THE EXTERIOR AND INTERIOR OF A PLANE CURVE.*

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THE theorem that a continuous closed curve without double points divides the plane into two regions, an exterior and an interior, has been discussed by several writers. Jordan † and Schoenflies ‡ assume the theorem for polygons and then extend it to apply to more general curves by processes which are extremely complicated. In the March number of the BULLETIN, Ames has sketched an apparently simpler method which applies to the so-called *regular* curves. The writer ventures to give below another proof for a class of curves which is more general in that the tangent is not assumed to exist, but which on the other hand may not include all of the regular curves. Polygons, curves consisting of a finite number of analytic pieces, and *regular* curves of the type treated by Schoenflies, are among those to which the method applies.

§ 1. *Hypotheses on the Curve.*

The curve C is taken in the form

$$x = \phi(t), \quad y = \psi(t), \tag{1}$$

where ϕ, ψ are considered for values of t in the interval

$$[T] \qquad t_0 \leqq t \leqq T$$

and are supposed to have the following properties:

1) ϕ, ψ are continuous on $[T]$;

2) the curve is closed, *i. e.*,

$$(\phi(t_0), \psi(t_0)) = (\phi(T), \psi(T)),$$

but otherwise has no multiple points;

3) the function ϕ is an increasing § or a decreasing function for all but a finite number of points of $[T]$.

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* Read before the Mathematical Club of the University of Chicago, October 23, 1903.

† Cours d'Analyse, vol. 1, pp. 91–99.

‡ *Göttinger Nachrichten*, 1896, p. 85.

§ A function is said to be increasing at a point t' when an interval $[t' - \delta, t' + \delta]$ can be found such that

The parameter representation can be so arranged that the t values mentioned in 3), the maxima and minima of ϕ, are

$$t_0, t_1, t_2, \cdots, t_n = T,$$

where $t_{k-1} < t_k$, $(k = 1, 2, \cdots, n)$, and t_0, t_n correspond to the smallest value of x on the curve.

Consider the interval $[t_{k-1}, t_k]$.* On account of 3) the function $x = \phi(t)$ is monotonic in this interval, and consequently the inverse function $t = t(x)$ is single-valued and continuous in the interval $[x_{k-1}, x_k]$, when x_{k-1} and x_k are the values of x corresponding to t_{k-1} and t_k respectively.

By substitution in (1),

$$C_k: \qquad y = \psi(t(x)) = \psi_k(x), \qquad [x_{k-1}, x_k],$$

which is also a single-valued continuous function of x in $[x_{k-1}, x_k]$.

The curve is therefore divided into n continuous pieces C_k $(k = 1, 2, \cdots, n)$. Any piece C_k has its endpoints (x_{k-1}, y_{k-1}), (x_k, y_k) in common with C_{k-1} and C_{k+1} respectively, but otherwise has no intersections with the other pieces.

§ 2. *Classification of the Points of the Plane by Means of a Continuous Function $g(x, y)$.*

Consider the piece C_1. The two pieces of horizontal lines, $y = y_0$ for $x \leqq x_0$, and $y = y_1$ for $x \geqq x_1$, adjoined to C_1 (Fig. 1), together define a continuous function

$$y = Y_1(x), \quad [-\infty < x < +\infty].$$

The equation

$$g_1(x, y) = y - Y_1(x)$$

$$\phi(t) < \phi(t') \quad \text{for } t' - \delta < t < t',$$

$$\phi(t) > \phi(t') \quad \text{for } t' < t < t' + \delta.$$

This third assumption is made also by Jordan, Cours d'Analyse, vol. 2, p. 132, in reducing to a line integral the integral

$$\iint \frac{\partial F}{\partial y} dxdy,$$

taken over a region bounded by a curve.

* The interval in which a function is to be considered will always be indicated as here by a square bracket.

defines g_1 as a function of x and y, continuous over the whole plane. The product

$$g(x, y) = \prod_{k=1}^{n} g_k(x, y)$$

has the same property, where g_k $(k = 1, 2, \cdots, n)$ is supposed to have been formed from C_k in a manner analogous to the formation of g_1.

Near a point of C, g can take both positive and negative

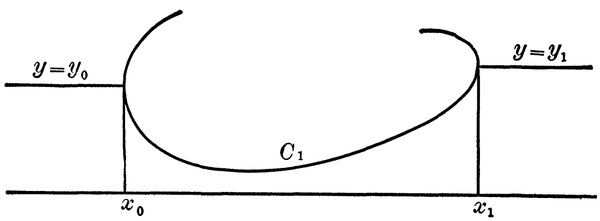

FIG. 1.

values. For if (ξ, η) is in C_k but not an end point, then for points on the line $x = \xi$, the function g has the form

$$(2) \qquad g(\xi, y) = (y - \eta)^{2\mu+1}\bar{g}(\xi, y),$$

where $\bar{g} \neq 0$ in the neighborhood of $y = \eta$. This follows because the factor $g_k(\xi, y)$ has the form

$$g_k(\xi, y) = y - \eta.$$

If any other factor g_λ has the value $y - \eta$ for $x = \xi$, then either $g_{\lambda-1}$ or $g_{\lambda+1}$ has the same value. Similarly g takes opposite signs near the end points of C_k because they are the limiting points of the interior points of the piece.

In the neighborhood of any point (ξ, η) not on C, g always takes values different from zero, but never takes values opposite in sign. For near such a point, g has the form

$$(3) \qquad g(x, y) = (y - \eta)^{2\mu}\bar{g}(x, y),$$

where $\bar{g} \neq 0$ in the neighborhood of (ξ, η).

The points of the plane can now be classified into :

1) *points of C, near which g takes both positive and negative values ;*

2) *interior points, near which g takes negative values but no positive ones ;*

3) *exterior points, near which g takes positive values but no negative ones.*

If $i(\xi, \eta)$ is an interior (exterior) point, then it is always possible to find a circle about i in which all the points are interior (exterior). This follows from (3) if the circle is taken so small that $\bar{g} \neq 0$ in it, and furthermore of radius smaller than the shortest distance from i to the pieces C_k which give rise to factors in the form $y - \eta$. For in such a circle these factors all retain the same form, and g has consequently the sign of $\bar{g}(\xi, \eta)$ at every point of the circle where it does not vanish.

§3. *Curves Joining an Interior with an Exterior Point.*

Suppose a curve

$$D: \qquad x = f(\tau), \quad y = g(\tau),$$

starting at an exterior point for $\tau = \tau_0$, and attaining an interior point for $\tau = \tau_1$ $(\tau_0 < \tau_1)$. Let τ' be the upper bound of the τ values such that the interval $[\tau_0, \tau]$ defines only exterior points on D. τ' can not correspond to an interior (exterior) point (x', y'), for then all points in a certain circle about (x', y') would be interior (exterior) points, and consequently τ' would be too large (too small) to be the upper limit in question. It must therefore define a point of the curve C.

If a continuous curve joins an exterior point with an interior point, it must have on it at least one point of C.

§ 4. *Construction of I-Curves and E-Curves Parallel to C.*

For purposes of construction two positive quantities δ and ϵ can be selected so that for $k = 1, 2, 3, \cdots, n$,

1) $\delta < \frac{1}{2} \mid x_k - x_{k-1} \mid$;

2_a) $\epsilon <$ smallest distance from C_k to any non-adjacent piece;

2_b) $2\epsilon <$ smallest distance from C_k on $[x_{k-1}, x_k + (-1)^k\delta]$ to C_{k+1}.

By means of the curve pieces

$$C'_k: \qquad y = x_k(x) + (-1)^k\epsilon, \qquad [x_{k-1}, x_k],$$

a closed continuous curve can be constructed which does not intersect C. Consider for example C'_1 and C'_2. On account of 2_a) the only pieces of C which C'_1 can intersect are C_n and

C_2, and from 2_b) it can intersect C_2 only on $[x_1 - \delta, x_1]$. If

$$\psi_2(x) - \psi_1(x) > 0$$

for all points common to $[x_0, x_1]$ and $[x_1, x_2]$ (see Fig. 2), then C_1' and C_2' have no points of intersection, but their ends on the

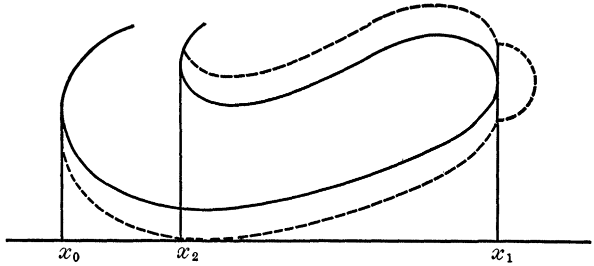

FIG. 2.

ordinate $x = x_1$ can be joined by a semicircle in the half-plane $x \geqq x_1$ with radius ϵ and center (x_1, y_1). The semicircle has no points in common with C on account of 2_a), and the resulting

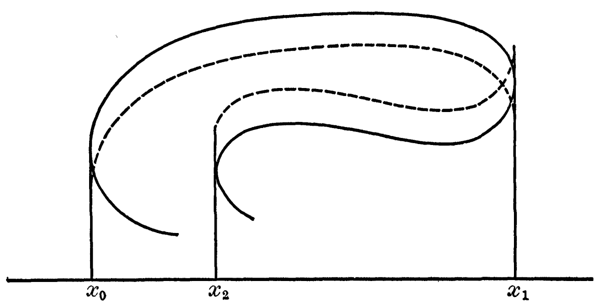

FIG. 3.

continuous curve formed by joining C_1', C_2' in this manner does not intersect C_1 or C_2.

On the other hand if

$$\psi_2(x) - \psi_1(x) < 0$$

(Fig. 3), then C_1' and C_2' will intersect for some smallest value x' between $x_1 - \delta$ and x_1. For at $x = x_1 - \delta$

$$(\psi_2 + \epsilon) - (\psi_1 - \epsilon) = \psi_2 - \psi_1 + 2\epsilon < 0$$

on account of 2_b), and at $x = x_1$

$$(\psi_2 + \epsilon) - (\psi_1 - \epsilon) = + 2\epsilon > 0.$$

If now the points of C_1' and C_2' between x' and x_1 are disregarded, then the resulting curve is continuous and does not intersect either C_1 or C_2.

Suppose the same process carried out for each pair C_k', C_{k+1}'. The result is a continuous closed curve D' which does not intersect C. For any C_k' can only intersect C_{k-1} or C_{k+1}, and it has been joined with C_{k-1}' so as to avoid intersection with C_{k-1}, and with C_{k+1}' so that it does not intersect C_{k+1}.

A similar curve D'' can be constructed out of the pieces

$$C_k'' : \quad y = \psi_k(x) + (-1)^{k-1}\epsilon, \quad [x_{k-1}, \quad x_k + (-1)^k\delta].$$

It is easy to show that D' and D'' lie on opposite sides of C. For the two points $(x_0 + \delta,\ \psi_1(x_0 + \delta) \pm \epsilon)$ lie on D' and D'' respectively, and are furthermore on opposite sides of C, as one discovers from the behavior of $g(x, y)$ (see equation (2)).

The result is then that *for a given* δ, ϵ *satisfying* 1), 2_a), 2_b), *two curves parallel to* C *can be constructed, the one entirely interior* (*I-curve*), *the other entirely exterior* (*E-curve*) to C. It should be noticed that the part of D' due to C_k' is cut by every ordinate on $[x_{k-1} + (-1)^{k-1}\delta,\ x_k + (-1)^k\delta]$.

§ 5. *Curves Joining Two Interior* (*Exterior*) *Points Without Intersecting the Curve C.*

Consider first a single interior point i (ξ, η). On $x = \xi$ there are at most a finite number of points of C because the number of pieces C_k is finite. There must be one at least with ordinate $> \eta$, for otherwise i could be joined to a point for which $g > 0$ by the ordinate $x = \xi$. Suppose the nearest to (ξ, η) lies on C_k, and let δ and ϵ be again restricted as follows:

a) when $\xi = x_k$ or $\xi = x_{k-1}$ take δ as in 1), and ϵ satisfying

2_c) $$\epsilon < \psi_k(\xi) - \eta$$

besides 2_a), 2_b);

b) when ξ lies between x_{k-1} and x_k, take δ satisfying

1_a) $$\delta < |\xi - x_{k-1}| \quad \text{and} \quad < |\xi - x_k|$$

besides 1), and ϵ as in *a*).

In either case the point $(\xi, \psi_k(\xi) - \epsilon)$ lies on the *I*-curve corresponding to these values of ϵ and δ, on account of § 3 and the fact that no point of C lies on the segment of $x = \xi$ which joins (ξ, η) and $(\xi, \psi_k(\xi) - \epsilon)$.

Any exterior point can be joined by straight line segments not intersecting C with another point of the same kind $e(\xi, \eta)$, for which ξ lies between the maximum and minimum values of x on C. The construction given above with slight modification can then be applied in order tc join e with an *E*-curve.

In a similar manner *an I-curve (E-curve) can be found with which any two interior (exterior) points can be connected by means of straight line segments not intersecting C.*

§6. *Some Classes of Curves to Which the Preceding Results Apply.*

A curve consisting of analytic * pieces can be represented in the form described in § 1 by choosing for the y-axis a direction a which is not parallel to any straight line piece. The only points where x can have maxima or minima are the ends of the pieces and points where the tangent is parallel to the y-axis. The latter are finite in number. For on any piece not a straight line, the fraction $\phi' / \sqrt{(\phi'^2 + \psi'^2)}$ is everywhere analytic and not constant. It would take the value $\cos a$ an infinity of times only if it were constant and equal to $\cos a$ on the whole piece.

In a similar manner the Schoenflies curves will have the desired form provided that the y-axis is not parallel to one of the straight pieces. For on the other pieces the direction of the tangent is presupposed to vary monotonically.

By properly choosing the coördinate axes the curves of character described in § 1 *can be made to include all those consisting of a finite number of analytic pieces, all the curves considered by Schoenflies, and all the so-called regular curves for which there exists a direction parallel to only a finite number of tangents to the curve.*

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* *I. e.*, on each piece ϕ and ψ are given by continuations of a single pair of analytic functions. Furthermore $\phi'^2 + \psi'^2$ is supposed $\neq 0$ except at perhaps a finite number of points, which can be made the end-points of pieces.