$$(21)\quad \begin{cases} p(n\,|\,0) = p(1^n\,|\,0) - \binom{n}{2}p(1^{n-2}\,|\,1^2) \\ \qquad\qquad + \binom{n}{4}p(1^{n-4}\,|\,1^4) - \cdots, \\ p(0\,|\,n) = \binom{n}{1}p(1^{n-1}\,|\,1) - \binom{n}{3}p(1^{n-3}\,|\,1^3) \\ \qquad\qquad + \binom{n}{5}p(1^{n-5}\,|\,1^5) - \cdots. \end{cases}$$

The q-forms of these are found in the same way by distributing the product, and equating, in

$$(22)\quad \prod_{j=1}^{n} \{q(\xi_j\,|\,0) + iq(0\,|\,\xi_j)\}$$
$$= q((\xi_1, \xi_2, \cdots, \xi_n)\,|\,0) + iq(0\,|\,(\xi_1, \xi_2, \cdots, \xi_n)).$$

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THE TRAINS FOR THE 36 GROUPLESS TRIAD SYSTEMS ON 15 ELEMENTS.

BY PROFESSOR LOUISE D. CUMMINGS.

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In the *Transactions* for January, 1913, Professor H. S. White derived a new method for the comparison of triad systems, and applied it to the two triad systems on 13 elements. One part of a memoir published in the *Memoirs of the National Academy of Sciences*, volume 14, second memoir, shows the efficacy of this method in determining the groups, and in establishing the noncongruency of the 44 systems Δ_{15} which have a group different from the identity. The present paper gives the results of this method of comparison applied to the 36 groupless systems Δ_{15}. In this method, the triad system is regarded as an operator, and certain covariants of that operator are deduced. These covariants can be represented graphically and are called the trains of the system.

A triad system of n elements consists of triads so selected that every pair of elements occurs once, and only once, in the chosen triads. From 15 elements 455 triads can be formed. Any system contains 35 of these; leaving 420 that are called extraneous triads. Applying the system to transform them all, we see that if the system contains the 3 triads $df8$, $de1$, $fg1$, then it will transform the triad $df1$ which is not in the system and which contains the pairs df, $d1$, $f1$ into the triad $8eg$, but it will transform the triad $df8$ which is in the system into $df8$. Under a given triad system as an operator a triad t_1 is transformed into t_2; t_2 into t_3; t_3 into t_4. Since only 455 triads exist, either a triad t_s of the system is reached, or else a triad that has already appeared is repeated, namely $t_{m+k} \equiv t_m$. In the former case the triad t_s recurs always, while in the latter case, the train beginning at t_m constitutes a recurring cycle. Triads that do not recur in the terminal cycle are designated as appendices, and a train consists of a recurrent cycle and all its appendices. The triad $df1$ as shown above transforms into $eg8$, similarly $eg8$ transforms into 257 which repeats indefinitely. The only other triad which transforms into 257 is 146. These 4 triads form the type of train which is exhibited graphically in the figure — ⋝ ⊣.

The system employed here as an operator is a Δ_{15} discovered by Professor F. N. Cole, and applied as a transformer on the 455 triads it yields the following covariants or trains: One class of trains terminating in a cycle of period 22; one class of trains terminating in a cycle of period 12; twelve classes of trains terminating in triads of the system. These trains separate the 35 triads of the system into 12 distinct classes given in the following table 1:

TABLE 1. COLE SYSTEM 36.

(1).	(2).	(3).	(4).	(5).	(6.)	(7).	(8).	(9.)	(10).	(11).	(12).
$ab4$ $bc1$ $cf2$ $ef4$ $af7$ $bf5$ $c37$ $fg1$ $a18$ $bg3$ $dg6$ $f36$ 348 $b28$ $d35$ $g47$ $df8$	$bd7$ $ad2$ 246 $cd4$ $c68$ $ae3$	acg $e78$ $ce5$	257	$eg2$	$de1$	167	$a56$	123	$g58$	$be6$	145

Some substitution may transform the triad system into itself. Such a substitution, evidently, must leave unchanged the totality of trains connected with the system. Every operation of the group that leaves the system invariant must transform any train into itself or into another train of the same class. Since only those elements may be permuted which occur the same number of times in a class, the enumeration of the appearances of each of the 15 elements in the 12 classes of trains terminating in triads of the system, as in the following table 2, shows the possible sets of transitive elements.

TABLE 2.

	a.	*b.*	*c.*	*d.*	*e.*	*f.*	*g.*	1.	2.	3.	4.	5.	6.	7.	8.
(1)	3	5	3	3	1	7	4	3	2	5	4	2	2	3	4
(2)	2	1	2	3	1				2	1	2		2	1	1
(3)	1		2		2		1					1		1	1
(4)									1			1		1	
(5)					1		1		1						
(6)				1	1			1							
(7)								1					1	1	
(8)	1											1	1		
(9)								1	1	1					
(10)							1					1			1
(11)		1			1								1		
(12)								1			1	1			

No two columns in table 2 are similar, therefore the system is not invariant under a single transposition, hence no substitution transforms the system into itself and the system is groupless. Similar investigations show the remaining 35 systems to be groupless.

Two systems are congruent only if their trains are identical both in type and in number. A comparison of the graphs* of the trains establishes conclusively the noncongruence of the 36 groupless systems.

The trains for the 44 systems Δ_{15} with a group furnished 216 types of covariants. This investigation shows that while many of these 216 types are found among the trains for the groupless systems, there are in addition 449 new types. Hence the 80 noncongruent systems Δ_{15} operating, as transformers, on the complete set of 455 triads formable from 15 elements, produce 665 distinct covariants. These covariants are of two kinds: (1) trains terminating in a one-term cycle, a triad of the system, and involving with the appendices from 1 to 299 triads; (2) trains ending in a terminal cycle forming a polygon

* A manuscript volume deposited in the Vassar College Library.

of 4, 6, 9, 10, 11, 12, 13, 14, 18, 20, 22, 24, 30, or 72 sides, respectively, with appendices ranging in number from 0 up to more than 100. The distinct covariants for the two noncongruent systems on 13 elements were only nine in number and much simpler in form. We see that an increase in the number n of elements, which is probably always accompanied by an increase in the number of distinct systems, produces greater complexity in the form and a very rapid increase in the number of distinct covariants connected with the noncongruent systems.

To extend this method of trains to systems on 19 or more elements would be evidently too laborious, if the object is only to classify the different triad systems. Here the analogy of invariants of algebraic forms under linear transformation is instructive; the complete calculation of systems of invariants is always possible, but only desirable when it involves finite time, as in forms of very low order. Beyond that, it is only particular forms with special invariant characters that are of general interest. So here, it is obviously most interesting to give detailed study first to triad systems which have covariant trains ending in polygon-cycles containing the largest possible number of extraneous triads. This recalls Professor E. H. Moore's study of systems whose groups are cyclic and those might probably be found again early in the proposed research.

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A THEOREM ON AREAS.

BY PROFESSOR TSURUICHI HAYASHI.

THE relative area of two given convex ovals in the same plane, swept out by moving the join of two points lying on the peripheries of the two ovals respectively, so that the point of the join dividing it in a given ratio traces out the periphery of the area containing the totality of all the points which divide the joins of two points lying on and within the two ovals respectively, satisfies the relation

$$\sqrt{S} \leqq \sqrt{A} \sim \sqrt{B},$$

independent of the ratio, A, B, S being the areas of the two ovals and their relative area, respectively.