BAIRE'S LEÇONS D'ANALYSE.

Leçons sur les Théories générales de l'Analyse. Par RENÉ BAIRE. Tome I: *Principes fondamentaux, Variables réelles.* 1907. 8vo. 17 figures, x + 232 pp. 8 fr. Tome II: *Fonctions analytiques, Équations différentielles, Applications géométriques, Fonctions elliptiques.* 1908. 8vo. 35 figures, x + 347 pp. 12 fr. Paris, Gauthier-Villars.

THE prefaces of these books express in vigorous and convincing language the author's beliefs and plans. Baire has abundantly demonstrated his right to strong opinions; what he thinks regarding comparatively elementary instruction is not to be despised. As a specimen, I quote the following to avoid loss of force in translation: Rigeur et simplicité ne sont nullement inconciliables, si l'on prend nettement le parti de faire pénétrer, dans l'enseignement des principes fondamentaux, certaines idées qui ont été acquises à la Science dans l'étude de questions d'ordre plus élevé. Pour en prendre un exemple frappant, la notion de bornes supérieures et inférieures d'un ensemble, qui commence seulement à être vulgarisée."

This bugle call to the standard of rigor may affright some to whom rigor and difficulty seem synonymous: precisely to such persons Baire's treatise will be a revelation — it *is* simple. That it is also reasonably accurate, Baire's name and the preceding quotation guarantee; indeed one's expectation outruns the author's intention, and one notes the careful avoidance of difficult questions far more than any tendency to finesse.

A very special interest attaches to the introductory work and to the treatment of the foundations of the subject, both because the presentation is somewhat novel, and because the main body of the work is strictly limited to rather usual topics which give rise to little comment.

The first chapter is a treatment of the fundamental concepts of irrational numbers, sets of points, limits, and continuity. As noted in the preface, this chapter is substantially a reproduction of Baire's Théorie des nombres irrationels, des limites et de la continuité, published in 1905 by Vuibert et Nony. As a whole it is clear, exact, and elegant, and may well serve any student as an introduction to this subject.

A consistent treatment of irrationals by the Dedekind cut process is followed by brief treatments of the bounds of an assemblage and of the limits of sequences. In the latter topics $\pm\infty$ are recognized as bounds, or as limits "in the extended sense"; whether such infinite limits are excluded or included either is stated or is determinable from the context, but the student might go astray occasionally, for example, in section 20, page 13.

In defining continuity, the sequence notion is used on page 21, but it is shown later, on page 46, that this definition coincides with the usual ε definition. The definition of uniform continuity is phrased in ε form immediately. These definitions are given for several variables, and essentially for any point set; they are then used to establish the four fundamental operations of algebra by means of the fundamental principle of extension of definition (page 28) from the rational to the real numbers.

There follow interesting treatments of concrete quantities, of relations of geometry to algebra, of sets of points, and of the properties of continuous functions. Among other fundamental topics, I shall mention the proof that the two definitions of continuity are equivalent, and the theorem on inverse functions (page 51). The latter, together with the fundamental theorem of extension, is used to establish such functions as x^y, $\log x$, and so on. The chapter ends with a very brief, but satisfactory, treatment of elementary theorems on series of constant terms.

The fundamental principles of the differential and integral calculus are welded in Chapter II into a single link in the mathematical chain; in fact, the traditional separation of those topics is characterized in the preface of the second volume as "superannuated." The sums which lead to the integral are introduced on page 74 via the law of the mean, which, together with other usual properties of derivatives, has been carefully proved. Though this does "lead us to study such sums a priori," the device lacks the ring of naturalness — the weld is obviously artificial. Still, the process leads directly to the fundamental properties of the integral without resort to geometry; the most important of these properties are derived here with characteristic brevity and clearness.

The remainder of the chapter contains rather usual theorems on elementary, improper, and parametric integrals, implicit functions and functional dependence, derivatives and differentials of higher order, differential equations and total differentials. Indeed, I need only mention one or two points to show that the author was looking rather toward broad presentation under somewhat easy restrictions than toward the extreme niceties. Thus the original definition of integral (page 74) is restricted to continuous functions (extended later in the case of improper integrals); the theorem on implicit functions assumes the existence and continuity of all the first derivatives; and so on.

There are even one or two cases of what might seem carelessness in another, but which are doubtless rather evidences of Baire's desire to avoid complicating detail. For example, the hypothesis that dy/dx or else dx/dy should exist and be different from zero is suppressed on page 65. On page 72, section 84, the condition that f'_x, etc., be continuous, which is necessary, *for this proof*, is omitted. On page 117, n is used in two senses, for it is necessary to assume, *for this proof*, that the derivatives of order $n+1$ are continuous. The hypothesis that f'_x, f''_{xx}, etc., be continuous is omitted on page 121 in the theorem on d^2u. Finally, in section 94, page 81, a proof of the formula for change of variable in an integral is given in new form; it is quite satisfactory, but to the author's hypothesis that $x = \phi(t)$ possess a continuous derivative may well be added the provision that t_0 and t_1 be so chosen that x remains in the interval $x_0 \leqq x \leqq x_1$.

The author's opinions are on the whole against the use of higher differentials. Even the one advantage recognized (preface, page vi) — that of writing relations valid for any later choice of independent variables — is by no means impossible with the derivative notations, and the fact mentioned in lines 18–19 of page 125 seems to rob this advantage of its force.

Among the most interesting and novel features of the whole work are the very elegant presentations of the ideas of length, area, and volume in Chapter III. A precise treatment of length is followed by an axiomatic development of area — which is not used in the original presentation of integration. While the ideas are not new, the presentations are so; they will be found interesting even to the initiated. A still simpler presentation seems possible, however, in the case of the area under a continuous curve $y = f(x) \geqq 0$. For let us assume the following axioms:

(1) The area of any rectangle is the product of its base and its height.

(2) The area of the figure formed by placing a finite number of rectangles in juxtaposition is the sum of the areas of those rectangles.

(3) The area of any region, if it exists, is a unique number.

(4) The area of any part of a region is less than or at most equal to the area of the whole region.

With these axioms, it follows immediately that the area between $x = a$, $x = b$, $y = 0$, $y = f(x)$ is necessarily $\int f(x)\,dx$, whenever that integral exists in Riemann's sense. Such a presentation, while admittedly not quite so far reaching as that of the text, would be wholly in keeping with the spirit of the work as a whole, in avoiding extreme cases.

Before passing to the question of volume, the double integral is defined, and its properties are discussed. The proof of the formula for change of variables is mentioned in the preface, and deserves notice here. The proof is given first for linear transformations, and is extended to the general case by use of the law of the mean. It is evident that Baire considered this presentation very carefully; the only objection appears to be its length, which seems out of proportion to some other parts in view of the fact that other proofs — artificial, to be sure — are available.

The treatment of improper double integrals (pages 177–179) is so brief that it may be misleading; the impression that circles may be used exclusively is not strictly justified by the text, but that impression will doubtless be received by a superficial reader.

The discussion of volume is exactly similar to that of area. It is followed by a treatment of triple integrals and a very extended treatment of the area of a surface. Especially the latter will be found interesting, but it is clearly out of proportion to the rest of the work. An excellent presentation of line and surface integrals, together with the theorems of Green and of Stokes, concludes this chapter and the first volume.

Volume II opens with a rather brief treatment of functions of complex variables. The presentation is general rather than an exposition of any one school; the demonstrations are conducted by methods of real variables whenever desired; emphasis is laid on the developability of any function which has a derivative; and only theorems of unquestionable prominence are given. Functions of several complex variables receive considerable attention; theorems are proved for several variables when convenient; and the results are used in proving such theorems as that for differentiating under the integral sign.

The most interesting portion of this chapter is the section on infinite series, which includes the theory of real series, since only series of constant terms were treated in Volume I. Baire makes extensive use of what he calls "normally convergent" series, i. e., series which satisfy the Weierstrass test for uniform convergence. It is shown that any uniformly convergent series may be transformed into a normally convergent series by properly grouping consecutive terms, and the usual forms of statement are deduced from the results for normally convergent series. The new phrase is certainly convenient; its use in this book abundantly justifies its introduction.

Chapter V contains a treatment of differential equations, confined for the most part to elementary methods of actual solution. It includes, however, an existence proof for a set of linear equations (pages 86–89), and a very excellent treatment of characteristics of differential equations of the first order (pages 148–164). Partial differential equations of higher order are barely mentioned.

Chapter VI deals with applications to geometry. Again the work is quite restricted to very usual theorems, the topics considered being the usual elementary theorems regarding such ideas as lines of curvature, asymptotic lines, applicability of surfaces, geodetic lines, and so on. The presentation is far more traditional than in the previous chapters.

The final chapter is entitled elliptic functions, but a considerable portion of it is devoted to infinite products. The treatments are quite elementary and traditional, as in several chapters of this volume. Here, as in the chapter on complex variables, little is said of geometric representation of the Riemann type — another example of the policy of rigid exclusion of all dispensable materials.

On the whole, Baire's work fulfils the promise of his prefaces; the work is simple, restricted to rather fundamental topics, and yet accurate. The manner of presentation is often quite new, and is always clear and effective. There is some lack of uniformity, some loss of proportion, of one topic as compared with another; but this must remain a question of opinion, possibly of tradition, even of prejudice. Thus the foundations are laid wide and strong and as if for eternity. That the superstructure is even less imposing than in many of the older Cours may be disappointing; it would seem reasonable and reassuring, however, to assume that Baire wished to build so as to leave for others not the task of rebuilding from the very ground what he has done here, but rather that of starting where he stops, assured that the foundations he has laid will stand the strain of an enormous superstructure.

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