Hence two contact transformations with the same transversality law will have a point transformation for alternant only when they are of the type

$$W = \sqrt{\alpha + 2\beta p + \gamma p^2}, \quad W_1 = \lambda \sqrt{\alpha + 2\beta p + \gamma p^2}.$$

Transversality is then expressed by a linear involutorial relation (15), *so that for each point the transversal of a given direction is the conjugate direction with respect to a conic with that point as center.*

8. A less important converse result, relating to the type considered in § 3, we state without proof. The only contact transformations which in combination with *every* transformation of type $W = \Omega \sqrt{1 + p^2}$ give a point transformation for alternant are those of the same type. The same is true even if Ω is restricted to the form $a(x^2 + y^2) + bx + cy + d$, a case of interest since then W converts circles into circles. When a vanishes the transformation belongs to the equilong class of Scheffers.

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ON AN INTEGRAL EQUATION WITH AN ADJOINED CONDITION.

BY ANNA J. PELL.

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IN his doctor dissertation * Professor Cairns develops for infinitely many variables the theory of a quadratic form with an associated linear form, in order to prove the existence of solutions of the following integral equation :

$$\phi(s) = \lambda \int_a^b K(s, t)\phi(t)dt + \mu p(s), \tag{1}$$

with the adjoined condition

$$\int_a^b \phi(s)p(s)ds = 0, \tag{2}$$

where $K(s, t)$ is a given continuous symmetric function of s and t, $p(s)$ a given continuous function of s, λ and μ are parameters, and $\phi(s)$ is the function to be determined.

* "Die Anwendung der Integralgleichungen auf die zweite Variation bei isoperimetrischen Problemen," Göttingen, 1907.

A geometrical consideration suggests the possibility of transforming the equations (1) and (2) into an equivalent homogeneous integral equation with a symmetric kernel, the existence of whose solutions has already been shown by Hilbert and others. In this paper such a transformation will be carried out.

If there is a solution $\phi(s)$, the corresponding μ may be expressed

$$\mu = \frac{-\lambda \int_a^b \int_a^b K(s, t)p(s)\phi(t)dsdt}{\int_a^b [p(s)]^2 ds},$$

and the solution $\phi(s)$ satisfies the integral equation

$$\phi(s) = \lambda \int_a^b L(s, t)\phi(t)dt, \tag{3}$$

where

$$L(s,t) = K(s, t) - \frac{p(s)\int_a^b K(t, t_1)p(t_1)dt_1}{\int_a^b [p(s)]^2 ds}.$$

Conversely any solution of (3) satisfies (1) and (2). The kernel $L(s, t)$, however, is not symmetric and we cannot make any definite conclusions about the existence and character of the characteristic number λ.

Consider now the integral equation

$$\phi(s) = \lambda \int_a^b M(s, t)\phi(t)dt, \tag{4}$$

where

$$M(s, t) = L(s, t) - p(t)\frac{\int_a^b K(s, t_1)p(t_1)dt_1}{\int_a^b [p(s)]^2 ds}.$$

The kernel $M(s, t)$ is symmetric in s and t, and therefore, unless $M(s, t)$ is identically equal to zero, there exists for at least one real value of λ a solution $\phi(s)$, not identically equal to zero, of the integral equation (4).

Any solution of the equations (1) and (2) satisfies the equation (4); we must investigate under what conditions a solution of (4) is also a solution of (1) and (2).

Case I. $M(s, t) \not\equiv 0$. There exists a solution $\phi(s)$ of the equation (4); multiply this equation by $p(s)$ and integrate from a to b and we see that $\phi(s)$ satisfies the condition (2), unless the corresponding characteristic number is given by

$$\lambda_0 = \frac{-\int_a^b [p(s)]^2 ds}{\int_a^b \int_a^b K(s, t) p(s) p(t) ds dt}.$$

Suppose first that $\lambda \neq \lambda_0$; then it can easily be verified that $\phi(s)$ satisfies both (1) and (2).

If λ_0 is a characteristic number of the kernel $M(s, t)$, the function $p(s)$ is always a corresponding solution, and $p(s)$ is clearly not a solution of (1) and (2). Let $\psi(s)$ be any other solution of (4) corresponding to λ_0; then the function

$$\psi(s) - \frac{p(s) \int_a^b p(s)\psi(s) ds}{\int_a^b [p(s)]^2 ds}$$

is a solution of (4) and also of (1) and (2). Hence there exists a solution of (1) and (2) unless λ_0 is the only characteristic number of $M(s, t)$ and $p(s)$ the only corresponding solution; in this case $M(s, t)$ has the form

$$(5) \qquad M(s, t) = kp(s)p(t),$$

where k is some constant not equal to zero.

Case II. $M(s, t) \equiv 0$. The equations (1) and (2) have no solution not identically equal to zero.

The final result is that the given integral equation always has a solution unless $K(s, t)$ and $p(s)$ satisfy the relation (5) * (where k may take the value zero). Further, the solutions of (1) and (2) form an orthogonal system of functions.

From the expansion theorem for the symmetric kernel $M(s, t)$ we obtain the following: any function $f(s)$ expressible in the form

* The exceptional cases are not indicated in the dissertation referred to.

$$f(s) = \int_a^b K(s, t) g(t) dt,$$

where $g(s)$ is any continuous function satisfying

$$\int_a^b p(s) g(s) ds = 0,$$

can be developed into the uniformly convergent series

$$f(s) = \frac{p(s) \int_a^b p(s) f(s) ds}{\int_a^b [p(s)]^2 ds} + \sum_i \phi_i(s) \int_a^b \phi_i(s) f(s) ds,$$

where $\phi_i(s)$ are the normalized solutions of (1) and (2).

That this expansion may not hold in case $g(s)$ is any continuous function (as Mr. Cairns states the theorem) is shown by the special example

$$K(s, t) = A(s)p(t) + A(t)p(s) + B(s)B(t),$$

$$A(s) \not\equiv cB(s), \quad \int_a^b A(s)p(s)ds = 0, \quad \int_a^b B(s)p(s)ds = 0,$$

$$g(s) = p(s).$$

THE UNIFICATION OF VECTORIAL NOTATIONS.

Elementi di Calcolo vettoriale con numerose Applicazioni. By C. BURALI-FORTI and R. MARCOLONGO. Bologna, Nicola Zanichelli, 1909. v + 174 pp.

Omografie vettoriali con Applicazioni. By C. BURALI-FORTI and R. MARCOLONGO. Torino, G. B. Petrini, 1909. xi + 115 pp.

1. IN view of the plan that the fourth international congress of mathematicians held at Rome in 1908 should discuss the notations of vector analysis and perhaps lend the weight of its recommendation to some particular system, Burali-Forti and Marcolongo awhile ago set themselves the laudable but somewhat thankless task of collecting and editing all the historical, critical, and scientific material which might be indispensable to a proper settlement of the question by the congress, and this material they published in a series of five notes beginning in