z - z_1} \right].$$

In calculating (u_0, v_0) we ignore the infinite velocity produced by the vortex itself, consequently

$$u_0 - iv_0 = U\left(1 - \frac{a^2}{z_0^2} \right) + \frac{ik}{z_0} - ic \frac{1}{z_0 - z_1}.$$

* *Mess. of Math.* (1878); Scientific Papers, vol. 1, p. 344. See also Lamb's Hydrodynamics, 4th ed., 1916, p. 77; Greenhill, *Mess. of Math.*, vol. 9 (1880), p. 113. Report on Gyroscopic Theory, Report of the Advisory Committee for Aeronautics, No. 146, p. 238.

Let us write

$$z_0 = r_0 e^{i\theta_0}, \qquad \zeta_0 = r_0 e^{-i\theta_0}, \qquad z_1 = \frac{a^2}{r_0} e^{+i\theta_0};$$

then since

$$\frac{d\zeta_0}{dt} = u_0 - iv_0,$$

we find that

$$\frac{dr_0}{dt} = U\left(1 - \frac{a^2}{r_0^2}\right)\cos\theta_0,$$

$$r_0 \frac{d\theta_0}{dt} = -U\left(1 + \frac{a^2}{r_0^2}\right)\sin\theta_0 - \frac{k}{r_0} + \frac{cr_0}{r_0^2 - a^2}.$$

In order that the vortex may be stationary we must have either $U = 0$ or $\cos\theta_0 = 0$. The former case has been discussed by Greenhill* both for a stationary and moving vortex. In the latter case U, k, c, and r_0 must be connected by the equation

$$(1) \qquad \mp U\left(1 + \frac{a^2}{r_0^2}\right) - \frac{k}{r_0} + \frac{cr_0}{r_0^2 - a^2} = 0.$$

To study the stability of this stationary vortex let R, Θ denote small displacements from the stationary position; then

$$\frac{dR}{dt} = \mp U\Theta\left(1 - \frac{a^2}{r_0^2}\right),$$

$$r_0 \frac{d\Theta}{dt} = \pm 2UR\frac{a^2}{r_0^3} + \frac{k}{r_0^2}R - cR\frac{r_0^2 + a^2}{(r_0^2 - a^2)^2}.$$

$$\therefore \frac{d^2R}{dt^2} = \mp U\left(1 - \frac{a^2}{r_0^2}\right)\left[\pm 2U\frac{a^2}{r_0^4} + \frac{k}{r_0^3} - \frac{c(r_0^2 + a^2)}{r_0(r_0^2 - a^2)^2}\right]R.$$

When the upper sign is taken $[\theta_0 = +(\pi/2)]$, the vortex is in stable equilibrium if

$$(2) \qquad 2U\frac{a^2}{r_0^3} + \frac{k}{r_0^2} > \frac{c(r_0^2 + a^2)}{(r_0^2 - a^2)^2}.$$

The velocity at the surface of the cylinder is determined from the formula

* Encyclopedia Britannica. Article on "Hydrodynamics." The present case may possibly also have been discussed by Greenhill in a paper, *Quart. J. Math.*, vol. 15 (1878), which is at present inaccessible to me.

$$(3)\quad u^2+v^2=\left[2U\sin\theta+\frac{k}{a}+\frac{c}{a}\,\frac{a^2-r_0^2}{a^2-2ar_0\cos(\theta-\theta_0)+r_0^2}\right]^2.$$

The component forces (X, Y) on the cylinder are given by the formula

$$X-iY=\tfrac{1}{2}\rho\int_0^{2\pi}(u^2+v^2)ae^{-i\theta}d\theta+\rho\int_0^{2\pi}\frac{\partial\phi}{\partial t}ae^{-i\theta}d\theta.$$

Expanding u^2+v^2 by Fourier's theorem, we find that the terms involving $\cos\theta$ and $\sin\theta$ are

$$4U\frac{k}{a}\sin\theta-4U\frac{c}{a}\sin\theta-4\frac{kc}{ar_0}\cos(\theta-\theta_0)$$
$$+4U\frac{ca}{r_0^2}\sin(\theta-2\theta_0)+4\frac{c^2r_0\cos(\theta-\theta_0)}{a(r_0^2-a^2)};$$

hence the value of the first integral is

$$X-iY=-2\pi\rho i[Uk-c(u_0-iv_0)].$$

On the other hand we have for $r=a$

$$\frac{\partial\phi}{\partial t}+i\frac{\partial\psi}{\partial t}=-ic\frac{u_0+iv_0}{ae^{i\theta}-r_0e^{i\theta_0}}+ic\frac{u_1+iv_1}{ae^{i\theta}-(a^2/r_0)e^{i\theta_0}},$$

$$\int_0^{2\pi}\frac{ae^{i\theta}d\theta}{ae^{i\theta}-r_0e^{i\theta_0}}=0,\qquad \int_0^{2\pi}\frac{ae^{-i\theta}d\theta}{ae^{i\theta}-r_0e^{i\theta_0}}=-2\pi\frac{a^2}{r_0^2}e^{-2i\theta_0},$$

$$\int_0^{2\pi}\frac{ae^{i\theta}d\theta}{ae^{i\theta}-(a^2/r_0)e^{i\theta_0}}=2\pi,\qquad \int_0^{2\pi}\frac{ae^{-i\theta}d\theta}{ae^{i\theta}-(a^2/r_0)e^{i\theta_0}}=0,$$

$$\int_0^{2\pi}a(\dot\phi+i\dot\psi)\cos\theta d\theta=\pi ic\frac{a^2}{r_0^2}(u_0+iv_0)e^{-2i\theta_0}+\pi ic(u_1+iv_1),$$

$$\int_0^{2\pi}a(\dot\phi+i\dot\psi)\sin\theta d\theta=-\pi c\frac{a^2}{r_0^2}(u_0+iv_0)e^{-2i\theta_0}+\pi c(u_1+iv_1),$$

$$\int_0^{2\pi}\frac{\partial\phi}{\partial t}a\cos\theta d\theta=\pi c\frac{a^2}{r_0^2}(u_0\sin 2\theta_0-v_0\cos 2\theta_0)$$
$$-\pi cv_1=-2\pi cv_1,$$

$$\int_0^{2\pi}\frac{\partial\phi}{\partial t}a\sin\theta d\theta=-\pi c\frac{a^2}{r_0^2}(u_0\cos 2\theta_0+v_0\sin 2\theta_0)$$
$$+\pi cu_1=2\pi cu_1.$$

Hence we finally obtain the formulas

$$X = 2\pi\rho c(v_0 - v_1), \qquad Y = 2\pi\rho(Uk - cu_0 + cu_1).$$

For a stationary vortex $X = 0$, $Y = 2\pi\rho Uk$ and the transverse force is the same as in Rayleigh's case when $c = 0$ and there is no vortex.*

For a stationary vortex (1) and (2) give the inequality

$$- U > \frac{2ca^2r_0^3}{(r_0^2 - a^2)^3};$$

hence if U is positive and $\theta_0 = \pi/2$, c must be negative and k negative. This means that the circulations around the vortex and cylinder are both in the counterclockwise direction and that the force $2\pi\rho Uk$ is negative. Hence this force tends to move the cylinder away from the region where the cylinder and stream of air are moving in opposite directions. This phenomenon attracted the attention of Newton in 1671 and was the subject of some experiments by Magnus in 1852. Many other experiments have been made since this time and in his beautiful lecture on† "The dynamics of a golf ball" Sir Joseph Thomson showed with the aid of a pressure gauge that there is indeed a difference of pressure on the two sides of a golf ball rotating in a stream of air and remarked that when a golf ball is in flight this difference in pressure may provide a lifting force greater than the weight of the ball, so that a rotating golf ball is a kind of flying machine. This

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* In this case the formula gives Rayleigh's law that the transverse force is proportional to the velocity of the stream relative to the cylinder and the velocity of spin. In the case of a ball this product must be multiplied by the sine of the angle between the direction of motion of the ball relative to the air and the axis of spin. Rayleigh's law was adopted by P. G. Tait (*Nature*, June 29, 1893; Papers, vol. 2, p. 386) in his mathematical calculation of the trajectories of a golf ball and by Sir J. J. Thomson in his experimental method (*Nature*, Dec. 22, 1910) of imitating these trajectories by means of the path of an electron in superposed electric and magnetic fields. A slightly different law is adopted by Appell (*Journ. de Physique*, vol. 7 (1917), p. 5) in his analysis of Carrière's experiments, ibid., vol. 5 (1916), p. 175. The formula has been established for a cylindrical surface of arbitrary shape in a stream of fluid on the assumption that there is a circulation round the cylinder. This is the basis of the theory of sustentation developed by Kutta, Joukowsky and others (see Joukowsky, Aérodynamique, Paris, 1916, and a paper by R. Jones, *Proc. Roy. Soc. London*, vol. 92, A (1916), p. 107) and of the theory of propeller action which has been developed by R. Grammel (*Jahrb. d. Schiffsbautechnik*, vol. 17 (1916), p. 367).

† Royal Institution, March 18 (1910); *Nature*, Dec. 22 (1910).

effect of rotation has been used to increase the range of spherical projectiles and in his experiments with golf balls* Tait conclusively proved that the great factor in long driving was the underspin communicated to the ball by the impact of the club.

Lafay† has recently determined the way in which the pressure varies over the surface of a cylinder rotating rapidly in a stream of air and finds that when the transverse force is in the direction indicated by Magnus it is produced chiefly by the suction on the side which is moving in the same direction as the stream of air.‡ The analogy with the wing of an aeroplane is thus complete.

The distribution of pressure is important, for it should indicate whether or not a rotating cylinder carries along with it one or more vortices that do not produce any circulation around the cylinder.

Lafay's results for a speed of rotation of 9,450 revolutions per minute and a velocity of the air stream of 19 meters per second are given below (p) and compared with the corresponding results for the case of no rotation (p_0). The pressures are in millimeters of water and represent deviations from the atmospheric pressure.

θ	0	15	30	45	60	75	90	105	120	135	150	165	180
p	−18	−16	−12	−9	−8	−5	−2	3	8	14	20	22	18
p_0	−10	−10	−10	−10	−11	−13	−17	−21	−21	−13	7	20	23

θ	−0	−15	−30	−45	−60	−75	−90	−105	−120	−135	−150	−165	−180
p	−18	−23	−32	−63	−93	−102	−104	−85	−63	−37	−14	7	18
p_0	−10	−10	−10	−11	−12	−14	−17	−20	−21	−13	7	19	23

* *Badminton Magazine*, March (1896).

† *Comptes Rendus*, vol. 153 (1911), p. 1472.

‡ The experiments and remarks made by W. S. Franklin, *Journ. Franklin Inst.*, vol. 177 (1914), p. 23, are of some interest in this connection.

To compare these results with formula (3), we must take into account the fact that the theory gives no drag X when $k = 0$ and $c = 0$. It seems reasonable to suppose, however, by analogy with (3) that the correct formula is of type

$$u^2 + v^2 = \left[f(\theta) + \frac{k}{a} + \frac{c}{a} \frac{a^2 - r_0^2}{a^2 - 2ar_0 \cos(\theta - \theta_0) + r_0^2} \right]^2,$$

where $f(\theta)$ is some function which gives the distribution of velocity in the case when there is no rotation of the cylinder. Now if c were zero the above formula would indicate that $\sqrt{P - p}$ should differ from $\sqrt{P - p_0}$ by a constant proportional to k/a, where P is some constant. This means that

$$(p - p_0)^2 + \lambda(p + p_0)$$

should be constant where λ is some constant. This, however, is far from the case as may be seen from some of the values of $(p - p_0)^2$ and $p + p_0$

$(p - p_0)^2$	64	36	4	1	9	8100	441	25
$p + p_0$	−28	−26	−22	−21	−29	−118	−7	41

It seems reasonable then to assume that the flow is modified by the presence of one or more vortices and if we wish to try to account for the drag on the cylinder with the aid of these vortices it is necessary to assume that they are in motion relative to the cylinder just as in Kármán's theory of resistance.

It should be noticed that the region of low pressure in Lafay's experiment occurs in the neighborhood of $\theta = -90°$, and so is directly opposite to the region where a vortex can remain in stable equilibrium. Hence if a vortex forms in the region of low pressure* it cannot remain stationary.

If c is negative and u_0 positive so that the vortex is carried away by the stream, Y may be decreased in magnitude and may even be positive instead of negative. A vortex with counterclockwise rotation may perhaps form when U is greater than the circumferential velocity of the rotating cylinder. As before, we suppose that the rotation of the

* Lord Kelvin pointed out that if the velocity of a spherical solid moving through a fluid exceeds a certain value the pressure becomes negative when calculated by the ordinary theory and so cavitation must commence at the back of the sphere; coreless vortices will be formed periodically and shed off behind the sphere during its motion through the fluid. *Phil. Mag.*, vol. 23 (1887).

cylinder is counterclockwise. If on the other hand c is positive and u_0 positive our formula shows that Y is negative and numerically greater than in the case when there is no vortex. A vortex with clockwise rotation may, perhaps, form in the neighborhood of $\theta = -90°$ when the circumferential velocity of the rotating cylinder is greater than the velocity of the stream.

To account for the force on the cylinder in the direction of the axis of x it is necessary to suppose that a vortex with counterclockwise rotation moves away from this axis and that a vortex with clockwise rotation moves towards this axis.

It should be mentioned that Lafay has found* that the direction of the transverse force in the Magnus experiment could be reversed. Experimenting with a smooth aluminum cylinder 35 cm. long and 10 cm. in diameter, he found that if the velocity of the air stream were kept constant at 18 or 19 meters per second and the velocity of rotation gradually increased, the direction of the force on the cylinder first swung to one side of the air stream, attained a maximum inclination to it of about 11°, then swung to the other side, attained a maximum inclination of about 57°, and finally appeared to approach asymptotically to a direction making an angle of 45° with the air stream.

The maximum inverse effect occurred when the cylinder was rotating at a speed of 1570 turns per minute giving a circumferential velocity of about 8.22 meters per second which is less than the velocity of the air stream. On the other hand the direct effect was quite marked when the speed of rotation was 9450 turns per minute, in which case the circumferential velocity is in the neighborhood of 50 meters per second and is greater than the velocity of the air stream. This is exactly in accordance with the above view that the transverse force is modified by the production of vortices in the neighborhood of the region $\theta = -90°$ and that the direction of rotation in the vortices depends upon whether the circumferential velocity is greater or less than that of the stream of air.

It is well known that the drift of a projectile fired from a rifled gun is exactly opposite to the direction of the transverse force which is indicated by the normal Magnus effect, but here we are dealing with a case in which the component velocity of translation of the bullet in a direction perpendicular to its axis is perhaps at some time greater than the circumferential

* *Comptes Rendus,* vol. 151 (1910), p. 867; vol. 153 (1911), p. 1472.

velocity due to its spin and so the transverse force may be reversed as in Lafay's experiments. In Kármán's theory of resistance* it is supposed that vortices with opposite senses of rotation are formed alternately behind a cylinder and move down the stream in two rows at a certain distance apart. These vortices occupy the region of the "wake" behind a cylinder in a stream of fluid. Now in Lanchester's theory of the Magnus effect† it is assumed that the wake behind the cylinder is displaced to one side on account of the rotation of the cylinder. If the vortices in this wake are produced somewhere in the neighborhood of $\theta = -90^\circ$, the displacement of the wake to one side would be accounted for by the previous remark that a vortex with counterclockwise rotation moves away from the axis of x and that a vortex with clockwise rotation moves towards‡ the axis of x while it is carried down the stream.§ It should be mentioned, however, that here we imagine the resistance of the cylinder to arise from the velocities of the vortices in a direction at right angles to the stream, while in Kármán's theory the resistance arises from the momentum which is carried away from the cylinder whenever a fresh pair of vortices is formed. This momentum is calculated from the circulations around the vortices and the distance between the two rows, while the rate at which the vortices are formed is calculated from their final distance apart in a row and their final velocity relative to the cylinder which is now parallel to the axis of x. The difference between the two points of view is probably the same as the difference between the initial and final stages of an action; for, when the motion of the vortex perpendicular to the stream is considered, we are dealing with the actual transfer of momentum from the cylinder to the fluid or vice-versa.

§ 2. *Two Rectilinear Vortices and a Rotating Circular Cylinder in a Stream of Fluid.*

An interesting attempt to throw light on the initial stages of the formation of the two rows of vortices in the Kelvin-

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* *Phys. Zeitschr.* (1912), *Gött. Nachr.* (1911). See also Joukowsky Aérodynamique, Paris (1916), Ch. 8; Lamb, Hydrodynamics, 4th ed. (1916), p. 219.

† Aerodynamics, vol. I.

‡ When vortices originate on opposite sides of the x-axis as in the case of no rotation they must both move away from the axis of x if they are to produce a positive x.

§ If it crosses the axis of x it must then move away from it.

Kármán theory of resistance has been made by L. Föppl* who has shown that when a cylinder is at rest in a stream of fluid two vortices which are images of one another in the axis of x can be in equilibrium in a stationary position behind the cylinder provided they lie on the curves

$$\pm 2r^2 \sin^2 \theta = r^2 - a^2.$$

Föppl found that the vortices are stable for symmetrical displacements but unstable for asymmetric displacements. The former result is of some meteorological interest in connection with the flow of air past a mountain or other obstacle which is shaped roughly like a half cylinder standing on a plane. It is well known in fact that an eddy can form behind a mountain over which a wind is blowing†; near the rock of Gibraltar the eddy motion is sometimes quite large.

The second result is of interest because it indicates that if two vortices form in symmetrical positions behind a cylinder as they do behind a flat plate, they will be in unstable equilibrium for asymmetric displacements; consequently one vortex may be imagined to get ahead of the other and a new one to be formed to take its place, thus giving rise to an alternate formation as imagined by Lord Kelvin and Kármán.‡ It should be mentioned that in Föppl's analysis the strength of the vortices when in equilibrium increases with their distance from the cylinder while in Kármán's analysis the strengths of all the vortices in one row are supposed to be the same and equal but opposite in sign to those of the vortices in the other parallel row.

If $z_0 = r_0 e^{i\theta_0}$ indicates the position of one of the vortices and c the strength, we have in Föppl's case

$$c = U\left(1 - \frac{a^2}{r_0^2}\right)^2 \left(r_0 + \frac{a^2}{r_0}\right)$$

* *München Sitzungsberichte* (1913).

† See, for instance, W. H. Dines, Report of the Advisory Committee for Aeronautics, No. 92, March (1913), W. N. Shaw, *Science Progress*, vol. 6, p. 345.

‡ This alternate formation of two rows of vortices has been observed on many occasions. See for instance Osborne Reynolds, *Phil. Trans.*, vol. 174 (1883); Ahlborn, "Ueber den Mechanismus des hydrodynamischen Widerstandes," Hamburg (1902); Mallock, *Proc. Roy. Soc.*, vol. 79 (1907), p. 262; vol. 84 (1910), p. 490; *Engineering*, April 19 (1912); H. Bénard, *Comptes Rendus*, vol. 147 (1908), pp. 839, 970; vol. 156 (1913), p. 1003; vol. 157, pp. 7, 89, 171; Rayleigh, *Phil. Mag.*, vol. 29 (1915), p. 433. Joukowsky, loc. cit.

and the velocity at the surface of the cylinder is given by the formula

$$u^2 + v^2 = \left[2U \sin\theta - \frac{4cr_0(r_0^2 - a^2)\sin\theta\sin\theta_0}{\{r_0^2+a^2-2ar_0\cos(\theta-\theta_0)\}\{r_0^2+a^2-2ar_0\cos(\theta+\theta_0)\}} \right]^2.$$

Since $2r_0^2 \sin\theta_0 = r_0^2 + a^2$ it appears that the velocity vanishes when $\theta = 0$, $\theta = \pi$ and also when

$$\cos\theta = \frac{1}{2a}\left(r_0 + \frac{a^2}{r_0}\right)\cos\theta_0 - \frac{1}{a}\sqrt{3r_0^2 + 4a^2}\,\sin^2\theta_0.$$

This equation can give a real value of θ, for when $r_0^2 = 3a^2$, we have

$$\cos\theta = \frac{4\sqrt{6} - \sqrt{13}}{9} < 1.$$

When the surface velocity q is plotted as a function of θ, q and θ being regarded as polar coordinates, a curve is obtained which is shaped like a butterfly with two wings. The measurements of J. T. Morris,* A. Thurston† and A. Lafay‡ indicate that for a cylinder in a stream of fluid the curve indicating the distribution of velocity should be shaped like a butterfly with only one pair of wings. The lack of agreement is to be expected on account of the instability of the two vortices behind the cylinder. A comparison of the theoretical formula ought to be made with some measurements of the velocity over the surface of a semi-cylinder standing on a smooth plane surface over which a wind is blowing. Of course the roughness of a rotating ball or cylinder has a great influence on the magnitude of the transverse force exerted by the wind as is clearly shown in the experiments of Sir Ralph Payne-Gallwey,§ Sir Joseph Thomson and Commandant Lafay. Curiously enough the former found that to obtain the best lifting effect with a golf ball the ball must not be too rough. This suggests that with a very rough ball vortices are produced which modify the Magnus effect.

* *Engineering*, Aug. 8, 1913.
† Ibid., Aug. 21, 1914.
‡ Loc. cit.
§ *The Times*, March 16, 23, 1909. *Nature*, April 22, 1909.

It may be of interest to indicate briefly the extension of Föppl's analysis for the case in which the cylinder rotates and there is no symmetry about the axis of x. Using z_0 and z_2 to indicate the positions of the vortices and z_1 and z_3 those of their images in the cylinder, the appropriate expressions for ϕ and ψ are given by

$$\phi+i\psi=U\left(z+\frac{a^2}{z}\right)+ik\log z+ic\log\frac{z-z_0}{z-z_1}-ic\log\frac{z-z_2}{z-z_3}.$$

Differentiating to obtain the velocities we have

$$u-iv=U\left(1-\frac{a^2}{z^2}\right)+\frac{ik}{z}+ic\left[\frac{1}{z-z_0}-\frac{1}{z-z_1}\right]-ic\left[\frac{1}{z-z_2}-\frac{1}{z-z_3}\right],$$

$$u_0-iv_0=U\left(1-\frac{a^2}{z_0^2}\right)+\frac{ik}{z_0}-ic\left[\frac{1}{z_0-z_1}+\frac{1}{z_0-z_2}-\frac{1}{z_0-z_3}\right],$$

$$u_2-iv_2=U\left(1-\frac{a^2}{z_2^2}\right)+\frac{ik}{z_2}+ic\left[\frac{1}{z_2-z_0}-\frac{1}{z_2-z_1}+\frac{1}{z_2-z_3}\right].$$

Writing

$$z_0=r_0e^{i\theta_0},\quad z_2=r_2e^{i\theta_2},\quad R^2=r_0^2+r_2^2-2r_0r_2\cos(\theta_0-\theta_2),$$
$$S^2=r_0^2r_2^2+a^4-2a^2r_0r_2\cos(\theta_0-\theta_2)$$

we find that when $u_0=v_0=u_2=v_2=0$,

$$0=U\left(1-\frac{a^2}{r_0^2}\cos 2\theta_0\right)+\left(\frac{k}{r_0}-\frac{cr_0}{r_0^2-a^2}\right)\sin\theta_0+\frac{c}{R^2}(r_2\sin\theta_2-r_0\sin\theta_0)+\frac{cr_2}{S^2}(r_0r_2\sin\theta_0-a^2\sin\theta_2),$$

$$0=U\frac{a^2}{r_0^2}\sin 2\theta_0+\left(\frac{k}{r_0}-\frac{cr_0}{r_0^2-a^2}\right)\cos\theta_0$$

$$+ \frac{c}{R^2}(r_2 \cos\theta_2 - r_0 \cos\theta_0) + \frac{cr_2}{S^2}(r_0 r_2 \cos\theta_0 - a^2 \cos\theta_2)$$

and two similar equations with the suffixes 0 and 2 interchanged and $-c$ written in place of c.

Multiplying the first of these equations by $\cos\theta_0$, the second by $\sin\theta_0$ and subtracting, we get

$$(r_0^2 - a^2)\left[\frac{U}{r_0^2}\cos\theta_0 - \frac{cr_2}{R^2S^2}(r_2^2 - a^2)\sin(\theta_0 - \theta_2)\right] = 0.$$

Similarly

$$(r_2^2 - a^2)\left[\frac{U}{r_2^2}\cos\theta_2 - \frac{cr_0}{R^2S^2}(r_0^2 - a^2)\sin(\theta_0 - \theta_2)\right] = 0,$$

$$\left[r_0 - \frac{a^2}{r_0}\right]\cos\theta_0 = \left(r_2 - \frac{a^2}{r_2}\right)\cos\theta_2.$$

This equation tells us that the projection on the axis of x of the interval between a vortex and its image in the cylinder is the same in both cases. It is clear then that the vortex which is furthest from the plane $y = 0$ is also furthest from the axis of the cylinder. There is another equation connecting r_0, θ_0, r_2, θ_2, but it is very complicated; the two equations show, however, that when one vortex is given the position of the other is determined. The component forces on the cylinder are easily found by an extension of the analysis of § 1. They are

$$X = 2\pi\rho c(v_0 - v_1 - v_2 + v_3),$$

$$Y = 2\pi\rho(Uk - cu_0 + cu_1 + cu_2 - cu_3).$$

When both vortices are stationary we have $X = 0$, $Y = 2\pi\rho Uk$ as before. This result seems to be true for any number of stationary vortices outside a cylinder, as is probably well known.*

We have seen that a rotation of the cylinder alters the possible stationary positions for a pair of vortices; it may also alter the period of formation of the vortices in Kármán's theory of resistance. This is not easy to settle mathematically but the matter may perhaps be tested experimentally.

Eiffel found during his measurements of the force exerted by a stream of air on a sphere that at a speed above twenty

* See for instance the remark in § 3 of Föppl's paper.

miles an hour the flow was smoother and the force more constant. The vortices behind the sphere are usually said to be flattened out* and presumably the period of formation which depends on the distance between the two rows of vortices is changed. The rate of formation probably increases with the velocity of the stream until the flow becomes practically steady at high speeds.†

Now a rotation of the cylinder or sphere may cause a change in the critical velocity above which the flow is practically steady, consequently it may be worth while to determine this critical velocity for a given velocity of the stream and different velocities of rotation of the cylinder. This critical velocity may be closely connected with Lafay's critical velocity at which the sign of the transverse force in the Magnus effect is reversed; it should be noticed, however, that Lafay made his experiments in a wind of 19 meters a second which had a velocity more than double Eiffel's critical velocity and noticed at which *speed of rotation* the change occurred.

It is known that at high speeds the periodic formation of vortices is responsible for the production of sound‡ and the period is presumably that of the sound. In the case of the aeolian harp§ the sound is most intense when the period is close to one of the natural periods of the stretched string or wire. An effect of rotation of the stretched string or wire on the pitch of the sound produced in a given type of wind might perhaps be determined experimentally but there would be difficulties. For the mathematical theory of the aeolian harp the sound produced by the oscillation of a vortex about a state of uniform motion ought also to be considered, for this may contribute to the observed sound as well as periodic formation of vortices, but the effect is probably negligible.

§ 3. *Vortices in a Compressible Fluid.*

It was shown by Lord Kelvin that vortex lines in a compressible fluid move with the fluid provided the density of

* Cf. Loening, Military Aeroplanes, p. 51.

† Cf. Cowley and Levy, Aeronautics in Theory-Experiment (1918), pp. 17–20.

‡ See for instance Mallock, *Proc. Roy. Soc. London*, vol. 84 (1910), p. 490.

§ Lord Rayleigh, Theory of Sound, vol. II, p. 412, vol. 1, p. 212; *Phil. Mag.*, vol. 29 (1915), p. 433.

the fluid is a function of the pressure only and the body forces have a single-valued potential. It is also true that the circulation in any circuit moving with the fluid remains constant. An accurate theory of vortex motion in a compressible fluid is difficult. To make progress, it seems worth while to make an assumption which is nearly true except in the immediate neighborhood of a vortex.

Let us consider a two-dimensional irrotational motion in which the velocity potential ϕ satisfies the wave equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2},$$

wherein c, the velocity of sound in the fluid, is supposed to be constant. The component velocities (u, v) will then satisfy the same equation. A solution appropriate for the representation of a rectilinear vortex moving with constant component velocities a, b may be derived from the well known solution for a stationary vortex by an application of the transformations of the theory of relativity. The result is

$$u = -\frac{\mu c(y-bt)\sqrt{c^2-a^2-b^2}}{(c^2-b^2)(x-at)^2+(c^2-a^2)(y-bt)^2+2ab(x-at)(y-bt)},$$

$$v = \frac{\mu c(x-at)\sqrt{c^2-a^2-b^2}}{(c^2-b^2)(x-at)^2+(c^2-a^2)(y-bt)^2+2ab(x-at)(y-bt)},$$

where $2\pi\mu$ is the strength of the vortex.*

To find the paths of the particles of fluid relative to the moving vortex, we write $X = x - at$, $Y = y - bt$, $X^2 + Y^2 = z$, $bX - aY = w$, $u = dx/dt$, $v = dy/dt$; then it is easy to see that

$$\frac{1}{2}\frac{dz}{dt} + aX + bY = 0, \qquad \frac{dw}{dt} + \mu\frac{c\sqrt{c^2-a^2-b^2}(aX+bY)}{c^2z - w^2} = 0,$$

$$\therefore \quad \frac{dz}{dw} = \frac{2(c^2z - w^2)}{\mu c\sqrt{c^2 - a^2 - b^2}},$$

$$\therefore \quad z = Ae^{2cw/\mu\sqrt{c^2-a^2-b^2}} + \frac{1}{c^2}\left[w^2 + \mu\frac{w}{c}\sqrt{c^2-a^2-b^2} + \frac{\mu^2}{2c^2}(a^2 - c^2 - b^2)\right],$$

where A is an arbitrary constant.

* μ is positive for counterclockwise rotation.

Thus relative to the moving vortex a particle of fluid appears to describe a curve whose equation is

$$X^2 + Y^2 = Ae^{2c(bX-aY)/\mu\sqrt{c^2-a^2-b^2}} + \frac{1}{c^2}\Big[(bX - aY)^2 + \frac{\mu}{c}\sqrt{c^2 - a^2 - b^2}\,(bX - aY) + \frac{\mu^2}{2c^2}(c^2 - a^2 - b^2)\Big].$$

When $A = -(\mu^2/c^2)(c^2 - a^2 - b^2)$, we obtain an equation which is satisfied by $X = 0$, $Y = 0$ and this represents a curve having an isolated point at the origin which is in accordance with the theorem that the vortex moves with the fluid. As A varies, we obtain first a number of ovals surrounding the origin, then a curve which touches itself on the axis of y,* after completing the circuit, and then goes to infinity. Finally we obtain a series of curves which lie outside the last one and like it go to infinity. Each curve is described by a point which starts moving almost in the direction of the x-axis, then swings around the origin in the clockwise direction and finally returns to a state of motion very nearly parallel to the x-axis.

Let us now consider two rectilinear vortices of strengths $2\pi\mu$ and $-2\pi\mu$ respectively moving parallel to the axis of x with constant velocity a and at a distance apart equal to $2y$. If the velocity of each vortex is that produced by the other, we have simply

$$a = \frac{\mu c}{2y\sqrt{c^2 - a^2}}$$

or

$$a^2 = \tfrac{1}{2}c^2 \pm \sqrt{\tfrac{1}{4}c^2 - \frac{\mu^2 c^2}{4y^2}}.$$

For a real value of a^2 we must have $c^2y^2 > \mu^2$. Hence when the circulations around the vortices are given they cannot move parallel to one another at a distance apart less than $2\mu/c$. If the two vortices are in a stream of fluid moving with velocity U the equation determining a is

$$a = U + \frac{\mu c}{2y\sqrt{c^2 - a^2}}$$

* We put $y = bX - aY$, $x = aX + bY$ to obtain a curve which is symmetrical with regard to the axis of y.

or

$$4y^2(c^2 - a^2)(a - U)^2 = \mu^2 c^2.$$

It should be noticed that a is numerically less than c whatever the value of U.

The velocity potential of a vortex moving with constant velocity differs by a constant from the function

$$\mu \tan^{-1} \frac{(ay - bx)\gamma}{a(x - at) + b(y - bt)} \equiv \mu \int_{-\infty}^{t} \frac{ds}{\sqrt{t - s}} B(x, y, s),$$

where

$$\gamma = \left(1 - \frac{a^2 + b^2}{c^2}\right)^{1/2}$$

and

$$A(x, y, s) + iB(x, y, s)$$
$$= \left[\frac{a^2 + b^2}{a(x - as) + b(y - bs) - i\gamma(ay - bx)}\right]^{1/2},$$

$$A(x, y, s) + iB(x, y, s)$$
$$= \left[\frac{a^2 + b^2}{a(x - as) + b(y - bs) - i\gamma(ay - bx)}\right]^{1/2}.$$

For a vortex which is moving with component velocities $\xi(\tau)$, $\eta(\tau)$ at time τ and is at an infinite distance from the origin at time $\tau = -\infty$, the natural generalization of the function $B(x, y, s)$ is obtained as follows:

Let τ be defined in terms of x, y, s by the equation

$$[x - \xi(\tau)]^2 + [y - \eta(\tau)]^2 = c^2(s - \tau)^2 \quad \tau \leqq s,$$

where $\xi'^2(\tau) + \eta'^2(\tau) < c^2$ and primes denote differentiations with respect to τ. Let

$$l(\tau) = \xi'(\tau) + i\sigma\eta'(\tau), \qquad c^2 p(\tau) = \xi'^2 + \eta'^2,$$
$$m(\tau) = \eta'(\tau) - i\sigma\xi'(\tau), \qquad c^2\sigma^2 = c^2 - \xi'^2 - \eta'^2;$$

then

$$A(x, y, s) + iB(x, y, s)$$
$$= \left[\frac{c^2 p(\tau)}{\{x - \xi(\tau)\}l(\tau) + \{y - \eta(\tau)\}m(\tau) - c^2(s - \tau)p(\tau)}\right]^{1/2}.$$

This result may be of interest for an analysis of the sound produced when a vortex oscillates about a state of uniform motion. It must be remembered, however, that the above analysis is only approximate, for velocities are treated as small in the derivation of the wave equation.

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SHORTER NOTICES.

Lectures on the Philosophy of Mathematics. By JAMES BYRNIE SHAW. Chicago, The Open Court Publishing Company, 1918. viii + 206 pp.

THE purpose of Professor Shaw's book is a discussion of the evergreen question: What is mathematics? While in his first chapter the author develops in a highly exalted style various aspects of this subject, the greater part of the subsequent chapters may be said to be essentially devoted to two more specific questions, viz., what influences operate and have operated in the development of mathematics, and how may existing mathematics be concisely described. With the treatment of these questions, perhaps not always recognized as explicit and distinct, Chapters II to XIII are taken up, together occupying 140 pages. To the first question the author gives a positive answer, viz.: "Mathematics is a creation of the mind and is not due to the generalization of experiences or to their analysis; nor is it due to an innate form or mold which the mind compels experience to assume, but is the outcome of an evolution, the determining factors of which are the creative ability of the mind and the environment in which it finds the problems which it has to solve in some manner and to some degree." The second question is answered in a negative sense; as the various fields and principles of mathematics are discussed, the conclusions are reached that mathematics is not wholly arithmetic, nor geometry, nor logistic; that mathematics can not be completely characterized as a theory of invariance, nor as a theory of functions, etc., however important each of these principles may be. The closest approach to a satisfactory answer to our second