POINTS OF VIEW OF CAUCHY AND WEIERSTRASS IN THE THEORY OF FUNCTIONS.

Leçons sur les Fonctions monogènes uniformes d'une Variable complexe. Par EMILE BOREL. Rédigées par GASTON JULIA. Gauthier-Villars, Paris, 1917. xii + 165 pp.

THE point of departure in this monograph is the opposition between the points of view of Cauchy and Weierstrass in the definition of functions of a complex variable called *monogenic* by Cauchy and *analytic* by Weierstrass. The conclusion which is reached—and it is supported by detailed investigation and significant results—is that the point of view of Cauchy is the more fundamental one. In fact, it is contended that the work of Weierstrass is confined to a class of functions more special than that which should be considered and that in a more general class the central theorems are maintained, so that there is no reason for confining attention entirely to the analytic functions of Weierstrass.

The researches of which we have here an exposition convenient for the reader go back as far as a quarter century. Borel entered upon some of the matters involved as early as 1892 and published the first results in 1894. But it was as late as 1912 when he made known the principal results of the theory of non-analytic monogenic functions. This was in his lecture before the Fifth International Congress of Mathematicians at Cambridge. This earlier exposition was incomplete in many respects; and it was only in the course of lectures at Paris of which this book is the reproduction that he has completely developed the theory, entering into detail in the demonstrations and leaving no delicate point without careful illumination.

It was intended that this book should appear at the end of 1914; but M. Gaston Julia, a student at the Ecole Normale Supérieure, who was charged with preparing the notes for publication, was called into the army and was wounded in January, 1915. Notwithstanding his sufferings, he continued with the labor of seeing the manuscript through the press even though he was also at the same time engaged in remarkable researches of his own suitable "for perpetuating French mathematical traditions." The difficulties under which the proofreading was done probably account for the rather large number of minor misprints, more than a score of which the reviewer detected. None of them is such as to cause the reader serious inconvenience.

The intimate connection between the theory of functions of a complex variable and the theory of the potential in mathematical physics has been an important guide to Borel in his researches on non-analytic monogenic functions, as he has pointed out in his Cambridge lecture; he has been helped particularly by the analogies of the new theory with the theories of molecular physics so much in consideration among scientists in very recent years. The non-analytic monogenic functions correspond to the case where the singular regions are at the same time extremely small and extremely numerous.

In the monograph under consideration attention has been confined to what may be called the purely mathematical theory of monogenic functions, and none of the physical applications just indicated are developed. In his preface the author states that it had originally been his intention to publish simultaneously with this monograph another in which were to be given the applications of these theories to the problems of molecular physics. But he finds himself compelled to leave off this study "until the moment when the victory of civilization . . . over a brutal aggression shall give us the right to take up again the thoughts and labors of times of peace." Some indication of what he has in mind in this further study is to be found in the second note in the monograph under review and in the seventh note in his Introduction géométrique à quelques Théories physiques. Perhaps one must wait until the appearance of a completed exposition of these applications before reaching a final judgment on the matters insisted upon by Borel; but the development already before us seems to indicate that the point of view of Cauchy is indeed more fundamental than that of Weierstrass and in a way which will require our thinking of the latter simply as the most important and most elegant case of the former.

Cauchy has laid down the conditions of monogeneity which lie at the base of the theory of functions of a complex variable and in his integral has created a matchless instrument of demonstration for dealing with this theory. In the study of differential equations he employed the process of analytic continuation. The rôle of Weierstrass was to make more precise certain points of the latter theory, notably that which pertains to the form of the domains. We may speak of the domains of Weierstrass, or the domains W, to mean those domains in which an analytic function may be defined by the process of analytic continuation as employed by Weierstrass. These domains may be defined geometrically and their properties be developed without reference to functions which are or may be defined in them.

The first chapter of this monograph (pages 1–23) is devoted to certain generalities (pages 1–5), the definition of a monogenic function (pages 5–10), geometric properties of domains W (pages 10–18), and the representation of an analytic function defined in a domain W (pages 19–23), the latter being given in the form of a Cauchy integral with the consequent series of Taylor and Laurent. Thus the chapter deals throughout with classic matter; the discussion of domains W may be singled out as particularly pleasing.

Chapter II (pages 24–54) is devoted to an application of the integral of Cauchy to the development in a series of polynomials of a function defined in a domain of Weierstrass. The first fourteen pages are given to an exposition of the method of Runge and the remainder of the chapter to applications to certain particular developments.

The third chapter (pages 55–72) is devoted to certain remarkable consequences of the development in series of polynomials and to an extension of the theory of analytic continuation. Here we have some preliminary aspects of Borel's reasons for insisting upon the inadequacy of the Weierstrass theory from the point of view of the more general theory of monogenic functions, the arguments here being presented in such way as to enable one to pass to the conclusion as directly as possible from first considerations. Then, after a necessary preliminary treatment of sets of points of measure zero (in Chapter IV, pages 73–124), he introduces in Chapter V (pages 125–150) the domains C, or domains of Cauchy, and develops the first elements of the general theory of non-analytic monogenic functions defined in these domains. There is nothing to indicate that these are the most general domains in which monogenic functions may be defined; but they are more general than the domains of Weierstrass.

Poincaré, with great ingenuity, constructed analytic expressions presenting certain remarkable singularities and believed that he had sufficient grounds for concluding to the impossibility of extending the theory of analytic functions beyond the bounds fixed by Weierstrass; that is, he believed in the impossibility of a process of analytic continuation by which one could extend a function beyond the domain of Weierstrass. With this negative result Borel has been for a long time in disagreement, at first thinking that it was not established and now finally justifying his position by means of the detailed results set forth in this monograph. Particularly, he has objected to the conclusion that the domain throughout which a function could be continued analytically by means of power series necessarily constitutes its *natural* domain of existence, that the function does not exist outside of this domain, and that every attempt to pierce this Unknown is destined to failure and contradiction. In this connection it is interesting to note the following from the preface to this monograph:

"A la suite des travaux de Poincaré que j'ai rappelés il y un instant, ce point de vue paraissait universellement admis; mais tandis que Poincaré accueillait avec bienveillance le premier essai dans lequel je montrais que les limites imposées par Weierstrass n'étaient pas aussi infranchissables qu'on l'avait cru, les disciples fidèles de Weierstrass ne consentaient même pas à discuter; je me rappellerai toujours l'étonnement avec lequel je vis M. Mittag-Leffler, auquel j'avais essayé d'exposer mes projets de recherches, ne faire aucun effort pour entrer dans ma pensée et se contenter de retirer de sa malle un Mémoire de Weierstrass pour me montrer une phrase qui devait clore définitivement toute discussion: Magister dixit."

In Chapter III we have an interesting example bearing on this matter of dispute and one which seems to settle the question definitively in favor of Borel's contention. Let a_1, a_2, a_3, $\cdots$ be the rational numbers between 0 and 1 arranged in enumerable sequence and consider the infinite series

$$(1) \qquad f(z) = \sum_{n=1}^{\infty} \frac{A_n}{z - e^{2\pi i a_n}}, \qquad i = \sqrt{-1},$$

where the A's are constants such that the series $A_1 + A_2 + \cdots$ converges absolutely. It is easy to see that this series defines a function which is holomorphic at every point in the interior of the unit circle about the point zero. It may be shown that this circle is a *natural* boundary, in the sense of

Weierstrass, of the function defined by the series at an interior point of this circle. Likewise the series defines a function which is holomorphic at every point exterior to this unit circle and the circle is a natural boundary of the function thus defined. In the theory of Weierstrass there is no means of continuation by which one may establish a connection between the function defined by (1) in the interior of the unit circle and that so defined in the exterior of this circle. In fact, from the point of view of Weierstrass they are to be treated as unrelated functions.

Nevertheless Borel shows that a suitable continuation does exist for establishing the connection between these functions, this being constituted by a series of polynomials. In fact, one can form a series of polynomials

$$\sum_{p=0}^{\infty} P_p(z)$$

which converges uniformly on every finite segment of every straight line of argument $m\sqrt{2} + n$, where m and n are integers and $m \neq 0$, an infinity of which lines lie in every angle issuing from the origin; on each of these lines the series converges to the sum of the series in (1).

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Elements of Optics for the Use of Schools and Colleges. By GEORGE W. PARKER, M. A. London, New York, and Bombay, Longmans, Green and Company, 1915. vi + 122 pp.

THE mathematical prerequisites necessary for reading this little book have been reduced to a minimum. A student whose knowledge of mathematics is limited to an acquaintance with elementary geometry, the solution of simple algebraic equations and a few fundamental propositions in trigonometry will be able to follow the treatment at all places. The knowledge of physical phenomena presupposed is also reduced to an extreme minimum. The book is therefore of a strictly elementary character. It is written in a satisfactory style and its material is arranged in interesting sequence, so that it may be