Beside the books and articles mentioned in this review, several of the recent treatises on the calculus contain a chapter on the subject. Among these should be mentioned at least the new edition of Serret-Bohlmann, Differential and Integralrechung, volume 3, and Goursat, Cours d'Analyse, volume 2. These books will doubtless be reviewed in their entirety in the BULLETIN, and I therefore satisfy myself here with a mere mention.

The works mentioned attest in the strongest possible manner the extraordinary vogue into which the calculus of variations has suddenly sprung. The cause is not far to seek: it is the revelation through the work of Weierstrass, Kneser, Hilbert and others that the calculus of variations is susceptible of the same exquisite rigor which had previously existed only in the theory of functions of a real variable, and that a wide field of research and rich discovery was opened by such methods.

Although the end of these investigations has by no means been reached in this single subject, it is not premature to suggest the analogous development of other mathematical theories along equally rigorous lines, and also the construction of a supplementary theory in each of them which shall be as rigorously applicable to general geometric problems as is the Weierstrass theory in the calculus of variations. E. R. HEDRICK.

COLUMBIA, MO.,
November, 1905.

GRANVILLE'S DIFFERENTIAL AND INTEGRAL CALCULUS.

Elements of the Differential and Integral Calculus. By W. A. GRANVILLE, Ph.D., with the editorial cooperation of PERCY F. SMITH, Ph.D. Ginn & Co., 1904. 463 pp.

So many text-books have been written upon the elementary branches of college mathematics that a raison d'etre can properly be asked upon the appearance of each new work. The great number of American text-books upon such subjects as college algebra, trigonometry and calculus, duplicating one another in aim and character, is in striking contrast with the paucity of our text-books upon more advanced mathematical subjects. What, then, we naturally ask, is the purpose of this new treatise, and what does it seek to accomplish which has not been already accomplished?

In the introduction the authors tell us that "the present volume is the result of an effort to write a *modern* text-book on the calculus which shall be essentially a drill book." In making this quotation I have taken the liberty to italicize the word "modern," inasmuch as it seems to me that the chief interest of the book, and the first as well as the last questions concerning it center in this word. The need of modern text-books on calculus is recognized by every one who is familiar with its modern developments and reconstruction. The older English text-books — as, for example, those of Todhunter and Williamson — fail altogether to satisfy the recent demands for precision and rigor. Some of the most fundamental concepts have not been formulated with mathematical definiteness, and hence the consequences of concepts and definitions can not be clearly traced. As a result, the theorems are commonly stated without restrictions. But, as universals, they are rarely true. Thus while the student gains a good knowledge of rules and formulas, he is left in ignorance both of the limits of their applicability and of the restrictions which must be imposed upon the functions to bring them under the theorems. Text-books of this nature are only too likely to produce thinkers of the same indefinite character; for like breeds like. Recently two or three English text-books, in particular Lamb's and Gibson's, have been written to remedy this defect. But notwithstanding manifold excellences these books are scarcely serviceable for a first course in our American colleges.

The composition of our classes often gives rise to grave pedagogical difficulties in teaching the calculus. Very rarely do these classes consist of a few choice spirits, for whom calculus is a preparation for the higher mathematics. The bulk of the class usually consists of men of very varied aims and ability. Some there will almost certainly be who are pursuing calculus for the sake of its applications in physics, chemistry and other sciences, as for example the engineer, by whom calculus is often used as a mechanical tool; some there may be who are preparing to teach mathematics in secondary schools, who do not care to advance into its higher branches and yet wish to gain a glimpse of modern methods and conceptions; others perchance who finish their mathematical education with calculus and aim merely to review and strengthen their previous knowledge; others still, who, under our elective system, take the course by instinct, without any definite object, because they have "a fondness for examples" and for working out the neat riddles and puzzles which mathematics affords. Such elements, and even others, may be included in a class. In such a constituency we find a "mathematisches Publicum" of the greatest interest and importance, from which we must continue to draw much of our best mathematical talent. It is, moreover, this Publicum which must be carefully cultivated to preserve for mathematics that wide influence on thought which it has had in the past, and which is so admirably traced in Merz's History of European thought in the nineteenth century.*

How shall calculus be taught to classes of such varied composition as has been described? How far is it possible, and, if possible, is it desirable and wise to introduce the modern developments into elementary teaching? The position assumed by the authors in their new treatise in answer to this question seems to the reviewer both a sane and a practical one.

First of all, it must be kept in mind that for an ordinary class the text-book must be primarily a drill book. Dr. Granville understands thoroughly this need, and it may be confidently predicted that his text-book will be a most decided success in the class room. Unusually attractive in appearance, the volume presents the subject matter clearly and bears the marks of a teacher who feels a student's difficulties. There is an unusually large and well-graded stock of examples, from which the student will gain an admirable review in the branches of mathematics preceding calculus. In this particular Granville's book resembles Osborne's treatise. It is provided also with good collections of formulas, of integrals, and of curves, for reference.

How to write such a drill book and keep it in touch with modern analysis is the chief problem of the author. This accord has been sought by drawing upon intuition for certain principles, of which the proof is postponed to a second course in the calculus. Upon these principles as a firm foundation the calculus can then be securely built. Such, at least, is my interpretation of what has been and should be sought.

Consider first the application of this method to the differential calculus. Here the foundations for a rigorous treatment

* This book is pervaded with a mathematical spirit. The influence of mathematics upon thought is considered, beginning with the seventeenth century.

are Rolle's theorem and the theorems of mean value. These are easily approached from the intuitional side. No attempt is made by the author to reproduce the delicate analysis by which they are established. He clearly recognizes that such an analysis is not suited for an elementary course. But his very brief intuitional and geometric treatment will be grasped at once by the student. Once obtained, these theorems serve as a sound basis for the theory of indeterminate forms, for Taylor's formula * and series, for the theory of envelopes, etc. This plan on the one hand has the advantage of avoiding considerations which are premature, and on the other hand removes the necessity of unlearning subsequently false demonstrations and methods. No such fallacious proofs need be foisted upon the learner as are found in many text-books for the expansion of a function into Taylor's series or for the theorem that in the partial differentiation of a function of two variables the order of the differentiations is immaterial. Such proofs confuse and enfeeble the perceptive powers of the learner.

Complete success in infusing an elementary text-book with the modern spirit could scarcely be expected. Let me point out one or two places where in pouring the new wine into the old bottles the skin has been badly cracked. One of these rents is found in the very theorem which was mentioned last; namely, that in obtaining $\partial^2 f(x,y)/\partial x \partial y$ the order of the two differentiations may be reversed. Here, in applying the mean value theorem to the two increments

$$f(x+\Delta x,\, y) - f(x,\, y) = \Delta x \cdot f'_x(x+\theta_1 \Delta x,\, y) \qquad (0 < \theta < 1),$$

$$f(x+\Delta x,\, y+\Delta y) - f(x,\, y+\Delta y) = \Delta x \cdot f'_x(x+\theta_1 \Delta x,\, y+\Delta y),$$

the fatal slip is made that a common value θ_1 is taken for θ in the two equations. (Cf. equation C, page 206.) The oversight contained in the proof (§ 124) of the rule for evaluating the indeterminate form ∞/∞ is certainly surprising.†

Let us turn now to the integral calculus. Here the first pedagogical difficulty relates to the definite integral

$$\int_a^b \phi(x)dx,$$

* Called by Granville "the extended theorem of mean value" (§ 118).

† When h approaches 0, $f(b)$ approaches ∞ so that $f(b)/F(x)$ itself tends to take the form ∞/∞ !

defined of course as the limit of

$$\sum_{x=a}^{x=b}\phi(x)dx.$$

Shall the existence of Darboux's upper and lower integrals be first demonstrated, and afterward their equality for a continuous function? Or shall the scope of the concept of an integral be contracted by taking all the increments dx equal to each other and multiplying each increment by the value of $\phi(x)$ at the beginning of the increments, as is done in most of our text-books? Granville and Smith wisely seek a middle course. Neither Darboux's integrals are introduced nor the above restricted concept of an integral. Appeal is again made to geometric intuition. By the consideration of the area under the arc of a continuous curve it is made clear that the two restrictions above described are unnecessary, and a corresponding analytic investigation is added, though of a somewhat incomplete character. I say "incomplete" because the investigation proves only that if $\phi(x)$ is continuous between $x = a$ and $x = b$, and *if there is a corresponding function* $f(x)$, *of which* $\phi(x)$ *is the derivative*, then there must be a limit for

$$\sum_{x=a}^{x=b}\phi(x)dx.$$

This is shown by the aid of the mean value theorem. Evidently the authors have thought it wisest to content themselves with this result. But in the opinion of the reviewer the all important fact should be blazed before the student that every function, continuous between a and b inclusive, has an integral between these limits. This theorem is not even stated, although obviously implied in the concept of the area under a curve. Moreover, it is needed by the author himself to complete his derivation (page 379) of the formula

$$s = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

for the length of a curve.* A suitable and simple proof of

* The theorem could have been used also as the basis of other formulas of the integral calculus, and is then preceded by a summation. This is the integral method of proof. The author chooses the differential method. Cf. §§ 223, 227, 228.

the theorem could be given readily, without entering upon all of the work necessary to establish Darboux's upper and lower integrals, with the aid of the geometric intuition that when $y = \phi(x)$ is continuous in an interval, a value δ can be found so small that the variation of $\phi(x)$ in any subinterval less than δ in length will be less than any arbitrarily assigned positive magnitude.* This very fundamental intuition is almost hidden by the author in the fine print of page 370; it should be thrown into bold relief.

A second deficiency is the failure to exhibit the two definitions of the definite integral — obtained (1) from the limit of a sum and (2) from the difference of two values of the indefinite integral — as two wholly independent and by no means coextensive definitions.

In these and some other particulars it is to be hoped that when a second edition of the book appears, it will be permeated yet further with the modern spirit. The restrictions respecting continuity imposed upon the derivatives in various theorems might be incorporated advantageously into the theorems themselves, instead of being relegated to footnotes or altogether omitted (e. g., § 118). Continued insistence that the student shall trace the effect of so primitive a concept as continuity and shall note the conditions introduced successively into a theorem or into a chain of theorems will help to impress upon him the fact, too often overlooked, that careful observance of the limitations imposed by the reasoning is fundamental in algebraic analysis as well as in synthetic geometry.

I have dwelt, perhaps unduly, upon certain aspects of an admirable text-book because of my great interest in the adaptation of calculus to the beginner and because of a belief that the most practicable plan is that followed by the authors, to use intuition when and only when it has been justified by exact analysis and then to build boldly upon it.

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* For first, it could be shown immediately from this intuition that the difference between the values of

$$\sum_{x=a}^{x=b} \phi(x) dx$$

for any two partitions whatsoever of the interval (ab) can be made as small as we choose by taking the maximum of $|dx|$ sufficiently small. Then it would remain only to prove that for some system of partitions $\Sigma\phi(x)dx$ converges to a limit. The latter fact is evident since for any infinity of values of $\Sigma\phi(x)dx$ there must be at least one point of condensation.

A few words should be added about the contents of the book. The usual range of topics is included, without neglecting applications in mechanics and with the addition of brief but suitable chapters upon ordinary differential equations and upon tangent lines and planes in space. Special attention is given to the parametric representation of curves. An unfortunate omission will be noted under the topic of differentials. It is not proved nor even remarked that the differential and increment of a function $f(x)$ differ from each other by an infinitesimal of higher order, although the proof of this important fact would occupy only a few lines. The application of differentials to the approximate computation of small increments of $f(x)$ is simultaneously excluded. The corresponding omissions in the case of the differential of a function of two or more variables are especially to be regretted, for the differential (§§ 136, 137) is left devoid of significance when the variables are independent.

In conclusion, generous recognition should be accorded to the care which has been bestowed upon the work. At many points improvements over our current text-books will be noticed, not in themselves sufficiently important to dilate upon but having together great cumulative force. As an instance, I shall cite the inclusion of a real proof that two functions which have a common derivative can differ only by a constant. The introductory chapter on the concepts continuity, function, and limits can also be especially commended, and the chapters on series and the expansion of functions. I know of no work which has greater promise of success in our college classes.

EDWARD B. VAN VLECK.

GENEVA,
September 20, 1905.

THE FOUNDATIONS OF SCIENCE.

Wissenschaft und Hypothese. Von HENRI POINCARÉ. Autorisierte deutsche Ausgabe mit erläuterenden Anmerkungen von F. und L. LINDEMANN. Leipzig, B. G. Teubner, 1904. xvi + 342 pp.

NOT logical enough for the logician, not mathematical enough for the mathematician, not physical enough for the physicist, not psychological enough for the psychologist, nor metaphysical enough for the metaphysician, Poincaré's Science and Hypothe-