REMARKS CONCERNING THE SECOND VARIATION FOR ISOPERIMETRIC PROBLEMS.

BY PROFESSOR OSKAR BOLZA.

IN his lectures on the calculus of variations (Göttingen, 1904–05), Hilbert has given an elegant modification of Weierstrass's proof of Euler's rule for isoperimetric problems, which reduces the proof to the consideration of an ordinary extremum with a condition. The object of the present note is to show how the same method can also be applied to the *second variation.*

§ 1. *Hilbert's Proof of Euler's Rule.**

We consider the problem of minimizing the integral

$$(1) \qquad J = \int_{x_1}^{x_2} f(x, y, y')dx, \qquad y' = \frac{dy}{dx},$$

with respect to the totality of curves $y = y(x)$ of class C' which join two given points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, lie in a certain region $\mathfrak{R}$ of the x, y-plane, and furnish for the integral

$$(2) \qquad K = \int_{x_1}^{x_2} g(x, y, y')dx$$

a given value l. The functions f and g are supposed of class C''' in the domain

$$(x, y) \text{ in } \mathfrak{R}, \qquad -\infty < y' < +\infty.$$

Let

$$\mathfrak{E}_0: \qquad y = y(x), \quad x_1 \leqq x \leqq x_2$$

be a solution of the problem which lies in the interior of the region $\mathfrak{R}$, and which is not at the same time an extremal for the integral K. Then we can obtain, according to Weierstrass, a set of admissible variations of $\mathfrak{E}_0$ as follows:

Let $\eta_1(x)$, $\eta_2(x)$ be two arbitrary functions of x of class C' in the interval (x_1x_2), and vanishing at x_1 and x_2; then the curve

$$y = y(x) + \epsilon_1\eta_1(x) + \epsilon_2\eta_2(x) \equiv Y(x, \epsilon_1, \epsilon_2), \qquad x_1 \leqq x \leqq x_2$$

*The same proof is given by Kneser, "Euler und die Variationsrechnung," *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, Bd. XXV (1907), p. 50.

will be an admissible variation of $\mathfrak{C}_0$ provided that the two constants ϵ_1, ϵ_2 satisfy the relation

$$(3) \qquad K(\epsilon_1, \epsilon_2) \equiv \int_{x_1}^{x_2} g(x, Y, Y')\, dx = l.$$

Hence,* *the function*

$$J(\epsilon_1, \epsilon_2) \equiv \int_{x_1}^{x_2} f(x, Y, Y') dx$$

of the two variables ϵ_1, ϵ_2 *must have, for* $\epsilon_1 = 0$, $\epsilon_2 = 0$, *a minimum restricted by condition* (3).

But if a function $f(x, y)$ has for $x = a$, $y = b$ a (relative) minimum restricted by the condition

$$(4) \qquad \phi(x, y) = 0,$$

and if the two derivatives $\phi_x(a, b)$, $\phi_y(a, b)$ are not both zero, there exists a quantity λ such that

$$(5) \quad f_x(a, b) + \lambda\phi_x(a, b) = 0, \qquad f_y(a, b) + \lambda\phi_y(a, b) = 0,$$

and moreover, if we put

$$F = f + \lambda\phi,$$

the inequality

$$(6) \qquad F_{xx}\phi_y^2 - 2F_{xy}\phi_x\phi_y + F_{yy}\phi_x^2 \Big|_{y=b}^{x=a} \geqq 0$$

must hold.†

Applying these results to the function $J(\epsilon_1, \epsilon_2)$, it follows in the first place that there must exist a constant λ such that the two equations

$$(7) \qquad J_1 + \lambda K_1 = 0, \quad J_2 + \lambda K_2 = 0$$

hold simultaneously, where

$$J_i = \left(\frac{\partial J}{\partial \epsilon_i}\right)_0 = \int_{x_1}^{x_2} (f_y \eta_i + f_{y'}\eta_i')dx,$$

$$K_i = \left(\frac{\partial K}{\partial \epsilon_i}\right)_0 = \int_{x_1}^{x_2} (g_y \eta_i + g_{y'}\eta_i')dx,$$

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* Here Hilbert's proof branches off from Weierstrass's; the latter proceeds by solving (3) with respect to ϵ_2. Compare for instance my Lectures, p. 208.

† In order to prove these statements, it is only necessary to solve, by means of Dini's theorem on implicit functions, equation (4) with respect to y, $y = \mathfrak{y}(x)$, and to apply the ordinary rule for an extremum of functions of one variable to the compound function $f(x, \mathfrak{y}(x))$. The proof presupposes that f and ϕ are of class C'' in the vicinity of (a, b).

provided, however, that K_1 and K_2 are not both zero. But since $\mathfrak{C}_0$ is not an extremal for the integral K, we can always choose η_2 so that $K_2 \neq 0$.

The function η_2 being so chosen, the second of the equations (7) determines λ and shows that the value of λ is certainly independent of the choice of the function η_1. Hence it follows from the first of the equations (7), according to Du Bois-Reymond's lemma, that $y(x)$ *must satisfy the differential equation*

$$h_y - \frac{d}{dx} h_{y'} = 0, \tag{8}$$

where

$$h = f + \lambda g. \tag{9}$$

§ 2. *Application of Hilbert's Method to the Second Variation.*

We proceed next to apply the second necessary condition (6) to the extremum of the function $J(\epsilon_1, \epsilon_2)$. The condition takes here the form

$$\begin{aligned} K_2^2(J_{11} + \lambda K_{11}) - 2K_1K_2(J_{12} + \lambda K_{12}) \\ + K_1^2(J_{22} + \lambda K_{22}) \geqq 0, \end{aligned} \tag{10}$$

where

$$J_{ij} = \left(\frac{\partial^2 J}{\partial\epsilon_i \partial\epsilon_j}\right)_0, \qquad K_{ij} = \left(\frac{\partial^2 K}{\partial\epsilon_i \partial\epsilon_j}\right)_0.$$

But by an easy computation the inequality (10) reduces to

$$\int_{x_1}^{x_2} (h_{yy}\eta^2 + 2h_{yy'}\eta\eta' + h_{y'y'}\eta'^2)dx \geqq 0, \tag{11}$$

where

$$\eta = K_2\eta_1 - K_1\eta_2. \tag{12}$$

The inequality (11) must hold for all functions η_1, η_2 of class C' which vanish at x_1 and x_2, and it must be remembered that K_1 and K_2 vary with η_1 and η_2 respectively.

From the definition of K_1 and K_2 it follows that

$$\int_{x_1}^{x_2} (g_y\eta + g_{y'}\eta')dx = 0. \tag{13}$$

Conversely, the equation (12) represents the most general function η of class C' which vanishes at x_1 and x_2 and satisfies (13). For let η be any function satisfying these three conditions. Then choose for η_2 any function which satisfies the two first of them and for which $K_2 \neq 0$. Let c be a constant arbitrarily chosen and determine η_1 by the equation

$$\eta = K_2\eta_1 - c\eta_2.$$

Then η_1 is of class C', vanishes at x_1 and x_2, and $c = K_1$, as follows from (13).

Hence we obtain the result: *The inequality* (11) *must be satisfied for all functions* η *of class* C' *which vanish at* x_1 *and* x_2 *and satisfy the relation* (*13*).

This is the theorem which we wished to establish by means of Hilbert's method. In order to recognize the advantages of the new proof, let us compare it with the ordinary proof which proceeds as follows:

Suppose we had found — no matter how — a one-parameter set of admissible variations of the curve $\mathfrak{C}_0$,

$$y = \bar{y}(x, \epsilon). \tag{14}$$

Then we must have for this set

$$\delta^2 J \geqq 0. \tag{15}$$

In the discussion of this inequality *two difficulties* arise which have no analogue in the corresponding discussion for the unconditioned problem :

1) The integrand in $\delta^2 J$ contains *the second variations* $\delta^2 y$, $\delta^2 y'$, whereas in the unconditioned problem only the first variations δy, $\delta y'$ occur, owing to the fact that in the latter problem it is sufficient to consider variations of the simplest type

$$\bar{y}(x, \epsilon) = y(x) + \epsilon\eta(x),$$

which is no longer possible in the isoperimetric problem.

This difficulty is removed by eliminating* $\delta^2 y$, $\delta^2 y'$ by means of the equation $\delta^2 K = 0$, the result being

$$\delta^2 J + \lambda\delta^2 K \geqq 0. \tag{1}$$

2) This inequality, which contains only the first variations δy, $\delta y'$, must hold for all functions δy which can be derived from an admissible variation (14) by the δ-process. But the totality of these functions is identical, according to a lemma† due

* Compare the remarks by Swift, BULLETIN, vol. 14 (1908), p. 373.

† Compare for instance my Lectures, p. 214.

to Weierstrass, with the totality of those functions δy of class C' which vanish at x_1 and x_2 and satisfy the relation $\delta K = 0$.

The proof of this lemma — which is an essential step in the chain of conclusions, and whose omission forms a serious gap in the older theory — constitutes the second difficulty.

Neither of these difficulties occurs in the proof which we have given above.

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NOTES ON THE SIMPLEX THEORY OF NUMBERS.

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I. *Continued Product of the Terms of an Arithmetical Series.*

1. Let a and c be two relatively prime positive integers and form the arithmetical series

$$xa + c, \quad (x = 0, 1, 2, \cdots, n - 1).$$

If we inquire what is the highest power of a prime p contained in the product

$$\prod_{x=0}^{x=n-1} (xa + c), \quad a \not\equiv 0 \pmod{p},$$

we shall find that the general result takes an interesting form. The solution of the problem may be effected in the following manner:

Evidently there exists some number x such that $xa + c$ is divisible by p. Let i be the smallest value of x for which this division is possible, and let c_1 be the quotient thus obtained. Using the notation

$$(1) \qquad H\{y\}$$

to represent the index of the highest power of p contained in y, we will show that

$$(2) \qquad H\left\{\prod_{x=0}^{x=n-1} (xa + c)\right\} = H\left\{\prod_{x=0}^{x=e_1} (xa + c_1)\right\} + e_1 + 1,$$