NOTE ON FOURIER'S CONSTANTS.

BY PROFESSOR E. H. MOORE

(Read at the Preliminary Meeting of the Southwestern Section of the American Mathematical Society, December 1, 1906.)

A FUNCTION $f(x)$ real and single valued, bounded, and in the sense of Riemann integrable from 0 to 2π gives rise to the Fourier's series

$$(1)\quad \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \tfrac{1}{2} \sum_{k=-\infty}^{+\infty} (a_k \cos kx + b_k \sin kx),$$

where

$$(2)\quad a_k = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos kx\, dx, \qquad b_k = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin kx\, dx. \qquad (k = 0, \pm 1, \cdots)$$

Hurwitz * calls the constants a_k, b_k the Fourier's constants of the integrable function $f(x)$, and for the realm of integrable functions he studies the theory of Fourier's constants instead of the theory of Fourier's series which are convergent only under conditions.

The realm of integrable functions is a realm of integrity, *i. e.*, the sum, the difference, and the product of two integrable functions is integrable. The constants a_k, b_k, or in more explicit notation a_{kf}, b_{kf}, depend in a linear distributive way on the function $f = f(x)$. Further, the constants a_{kh}, b_{kh}, for the product $h(x) = f(x)\, g(x)$ of two functions $f(x)$, $g(x)$ are determinable from the constants a_{kf}, b_{kf} and a_{kg}, b_{kg} of these functions. In fact, from (2) we have

$$(3)\quad a_{0f} = \frac{1}{\pi}\int_0^{2\pi} f(x) dx,$$

$$(4)\quad a_{kf} = a_{0,f(x)\cos kx}, \quad b_{kf} = a_{0,f(x)\sin kx},$$

* Hurwitz, "Ueber die Fourierschen Konstanten integrierbarer Funktionen," *Math. Annalen*, vol. 57 (1903) pp. 425–446, and vol. 59 (1904), p. 553.

so that

$$(5) \qquad a_{kh} = a_{0,fg\cos kx}, \quad b_{kh} = a_{0,fg\sin kx}.$$

These formulas in effect make the expression of a_{kh}, b_{kh} depend simply upon the expression of a_{0h} in terms of the constants of f and g. The fundamental theorem of the theory of Fourier's constants states that

$$(6) \quad a_{0h} = \frac{1}{\pi}\int_0^{2\pi} f(x)g(x)dx = \tfrac{1}{2}a_{0f}a_{0g} + \sum_{k=1}^{\infty}(a_{kf}a_{kg} + b_{kf}b_{kg}).$$

If either $f(x)$ or $g(x)$ is expressible as a uniformly convergent Fourier series, the relation (6) follows by termwise integration. In its generality the fundamental theorem with important applications is due to de la Vallée Poussin * (1893), and it has been extended (1903) by Stekloff to all the types of Fourier constants frequently employed in analysis in connection with the representation of arbitrary functions of one or of several variables.

Hurwitz (loc. cit., volume 57, page 437) expresses the constants of $h(x) = f(x)\,g(x)$ in terms of those of $f(x)$, $g(x)$, apart from notation, as follows:

$$a_{0h} = \tfrac{1}{2}a_{0f}a_{0g} + \sum_{k=1}^{\infty}(a_{kf}a_{kg} + b_{kf}b_{kg}),$$

$$(7) \quad a_{nh} = \tfrac{1}{2}a_{0f}a_{ng} + \tfrac{1}{2}\sum_{k=1}^{\infty}\{a_{kf}(a_{k+n\,g} + a_{k-n\,g}) + b_{kf}(b_{k+n\,g} + b_{k-n\,g})\},$$

$$b_{nh} = \tfrac{1}{2}a_{0f}b_{ng} + \tfrac{1}{2}\sum_{k=1}^{\infty}\{a_{kf}(b_{k+n\,g} - b_{k-n\,g}) - b_{kf}(a_{k+n\,g} - a_{k-n\,g})\}.$$

Hurwitz remarks further that these series converge absolutely, even when the terms at present collected into binomial expressions are taken as the constituent terms of the series. In the formulas (7) the rôles of f and g may of course be interchanged.

The series on the right of the equations (7) should however be expressed symmetrically with respect to f and g.

Let us use as Fourier constants

$$(\bar{2}) \quad A_{kf} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\cos kx\,dx, \qquad B_{kf} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\sin kx\,dx,$$

* For references see the memoirs of Hurwitz

viz., the halves of the constants according to Hurwitz. Then with use of the relations

$$(\overline{2}^\circ) \qquad A_{-kf} = A_{kf}, \qquad B_{-kf} = -B_{kf}$$

one readily finds from (7)

$$(\overline{7}) \qquad \begin{aligned} A_{nh} &= \sum_{k+l=n}^{k,\, l=0,\, \pm 1, \dots} (A_{kf} A_{lg} - B_{kf} B_{lg}), \\ B_{nh} &= \sum_{k+l=n}^{k,\, l=0,\, \pm 1, \dots} (A_{kf} B_{lg} + B_{kf} A_{lg}). \end{aligned} \qquad (n = 0, \pm 1, \cdots)$$

These equations $(\overline{7})$ have the desired symmetric form. The form suggests moreover the introduction of the complex Fourier constants

$$(\overline{2}_a) \qquad C_{kf} = A_{kf} + jB_{kf} \qquad (j^2 = -1),$$

in terms of which we have

$$(\overline{7}_a) \qquad C_{nh} = \sum_{k+l=n}^{k,\, l=0,\, \pm 1, \dots} C_{kf} C_{lg} \quad (n = 0, \pm 1, \cdots).$$

This result may be expressed thus:

To every integrable function $f(x)$ there corresponds a system of Fourier constants A_{kf}, B_{kf}, C_{kf} $(\overline{2}, \overline{2}_a)$ and a Laurent series with coefficients C_{kf}

$$f(x) \sim \sum_{k=-\infty}^{+\infty} C_{kf} Z^k.$$

This Laurent series is to be considered formally. Then the Laurent series corresponding to the product of two functions is the formal product of the two Laurent series corresponding respectively to the two functions.

As thus expressed, the fundamental theorem of the theory of Fourier's constants points to the known expression of Fourier series for suitably conditioned functions as convergent Laurent series.

THE UNIVERSITY OF CHICAGO,
December 7, 1906.