order 3 into its inverse and the non-invariant operators of order 2 in the dihedral rotation group of order 12 into themselves multiplied by the invariant operator of order 2. In case of the groups of order 40, 48 and 72 all the possible groups are obtained by forming the direct product of these groups and some group of odd order.

STANFORD UNIVERSITY,
November, 1905.

TYPES OF SERIAL ORDER.

*The Continuum as a Type of Order: An Exposition of the Modern Theory. With an Appendix on the Transfinite Numbers.** By EDWARD V. HUNTINGTON. Cambridge, Mass., The Publication Office of Harvard University, 1905. 4to. 63 pp. Price, 50 cents.

THE *Annals of Mathematics* has for some time followed the plan of printing articles expository of subjects which are little known or not easily accessible in the English language. Reprints of these articles are then placed on sale with the double and laudable purpose of making the circulation of the article wider than it would otherwise be and of helping solve the difficult problem of financing a mathematical journal.

The plan can hardly fail to succeed if all the articles are as clear in style and as just in the balance between generality and detail as is that of Professor Huntington. In point of readableness, we are inclined to think that the only other exposition of subjects connected with the foundations of mathematics which can be compared with Huntington's is that (in French) of L. Couturat.

The principal contents of the paper are the ordinal theory of integers, rational numbers, and the continuum, together with an appendix on the transfinite numbers of Cantor. It is intended for non-mathematical readers as well as for mathematicians, and therefore presupposes very little in the way of detailed knowledge, though of course it requires for complete comprehension a considerable maturity in abstract reasoning. We have noticed only one error of any consequence. It is

* Reprinted from the *Annals of Mathematics*, second series, vol. 6, No. 4 (July, 1905), and vol. 7, No. 1 (October, 1905).

stated in theorem 3, section 62, that "if a series is dense, it will also be dense in itself." That this is incorrect may be shown by an example (involving Cantor's order type Ω) similar to one given on page 169, volume 5, of the *Transactions*.

On page 184 appears an expression of opinion which is repeated in one form or another in several other places: "It is not possible to give purely ordinal definitions for the sums and products of the elements of such a series. All that we could do in this direction would be to define the sums and products of some particular dense series, say the series of the rational numbers in the usual order, by the use of some extraordinal properties peculiar to that series; then since all series of the type η are ordinarily similar, the definitions set up in the standard series could be transferred to any other series of the same type by a one-to-one correspondence. This method would be wholly inadequate, however, since the ordinal correspondence could be set up in an infinite number of ways." The reviewer is unable to see why it is incorrect to fix attention on a single correspondence and to base the definition on that one. The fact that there exist an infinitude of other correspondences giving similar definitions is a theorem which has no particular bearing on the validity of the method pursued in regard to the particular one with which we work. On the other hand, we do not wish to controvert the author's opinion that, from a didactic point of view, it is better to introduce such fundamental concepts as addition and multiplication at the outset of the theory.

In this connection it seems proper to refer to a logical distinction recently stated by Huntington. The reference to it here, however, must not suggest that Huntington allows such logical subtleties to interfere with the clearness of his elementary exposition. A mathematical science is said to be determined categorically by a set of postulates or conditions if it can be proved that any two classes of objects satisfying the conditions are capable of a one-to-one correspondence preserving the relations described in the postulates. For example, the Cantor definition of the linear continuum in terms of ordinal relations is such that any two sets of objects satisfying the conditions imposed (for example, the points of a parabola and the points of a straight line) are capable of a one-to-one correspondence preserving order. Huntington proposes to distinguish among the ways in which a set of conditions may be categorical.* In the case just mentioned the one-to-one correspondence can be set up in an infinitude of ways, whereas if the continuum is defined in terms of postulates about addition and multiplication as well as order, the correspondence can occur in only one way. This is in part because there now are specified in any set of elements satisfying the postulates two singular elements, zero and unity, which must correspond to the zero and unity elements of any other set which satisfies the postulates.

The distinction suggested by Huntington can be carried a step further by counting parameters. Thus in the case of the linear continuum, if the postulates are stated in terms of order and addition alone there is only one singular element, zero, and the unity element may be chosen arbitrarily. Thus we have ∞^1 correspondences. If the zero is also allowed to be arbitrary but a relation of equality of segments (such as the additive operation gives by the equation $b - a = b' - a'$) be retained, then we have ∞^2 correspondences. If order alone is used in the definition then Cantor's method of showing the correspondence gives ∞^∞ correspondences. That is, there exists no n such that ∞^n will describe the possibilities of choice. We seem to have arrived at a criterion by which, given two systems of axioms, one may determine which system is, as it were, the more categorical.

In plane geometry there is an interesting question which can be decided in the same way. Hilbert's Festschrift, in addition to its axioms of connection and order and its archimedean axiom, makes use of assumptions about congruence of segments and angles, a relation which it has long been known may be introduced by definition. This definition involves the choice of an arbitrary elliptic involution on the line at infinity (*i. e.*, of two conjugate imaginary points, the circular points) and thus may be made in ∞^2 ways. The question, therefore, arises whether congruence, which as an undefined symbol in the presence of order and continuity axioms is clearly redundant, does not after all make the system of axioms more categorical?

In setting up the one-to-one correspondence between any two classes satisfying the axioms of plane geometry a pair of coördinate axes for an analytic geometry is selected in each class.

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* Cf. page 159 of the paper under review and also *Transactions Amer. Math. Society*, vol. 6 (1905), p. 41.

If congruence axioms are present, a given line a of one class may correspond to any line a' of the ∞^2 lines in the other class. A given point A of a may correspond to any one of the ∞^1 points of a'. A line perpendicular to a at A, however, must correspond to a line perpendicular to a' at A'. The whole correspondence is therefore fixed when a unit of length $A'B'$ is chosen on a' to correspond to AB on a. There are thus ∞^4 ways in which the correspondence between two planes defined by order and congruence axioms may be set up. If order axioms alone are used, it is clear that the number of correspondences must be the same as the number of collineations of a cartesian plane into itself leaving the line at infinity invariant, and this count is ∞^6. A system of axioms stated in terms of order and congruence combined is therefore more categorical than one in terms of order alone.

OSWALD VEBLEN.

COLLEGE ALGEBRA.

A College Algebra. By HENRY BURCHARD FINE. Ginn and Company, 1905. viii + 595 pp.

AMONG all the text-books of college algebra, this book by Professor Fine, of Princeton, is distinguished by its broad grasp of the subject as a branch of mathematical science and by its entire freedom from the misleading statements and positive blunders which are all too common in our current text-books. What measure of success it will obtain in the class-room it would be hard to prophesy; my own impression is that on account of the close logical interdependence of its various parts the book will be found less available for those classes which wish to take up only a few detached chapters on separate topics than for those classes which can go through the whole subject in a systematic manner. However this may be, it is certainly a book which every teacher of mathematics, whether he happens to have a class in algebra or not, should own and read. In fact, if a mathematical library were to contain only one English text-book in algebra (besides the two-volume treatise of Chrystal) I should unhesitatingly recommend the present book.

The book is divided into two parts: "a preliminary part devoted to the number system of algebra, and a principal part devoted to algebra itself."