MULTIPLY PERIODIC FUNCTIONS.

An Introduction to the Theory of Multiply Periodic Functions. By H. F. BAKER, Sc.D., F.R.S., Fellow of St. John's College and Lecturer in Mathematics in the University of Cambridge. Cambridge University Press, 1907. Royal 8vo. xv + 335 pp.

THIS is a highly interesting and suggestive contribution to a field which has engaged the attention of numerous mathematicians since the time of Abel. Except for the first chapter, the present work has little in common with other treatises relating to the same subject, while a considerable portion of the material is drawn from the author's own investigations.

The book is divided into two parts, the first dealing with hyperelliptic functions of two variables, the second with periodic functions of n variables with reference to the fundamental problem of their connection with the theory of algebraic functions and their expression in terms of the Riemann theta functions.

"The first part is centered round some remarkable differential equations satisfied by the functions, which appear to be equally illuminative both of the analytic and the geometric aspects of the theory; it was, in fact, to explain this that the book was originally entered upon." Chapter I is introductory and is chiefly concerned with a deduction of the fundamental formulas connected with a Riemann surface of two sheets and six branch points. It contains a brief and condensed account of the hyperelliptic integrals of the first, second, and third kinds, and their behavior on the surface. After developing the properties of the theta functions, a notable departure is made from the usual treatment. Following closely the analogy of the Weierstrassian theory of elliptic functions, a single theta function is retained out of the sixteen with half-integer characteristics. This is multiplied by an exponential factor and the product regarded as a function of the unnormalized integrals u_1, u_2 of the first kind and of their homogeneous table of moduli. The function so obtained is denoted by $\vartheta(u_1, u_2)$ and later on by σ. The two first derivatives of ϑ with respect to u_i are the ζ-functions, and the three second derivatives with changed signs are the $\wp$-functions. All of these have properties strictly analogous to the corresponding functions in the elliptic case. The $\wp$-functions are expressible in a simple manner in terms of two positions on the Riemann surface. The author hopes "that the treatment here followed, which reduces the theory in a very practical way to that of one theta function and three periodic functions, may serve the purpose of encouraging a wider use of these functions in other branches of mathematics."

This would seem on the face of it to be a considerable simplification since the sixteen Riemann theta functions, connected by numerous relations, have been replaced by a single function together with the properties deducible from it and its derivatives. The advantage is not acquired, however, without sacrifice. For example, the three $\wp$-functions satisfy an algebraic relation of degree four, the equation of the Kummer surface. But in order to prove the existence of the sixteen singular tangent planes it is necessary to expand the original determinant equation of the surface and show that it can be put in the necessary algebraic form. This illustration is somewhat typical of much of the long discussion of the Kummer and Weddle surfaces, that is, considerably more algebraic manipulation (although of an elementary character) is required and not so easy an oversight of the geometrical properties is obtained as in using the sixteen theta functions and their properties. Moreover, in the field of geometric applications, to which the author devotes so much space, it does not seem possible to treat the important class of hyperelliptic surfaces in a general and comprehensive way, as Humbert* has so beautifully and clearly done, without using the Riemann theta functions and the relations among them. The author's choice is, however, deliberate, and he expresses regret that his desire to keep the work as elementary and as self-contained as possible leads to the exclusion of important methods and material.

The first chapter strikes the reviewer as somewhat less carefully worked out than the rest of the book. There seems to be an occasional uncertainty and inconsistency in the choice of notation which is not conducive to ease and economy of effort in reading the book. On page 25, for example, we start in with theta characteristic numbers q', q and period characteristics p', p. In the course of the deduction a previous formula is referred to in which the roles of the p and q numbers are interchanged, and in writing down the final result this interchange is retained, so that at the end the letters do not mean the same that they

* Théorie générale des surfaces hyperelliptiques, *Journal de Mathématiques*, 1893.

did at the start. On page 37, λ_1 and λ_2 are used in two different senses. In the second usage they represent the limiting values of the two integrals of the second kind. As these two integrals already contain parameters previously denoted by λ_1, λ_2, a different notation would have been more suitable and more conducive to clearness. As an example of lack of symmetry in the notation we might refer to the departure from customary usage on page 13. The four cross-cuts on the Riemann surface are denoted by A_1, A_2, A_3, A_4 while the periods of the integrals of the first and second kinds at these cuts are distinguished by subscripts 1, 2 with the addition of an accent for the second pair. We do not see that the lack of association of ideas produced by an unsymmetrical notation is compensated for by any new advantage.

The desire for compactness and condensation frequently leads the author to crowd too many things into one statement, as for example the long sentence which covers nearly the whole of page 21; and the still longer one beginning on page 25, ending exactly one page later, and including a number of different theta relations. In this last case the attempt to include so much in one sentence forces the last formula of page 25 out of its logical connection, since it is not a deduction from the one immediately preceding it, but is a special case of the formula given in the eighth line above.

Chapter II is in some sense a converse to part of the preceding chapter. Namely, starting with Kummer's quartic relation previously derived, it is shown how the variables x, y, z of this equation may be expressed as hyperelliptic functions of two variables w_1, w_2. Not only does this lead to the identification of these variables with the $\wp$-functions, but of especial significance is the culmination of the developments in a set of five partial differential equations whose general integral is

$$e^{a_1w_1+a_2w_2+b}\vartheta(w_1+c_1,\ w_2+c_2),$$

involving five arbitrary constants. A particular integral, denoted by σ, gives x, y, z as the second logarithmic derivatives with changed signs. It would at first glance seem quite out of the question to use these equations in order to obtain expansions of the σ-functions. The successful accomplishment of this step is one of the most brilliant achievements of the book. This is done by multiplying each equation by an arbitrary parameter and adding together. The result is then written in a condensed form by the aid of the Clebsch-Aronhold symbolic algebra. By means of this composite differential equation a recurring formula is obtained for the terms in the expansion of the σ-functions, of which six odd and ten even functions are introduced. "These expansions in their turn enable us to prove succinctly various relations involving the $\wp$-functions." The processes are quite analogous to those which are well known in the case of the elliptic σ-functions.

The remainder of Part I is devoted to a detailed study of the Kummer and Weddle surfaces, the geometrical relations developed being largely restricted to those which illustrate properties of the $\wp$-functions.

Part II is confined to a general investigation of periodic functions of n variables with the object of reducing the theory to that of algebraic functions. The method is admirable for its logical simplicity and directness. There is but one other book,* as far as we know, that gives this matter a systematic treatment, and that too from a transcendental point of view. The method of the present work is algebraic. Starting in Chapter VI with some introductory theorems on power series in several variables, we are led in Chapter VII to a study of periodic functions in the neighborhood of a given point. Particular consideration is given to their behavior when the values of the arguments u_i are restricted so as to depend on a single complex variable x as follows: Let $(a_1^{(r)}, \cdots, a_n^{(r)})$ $(r = 1, \cdots, n)$ be n sets of values of the arguments $u_1, \cdots, u_n$ at which the periodic functions $\phi(u_1, \cdots, u_n)$ is regular; define n functions $\phi_r(u)$ by means of the equations

$$\phi_r(u) = \phi(u_1 + a_1^{(r)}, \cdots, u_n + a_n^{(r)}) - \phi(a_1^{(r)}, \cdots, a_n^{(r)}) \qquad (r = 1, \cdots, n);$$

then form the n equations

$$\phi_r(u) = c_{r1}x + \cdots + c_{rn}x^n,$$

of which the determinant $|c_{rs}|$ is different from zero. Regarding the values of the n variables u_i as corresponding to the points of a real space S of $2n$ dimensions, the equations last written determine a locus C (or *construct*) of two degrees of freedom in S on which the functions ϕ_r are periodic. Any

* A. Krazer, Lehrbuch der Thetafunktionen, Leipzig, 1903, Chapter XI.

given branch Γ of C may be divided up into fundamental regions such that a particular value of x occurs the same number of times in each region. The values of the u_i at points in different regions corresponding to the same value of x differ by the addition of a set of periods. The derivative $y = du/dx$ of the function $u = \Sigma\lambda_i u_i$ is shown, from its behavior in Γ, to satisfy an algebraic equation $f(y, x, \lambda_1, \cdots, \lambda_n) = 0$ with coefficients which are rational in x, the degree in y being the number of values this derivative takes for any given value of x. To each value of x are thus associated values of $u_1, \cdots, u_n$ which are determined by the equations

$$u_s = -\int \frac{\partial f/\partial\lambda_s}{\partial f/\partial y} dx,$$

these integrals being of the first kind. It is in this way shown that the arguments of any multiply periodic function ϕ can be expressed as abelian integrals of the first kind associated with a Riemann surface, and that the periods of ϕ are the moduli of periodicity of these integrals on the surface. The integrals u_s are in general defective integrals on a Riemann surface of genus $p > n$. A long chapter, VIII, is then given to the study of defective integrals, including a detailed treatment of several simple cases.

Chapter IX is devoted to establishing the theorem which is one of the main objects of Part II, viz., the most general single-valued multiply periodic meromorphic function is expressible by theta functions whose arguments are written in the Jacobi form as linear functions of n integrals of the first kind.

The tenth and last chapter is a discussion of the number and sum of zeros of a set of Jacobian functions, that is, functions whose second logarithmic derivatives are periodic. As the Jacobian functions are expressible in terms of theta functions, we have (in a generalized form) Poincaré's theorem concerning the number of zeros common to a system of theta functions.

Several appendices are added for the elaboration of certain topics associated with the subject in hand. Four of these relate to the algebra of matrices, of which free use is made throughout the book; one note gives a proof of Abel's theorem and its converse; and a final note considers some examples of algebraic curves on the Kummer surface having defective integrals.

The book has one noticeable feature which will commend itself to all readers, namely, a separate caption for each page intended to indicate briefly the contents of that page. This is especially useful as the author seldom sums up results in a way readily to catch the reader's attention.

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BÔCHER'S HIGHER ALGEBRA.

Introduction to Higher Algebra. By MAXIME BÔCHER, Professor of Mathematics in Harvard University; prepared for publication with the cooperation of E. P. R. DUVAL, Instructor in Mathematics in the University of Wisconsin. New York, Macmillan, 1907. xi + 321 pp.

Einführung in die höhere Algebra. Von MAXIME BÔCHER, Deutsch von HANS BECK, mit einem Geleitwort von E. STUDY. Leipzig, Teubner, 1910. xii + 348 pp.

THE term "higher algebra" has been so often used in America to denote a very low type of merely formal algebra and to include subjects like infinite series, which are not properly algebraic at all, that it is refreshing to find a book like this one of Professor Bôcher's, which really corresponds to its title. It does so, not only by reason of the purely algebraic character of its material, but also because this material is worked up in a strictly logical as well as systematic manner.

The amount of available algebraic material is so enormous, and it branches out in so many different directions, that some selection is inevitable; even the extensive two-volume works of Weber and Netto are confined to certain special lines. The volume under review aims to furnish the reader with an *introduction* to the whole field, to lay a broad and deep foundation for further study, and in particular, to give an adequate algebraic preparation for the study of modern analytic geometry. This aim has been accomplished with remarkable success.

There is one special topic, however, to which the author gives more than an introduction, and that is the theory of elementary divisors (Elementarteiler). In the last three chapters he not only introduces elementary divisors in a most expeditious and satisfactory manner, but carries their theory through to a fair degree of completeness, so far as the more important applications are concerned.