THE INTERNATIONAL CONGRESS OF MATHEMATICIANS IN PARIS.

At the Zürich Congress of 1897 it was agreed to hold the next congress in Paris in 1900, the French Mathematical Society being charged with the preparations. Circulars have been issued at intervals during the last eighteen months, calling the attention of mathematicians to the arrangements in progress. The congress was finally announced for August 6th–11th, and the opening general meeting was held in the Palais des Congrès, in the Exhibition grounds, at 9.30 on the morning of Monday, August 6th. M. Poincaré was elected President, M. Hermite, who of course was not present, being the *Président d'honneur*. The executive board was constituted as follows: vice-presidents, MM. Czuber (Vienna), Geiser (Zürich), Gordan (Erlangen), Greenhill (London), Lindelöf (Helsingfors), Lindemann (Munich), Mittag-Leffler (Stockholm), Moore (Chicago, absent), Tikhomandritzky (Kharkoff), Volterra (Turin), Zeuthen (Copenhagen) ; secretaries, MM. Bendixson (Stockholm), Capelli (Naples), Minkowski (Zürich), Ptaszycki (St. Petersburg), Whitehead (Cambridge, absent); general secretary, M. Duporcq (Paris). After the announcement of the officers of the sections and the names of the official delegates, and a very few words from the President, the two addresses of the day, both in French, were delivered by MM. M. Cantor (Heidelberg) and Volterra (Turin); each occupied about three-quarters of an hour.

M. Cantor : *Sur l'historiographie des mathématiques.*

During the century drawing to its close the character of mathematics has changed ; its devotees are now differentiated into geometers, analysts, algebraists, arithmeticians, astronomers, theoretical physicists, and historiographers. These last make no claim to advancing the science itself ; they press neither towards the arctic pole of the theory of functions, nor towards the antarctic pole of algebra ; they explore neither the steep surfaces of geometry nor the depths of differential equations. Their task is rather to draw up guides and charts, to indicate by what routes the results have been obtained, and what important points have been passed by without sufficient exploration. This work began with the History of Eudemus of Rhodes, B. C.

300, of which only a fragment has been preserved, just sufficient to excite lively regret for the loss of the whole. During the next two thousand years there were many bald chronicles of mathematics, but historiography as a science begins with Montucla. Notwithstanding the errors, unavoidable at that time, to be found in the two volumes of his Histoire des mathématiques (1st edition, 1758, 2d edition with two volumes by Lalande, 1799), Montucla "est encore et restera peut-être toujours un modèle que tout historiographe des sciences doit suivre." Kästner published four volumes of his Geschichte der Mathematik in the last four years of his life, 1796–1800. He has been alternately over-praised and depreciated ; Gauss referred to him as the best poet among the mathematicians, the best mathematician among the poets of his day. His history is no real history, it is rather a catalogue *raisonné,* but it is nevertheless invaluable, on account of its conscientious analysis of a number of works, which, with their authors, would be otherwise absolutely unknown to us now. At about the same date, 1797–1799, there appeared the two volumes of Cossali's Storia critica dell'algebra, dealing exhaustively with the period 1200–1600 ; as regards Italy only, it is true, but then during this period the Italian algebra was of importance far surpassing that of any other country. Cossali's labors for the elucidation of Leonard of Pisa and Cardan are of special merit.

Bossut published in 1810 his Histoire des mathématiques ; in this he gives only rapid *aperçus* of the general development, interesting to those that know already, useless to those that need to learn. In the present century we have first Chasles, to whom the speaker paid a warm personal tribute. In his Aperçu historique published in 1837, the notes, dealing with geometry, calculation, algebra, mechanics, which attain the dimensions of memoirs, form the model part of the volume, the text, the actual "Apercu," being but a very condensed statement of the history of synthetic geometry. The other historical work of Chasles, the Rapport sur les progrès de la géométrie of 1870, is seriously affected by his ignorance of the German language. The years 1837–1841 saw the publication of Libri's Histoire des sciences mathématiques en Italie, from the earliest times up to the middle of the 17th century, a work which owing to the author's admirable style "se lit comme un roman, même dans les parties où elle n'en est pas un." Notwithstanding Libri's immense services in the study of manuscripts, his history is vitiated, as a historical

work, by his misplaced patriotism ; according to him all progress in mathematics is due to the Italians, with perhaps a few scattered French writers. When he finds an Italian in possession of any ideas or methods, no matter whence derived, he at once credits him with their discovery. In any case, it is not possible to give any true idea of the history of mathematics by tracing it in one country only. If there is an international science, it is mathematics ; it bears no stamp of nationality. In considering the earliest times, it is impossible to understand the course of mathematics in one country without following it in others also ; to understand Greek mathematics, we must know something of Egypt and Babylonia ; the mathematics of the Arabs cannot be explained without reference to Egypt, Greece, and India. After the invention of printing, so long as Latin was in use, mathematics had no country ; and even when the frontiers were faintly marked by the use of different languages, they were speedily obliterated for most mathematicians.

Passing rapidly over Gerhardt and Quetelet, with a few words of recognition, M. Cantor spoke of Nesselmann's Die Algebra der Griechen, 1842, "un chef d'œuvre digne d'être mis a côté de l'Aperçu historique de Chasles"; of Arneth's Geschichte der reinen Mathematik, 1852, which would have been an excellent book, if the author had made a better apportionment of his space to his material—parts of the work "fourmillent de remarques aussi spirituelles que profondes"; of Hankel's posthumous fragment, 1876, "un torse d'une telle beauté qu'il eut été pitié de ne pas le mettre au grand jour"; and of the prince Baldassare Boncompagni's disinterested labors on behalf of historiography. In this sketch he passed over many authors "tous aussi morts que leurs livres ; gardons-nous de les ressusciter"; and avoided all mention of living authors for very obvious reasons. He brought his address to a close by a forecast of the mode in which the history of more recent mathematics must be written. Regarding Lagrange as the founder of modern mathematics. this gives 1759 as the starting point ; and from this year on, the different subjects will have to be treated in special volumes. This however will be insufficient ; the development of the lines of thought that run through all these different branches of mathematics must be traced in one final volume, the History of Ideas ; difficult to write, certainly, but indispensable, for as Jacobi said, "Mathematics is a science of which it is impossible to understand any one part without knowing all the others."

V. VOLTERRA: *Trois analystes italiens; Betti, Brioschi, Casorati.*

The scientific existence of Italy as a nation dates from a journey which Betti, Brioschi, and Casorati took together in the autumn of 1858, with the object of entering into relations with the foremost mathematicians of France and Germany. It is to the teaching, labors, and devotion of these three, to their influence in the organization of advanced studies, to the friendly scientific relations that they instituted between Italy and foreign countries, that the existence of a school of analysts in Italy is due.

The extent of their joint influence, affecting minds of many diverse casts, is largely due to the differences in their natural faculties, in the circumstances of their lives, and in their acquired tendencies. Brioschi, "toujours jeune par son caractère et toujours mûr par son esprit," a Lombard by birth, was at first an engineer; but at an early age he acquired a profound knowledge of the classical mathematical works, and was called to the chair of mechanics at Pavia at the age of 25. He founded the Polytechnic School at Milan, and held the directorship until his death; in his capacity of Senator, he was active in public affairs; he found time to engage in public works and in engineering; and up to the last, as Director of the *Annali di Matematica,* and President of the Accademia dei Lincei, he was one of the leaders of the mathematical movement in Italy. A great contrast to this active life is offered by the calm existence of Betti. He was born in a mountain village in Tuscany; at 34 he became a professor in the University of Pisa, and at 41 Director of the Scuola normale superiore of Pisa, whose organization is much like that of the École normale supérieure of Paris; he took no part in political movements. He loved scientific researches for their own sake exclusively, without regard to the results they might produce in the scientific world, or to their importance in teaching. He did not care for publishing his researches; and even when he did undertake this, he was apt to push it aside, attracted by new ideas. The knowledge that his intellectual conception could be realized was all-sufficient for him; he did not give himself the trouble of carrying it out in detail. When once he had obtained a clear vision of hidden truths, and had constructed in his own mind a system in which they proceeded directly from the simplest principles, "tout était fait pour Betti."

Casorati was born and lived at Pavia; he passed through the various grades in the University, where at the time of

his death he was professor of infinitesimal analysis.   He lived and worked almost exclusively for his pupils ; all his works bear the stamp of the practical teacher, bent on elucidating some obscurity, correcting some error, expounding some theory.   All his writings were in a definite relation to his university teaching ; in his mind there was no distinction between the work of the savant and the work of the professor.

The fundamental differences in the three can be brought out most clearly by a comparison of their attitude towards the theory of functions.   The development of this theory exhibits three well-marked periods, corresponding to the three distinct phases that can be recognized in the history of any mathematical subject ; these three phases, however, correspond also to three distinct modes of regarding questions in analysis, each of which has its advocates.   In the first instance, the discovery of facts is all-important, and particular theories are elaborated.   There are no uniform methods ; every question is attacked on its own merits and methods are created as occasion arises ; the ideas and results disengage themselves finally from long calculations.   In the theory of functions this manifests itself in the heroic period, personified in Euler, Jacobi, Abel ; and this manner of approaching questions is natural to Brioschi, the engineer and practical man, with his extraordinary gift for dealing with formidable calculations.   He remained faithful to the classical method, never attracted by the second phase, which he even scorned somewhat.   In this second phase, ideas replace calculations ; the philosophic spirit takes control and demands a general method including the whole subject in one body of doctrine.   This desire found its fulfilment in the second period of the theory of functions, in the works of Cauchy, Weierstrass, and Riemann, who derive everything from the very sources of the fundamental conceptions.   To this period belongs Betti the philosopher.   His broad and cultivated mind loved philosophic systems ; his Tuscan indolence (which is not intellectual idleness) caused him to delight in meditation rather than in mechanical labor.   Curiously enough, his name is associated with the theory of Weierstrass just as surely as with that of Riemann ; his education had made him an algebraist while nature meant him for a physicist.

In the final period the theories find their appropriate applications, their most suitable forms ; they are refined by criticism, and cast into a didactic mould.   The name of Casorati, critic and teacher, is associated with this third

phase. His work, Teorica delle funzioni di variabili complesse, has served more than any other one book to popularize in Italy the fundamental conceptions of the theory of functions, for the reason that, while reading it, difficulties disappear. The influence of this book is not confined to professed analysts ; anyone attempting to trace the development of mathematics in Italy during this half century will find that analysts and pure geometers have influenced one another. For instance, the ideas of Riemann are at the foundation of many of the works of Italian geometers, and while the actual introduction of these ideas was due to Betti, it is this book of Casorati's that has carried them everywhere and attracted the attention of geometers.

This comparison of the work of these three analysts in the region that they had in common gives no idea, however, of the extent of the labors and influence of each one. For this it would be necessary to dwell on the work of Casorati in the theory of differential equations, in analytical and infinitesimal geometry; of Betti in mathematical physics and algebra, he being one of the first to accept the new ideas of Galois ; of Brioschi in mechanics, algebra, and geometry. The field in which Betti and Brioschi first obtained renown was in fact that of algebra ; their names will always be associated with that of Kronecker as second only to Hermite in their work on the equation of the fifth degree, an equation whose complete solution was due to and secured immortality for M. Hermite.

This concluded the business of the first general meeting, with the exception of one or two formal announcements relating to secretarial matters. This was the only one of the meetings to be held in the Exhibition grounds ; all the others were held at the Sorbonne. Six sections were organized for the presentation of special papers, to meet on the 7th, 8th, 9th, and 10th of August, as follows :

Section I, Arithmetic and Algebra ; Tuesday, Thursday and Friday mornings ; president, M. Hilbert, secretary, M. Cartan.

Section II, Analysis ; Tuesday and Thursday mornings ; president, M. Painlevé, secretary, M. Hadamard.

Section III, Geometry ; Tuesday and Thursday afternoons ; president, M. Darboux, secretary, M. Niewenglowski.

Section IV, Mechanics and Mathematical Physics ; Tuesday and Thursday afternoons ; president, M. Larmor, secretary, M. Levi-Cività.

Section V, Bibliography and History; Wednesday morning and afternoon and Friday morning; president, Prince Roland Bonaparte, secretary, M. d'Ocagne.

Section VI, Teaching and Methods; Wednesday morning and afternoon and Friday morning; president, M. Cantor, secretary, M. Laisant.

Sections V and VI, however, amalgamated and sat as one section, thus making the sections the same as those at Zürich. It hardly seems advisable to give a complete list of the papers, as all will be given in the full official report, which will appear shortly; it seems better to give some account of the most interesting.

In Section I the most noticeable communication was that of M. Henri Padé, of Lille, "Aperçu sur les développements récents de la théorie des fractions continues." The object of this communication was the discussion of the question as to what is to be understood by the development of a function as a continued fraction, and the examination of the consequences of the answer obtained. For the function e^x, for instance, five such developments are already known, due to Euler, Lagrange, and Gauss. Why five? By what are the five characterized? How are they related to one another? The bond that unites them consists in a certain property of approximation common to their convergents, and thus there arises the more general question of the study of rational fractions satisfying this condition of approximation for a given function.* This investigation yields the following results. Every function that is developable by Maclaurin's formula gives rise to an infinity of developments as a continued fraction; as to form, these fractions present the common characteristics that all the partial numerators are monomials, with coefficient and exponent different from zero; as to matter, they are characterized by the property that all the convergents satisfy the condition of approximation mentioned above, and give an approximation whose order increases constantly as we pass from any convergent to the following one.

Among these fractions, called *fractions continues holoïdes* of the function, are included the regular continued fractions, and it is among these last that are found, as a very particular case, the developments in continued fractions already known for some special functions, for example, the five developments of e^x.

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* "Sur la représentation approchée d'une fonction par des fractions rationnelles," *Ann. scient. de l' École norm. sup.*, vol. 9 (1892).

M. Padé then indicated the two ways in which these results can be generalized, the extension they involve in all the applications of continued fractions hitherto made, and the important consequences to which they lead, both in the theory of functions, where they have already introduced the question of the use of divergent power-series, and in the theory of numbers.

In Section II, the first paper read on Tuesday morning was M. Tikhomandritzky's "L'évanouissement des fonctions O de plusieurs variables indépendantes." The function

$\theta\left(u_h - \overset{\xi p}{\underset{1 \ x_0}{I_h}}\right)$ of p independent variables $u_h = \overset{p}{\underset{i=1}{\Sigma}} \overset{x^i}{\underset{a_i}{I_h}}$ vanishes

1° when some of the points $\left(\overset{p}{\underset{1}{x_i, y_i}}\right)$ fall at (ξ, y_ξ) ; 2° when

they are on an adjoint curve of the first kind, $\varphi(x^{m-2}, y^{n-2}) = 0$. If with Weierstrass we define this function by the equation

$$\theta\left(u_h - \overset{p \ \xi}{\underset{1 \ x_0}{I_h}}\right) = e^{\int \Sigma\left[\sigma_k + J\left(u_h \div \overset{p}{\underset{1}{I_h}}\right)_k\right]du_k}, \tag{1}$$

where $$J\left(\overset{p}{\underset{1}{u_h}}\right)_k = \overset{p}{\underset{i=1}{\Sigma}} \overset{x_1}{\underset{x_0}{\Pi_k}} \tag{2}$$

$\left(\overset{x_i}{\underset{x_0}{\Pi_k}}\right.$ denoting the integral of the second species, which becomes infinite when (x_i, y_i) falls at (a_k, b_k)), this property of O must be derived from those of the function (2). For this purpose, considering in the first place the function

$$J\left(u_h + \overset{p \ x'}{\underset{1 \ \xi}{I_h}}\right)_k = \overset{p}{\underset{i=1}{\Sigma}} \overset{a_i}{\underset{x_0}{\Pi_k}} \tag{3}$$

$((x', y')$ denoting a point very near to $(a_k, b_k))$, where the points (x', y'), $\left(\overset{p}{\underset{1}{x_i, y_i}}\right)$ are the infinities, and (ξ, y_ξ), $\left(\overset{p}{\underset{1}{a_i, y_{ai}}}\right)$ the zeros, of the principal function of (z, y_z)

$$P_{x,\xi}\left(x', y' ; \overset{p}{\underset{1}{x_i, y_i}}\right) = \frac{P_{x', x_p}\left(z, y ; \overset{p-1}{\underset{1}{x_i', y_i'}} ; \xi, y_\xi \Big| \overset{p}{\underset{1}{a_i, y_{ai}}}\right)}{\varphi\left(\overset{m-2 \ n-2}{z, \ y_z} ; x_i, y_i \Big| \overset{p-1}{\underset{1}{x_i', y_i'}}\right)} \tag{4}$$

we see, by the second form of this function, that in the two cases one of the infinities of the function in the numerator being absorbed by one of its arbitrary zeros ($1°$ (x_p, y_p) by (ξ, y_ξ) ; $2°$ (x_p, y_p) by (x'_{p-1}, y'_{p-1})), the other will be absorbed by one of its non-arbitrary zeros $\left(a_i, y_{a_i} \atop 1 \right)^p$. Hence in the two cases, at the limit, for $(x', y') = (a_k, b_k)$, one of the integrals in (3) will become infinite like $\dfrac{1}{x - a_k}$ for $x = a_k$; thus θ will vanish.

More general interest was taken in M. Mittag-Leffler's papers, which followed. ''Sur fonction analytique et expression analytique.'' '' Une application de la théorie des séries n-fois infinies.'' '' Sur une extension de la série de Taylor.'' In these M. Mittag-Leffler reported on his recent researches* in the theory of functions. Let $f(a)$, $f'(a)$,

$f''(a), \cdots$ determine an element $P(x \mid a) = \sum\limits_{n=0}^{\infty} \dfrac{1}{n!} f^n(a) \cdot (x - a)^n$

of an analytic function $f(x)$. This analytic expression is valid only for its circular domain of convergence ; for a more extended region $P(x \mid a)$ must be supplemented by certain of its continuations. Let a lie within a continuous region which does not overlap itself, and let the branch of $f(x)$ derived from $P(x \mid a)$ by continuation throughout K be one-valued and regular ; is it possible to find a *single* analytic expression for this branch $fK(x)$, where K is given its maximum extension, the formula to be based only on the primary quantities $f(a)$, $f'(a)$, $\cdots$?   M. Mittag-Leffler has shown that such an expression can be built up in a comparatively simple manner. In doing this he has employed a new geometrical notion, that of the *star* (étoile). If a ray revolves about a, proceeding in each position to the nearest singular point of $f(x)$, this point lying it may be at infinity, the collection of points on the totality of such rays forms for $f(x)$ the star A ; it is assumed that the lower limit of these rays is not zero. For A we have the theorem that

$$fA(x) = \sum\limits_{m=0}^{\infty} G_m(x),$$

where $G_m(x)$ denotes a polynomial $\sum c_{n, m} f^n(a)(x - a)^n$, in

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which the coefficients $c_{n,m}$ are given initially and do not depend on a or on $f(a)$, $f'(a)$, $\cdots$.

The expression

$$G_n(x \mid a)$$

$$= \sum_{\lambda_1=0}^{n^2} \sum_{\lambda_2=0}^{n^4} \cdots \sum_{\lambda_n=0}^{n^{2n}} \frac{1}{\lambda_1! \cdots \lambda_n!} f^{(\lambda_1 + \cdots + \lambda_n)}(a) \left(\frac{x-a}{n}\right)^{\lambda_1 + \cdots + \lambda_n}$$

leads to a limiting expression $\lim_{n=\infty} G_n(x \mid a)$ with the following properties : It is uniformly convergent for every region interior to the star A, but never uniformly convergent for a region containing a vertex of A. Within A it represents the branch $fA(x)$ of $f(x)$.

It is perfectly possible that $\lim_{n=\infty} G_n(x \mid a)$ may converge outside A ; the star A is not a star of convergence for $\lim_{n=\infty} G_n(x \mid a)$. M. Mittag-Leffler has shown that it is possible to replace $\lim_{n=\infty} G_n(x \mid a)$ by another expression for which A is a star of convergence. $\lim_{n=\infty} G_n(x \mid a)$ was obtained from an n-fold series in x by making the maximum values of λ_1, λ_2, $\cdots$, λ_n proceed simultaneously to the limit ∞. If the passage to the limit is performed in another way, viz., by taking $\sum_{\lambda_1=0}^{\lambda_1} \sum_{\lambda_2=0}^{\lambda_2} \cdots \sum_{\lambda_n=0}^{\lambda_n}$ in place of $\sum_{\lambda_1=0}^{n^2} \sum_{\lambda_2=0}^{n^4} \cdots \sum_{\lambda_n=0}^{n^{2n}}$ and then making λ_n, λ_{n-1}, $\cdots$, λ_1 tend *successively*, in the order named, to infinity, the expression

$$S_n(x \mid a) = \sum_{\lambda_1=0}^{\lambda_1} \sum_{\lambda_2=0}^{\lambda_2} \cdots \sum_{\lambda_n=0}^{\lambda_n} c_{\lambda_1 \cdots \lambda_n} f^{\lambda_1 + \cdots + \lambda_n}(a) \left(\frac{x-a}{n}\right)^{\lambda_1 + \cdots + \lambda_n}$$

(where the c's are given numerical constants of which

$$c_{\lambda_1} = \frac{1}{\lambda_1!}, \quad c_{\lambda_1 \lambda_2} = \frac{1}{\lambda_1! \lambda_2!} \left(\frac{1}{2}\right)^{\lambda_1 + \lambda_2}, \quad c_{\lambda_1 \lambda_2 \lambda_3} = \frac{1}{\lambda_1! \lambda_2! \lambda_3!} \left(\frac{1}{3}\right)^{\lambda_1 + \lambda_2 + \lambda_3},$$

while for values of $n > 3$ they are algebraic irrationals) yields th$\jmath$ desired result; $\lim_{\infty} S_n(x \mid a)$ has the star A as a star of convergence, and represents $fA(x)$ within A. Writing $n = 1$, the series is seen to be simply Taylor's series ; in general it is an extension of Taylor's series.

In the course of his remarks M. Mittag-Leffler referred to recent researches of M. Borel ; this led to a discussion in

which MM. Borel, Hadamard, and Painlevé took part, on
the nature of the connection between "analytic expression
in a complex variable x" and "analytic function in x."

In Section III papers were presented by MM. Lovett,
"On contact transformations between the essential elements
of space"; Macfarlane, "Applications of space analysis to
curvilinear coördinates"; Stringham, "Orthogonal trans-
formations in elliptic or in hyperbolic space"; Amodeo,
and others. In Section IV very few papers were read;
one appointed meeting of the section was not held, and
some of the papers intended for the section were presented
at the joint sitting of Sections V and VI, which was trans-
formed momentarily into a sitting of Section IV, to hear
MM. Hadamard: "Relations entre les caractéristiques
réelles et les caractéristiques imaginaires pour les équations
différentielles à plusieurs variables indépendantes"; and
Volterra: "Comment on passe de l'équation de Poisson à
caractéristique imaginaire à une équation semblable à
caractéristique réelle."

The communications made in Sections V and VI, while
not necessarily the most valuable mathematically, were yet
of the most general interest, and lend themselves best to
any general report. The sitting was opened by M. Hil-
bert's address, in German, on the future problems of
mathematics. The lines along which we may expect the
development of any science which is progressing in a con-
tinuous manner can be detected by an examination of the
problems to which attention is specially paid. Among
these most importance is to be attached to those that are
sharply defined and stand out well; such, for example, as
the problem of three bodies. In the earlier stages of any
science, problems present themselves naturally through ex-
perience, as is exemplified in mathematics by the duplica-
tion of the cube and the quadrature of the circle, and at a
later date by the questions arising with reference to in-
finitesimal analysis and the theory of the potential; but as
the science progresses, it is the logical faculty of the intel-
lect that imposes on us problems such as are found in the
theories of prime numbers, elliptic functions, etc.

As to our aim with regard to any problem, there must be
a definite result of some kind, it cannot be laid aside until
we have obtained either a satisfactory solution or a rigorous
demonstration of the impossibility of a solution. The
mathematical rigor that is essential in the treatment of a

problem does not require complicated demonstrations; it requires only that the result be obtained by a finite number of logical steps from a finite number of hypotheses furnished by the problem itself; in seeking this rigor we may find simplicity. The proper treatment of any problem depends on $1°$ a complete system of axioms, by means of which the conceptions are defined, $2°$ a system of symbols appropriate to the conceptions with which the problem deals; thus a demonstration by means of geometrical symbols is as legitimate as an arithmetical one, provided that the axioms on which it is based are perfectly understood. The mere formulation of these axioms is in some cases itself the problem, as for instance in arithmetic and physics. Among the ten problems that M. Hilbert specified in particular as fitted to advance mathematics, No. 2 is that of finding some one system of independent compatible axioms governing and defining arithmetical conceptions, and No. 3 is the same question for the calculus of probabilities, rational mechanics, and physics. Other problems are to prove that $e^{i\pi z}$ is transcendental when z is an algebraic irrational; and that the solution of the general equation of the 7th degree cannot be obtained by means of a finite number of operations involving only two parameters. In geometry, the relative situation of the circuits that a plane curve of assigned order can possess, with the corresponding question as regards surfaces, demands investigation; in the theory of functions there is the question of the expression of two variables, connected by any analytic relation whatever, as uniform functions of a single parameter z—for Poincaré's theorem (*Bulletin de la Société mathématique de France*, volume 11 (1883)) is subject to some limitations. These are but a few of the problems that M. Hilbert mentioned, and these were a selection from a much longer list for which he referred to an article about to appear in the *Nachrichten der Kgl. Gesellschaft der Wissenschaften zu Göttingen,* 1900. In the course of a rather desultory discussion that followed the reading of this paper, the claim was made, though apparently without adequate grounds, that more had been done as regards the equation of the 7th degree (by some German writer) than the author of the paper was willing to allow. A more precise objection was taken to M. Hilbert's remarks on the axioms of arithmetic by M. Peano, who claimed that such a system as that specified as desirable has already been established by his compatriots MM. Burali-Forti, Padoa, Pieri, in memoirs referred to on pp. 3–5 of no. 1, volume 7 of the *Rivista di Matematica*.

M. Hilbert was followed by M. Fujisawa, the official dele-
gate from Japan, who gave, in English, a very interesting
account of the mathematics of the older Japanese school.
It is difficult to follow the course of Japanese mathematics ;
there are some two thousand manuscript volumes still to be
transcribed ; in these much valuable work is mixed up with
what is purely elementary and even trivial.   The difficulty
of arriving at any clear idea is greatly increased by the fact
that publication of results was not customary; they were
preserved to a great extent only by oral transmission.   So
far as the books have been deciphered and collated, one fact
stands out with ever-increasing clearness, and that is that
side by side with one less important school of Japanese
mathematics there exists another earlier system of mathe-
matics of a peculiar kind, which had its origin in Japan,
and was developed there entirely free from any external
influences.

The mathematics of the first kind, probably derived from
the Chinese at a very early date, displays a noticeable lack
of rigor ; for instance, $\sqrt{10}$ is accepted as the value of π.
As to content ; in algebra, the solution of simple equations
and the formulæ for the sum of an arithmetical or geomet-
rical progression were known ; in geometry, the right-
angled triangle with sides proportional to 3, 4, 5 was used,
with some propositions regarding regular polygons ; magic
squares were discussed, even so far as those containing the
first 400 numbers.   Bamboo rods were used for purposes of
calculation ; these were placed one above another to in-
dicate addition, side by side to indicate multiplication,
diagonally to denote subtraction.

The other part of Japanese mathematics, that indigenous
to the country, is of more importance and interest.   It ap-
pears that the mathematicians of this school made use of
local value in expressing numbers, invented zero for them-
selves, and used the circle as the symbol for zero.   They
were familiar with imaginaries and complex numbers ; and
were such adepts at calculation that they found the value
of π correct to 49 places of decimals.   M. Fujisawa ex-
plained that the knowledge of this part of Japanese mathe-
matics so far obtained is very fragmentary, the unexplored
part offers an attractive field of research for Japanese who
may care to devote themselves to it.   It is a matter of
purely historical interest, as the present teaching of mathe-
matics in Japan is in no sense founded on it; for, very
wisely as he thinks, the Japanese educational authorities
made an entirely fresh start, sweeping away all trace of this
older educational system.

The president of the section, M. Cantor, then spoke of the difficulties he encountered, when writing his Geschichte der Mathematik, in finding out anything about the earlier Japanese mathematics. When he did finally hear of a work of reference it turned out to be written in Japanese. With reference to the earliest use of zero. he expressed the opinion that it was probably due to the Babylonians, about 1700 B. C.

Another paper of interest in these sections was that of M. Padoa (Rome) on Friday morning : " Un nouveau système irréductible de postulats pour l'algèbre." Naming the object, *entier* (integer), two undefined derivatives, *successif* and *symétrique*, are considered. The seven postulates are

1°. If a is an integer, then suc. a is an integer.

2°. If a is an integer, then sym. a. is an integer.

3°. If a is an integer, then sym. (sym. a) = a.

4°. If a is an integer, then sym. {suc. [sym. (suc. a)]} = a.

5°. There exists an integer x such that sym. $x = x$.

6°. There do not exist two different integers x, y, such that sym. $x = x$, and sym. $y = y$.

7°. If a class u of objects satisfies the conditions

    (i) it contains some one integer,

    (ii) if it contains an integer x it contains also suc. x,

    (iii) if it contains suc. x it contains also x,

then every integer belongs to the class u.

These postulates define an algebraic field, whose nature is at once seen to agree with that of the natural field, suc. x being interpreted as $1 + x$, and sym. x as $- x$. M. Padoa did not get beyond this definition, possibly because he had entered so minutely into the details of the proof of the independence of the seven postulates that he had exhausted his allowance of time.

A great part of the Friday morning sitting of these two sections was devoted to the discussion of a resolution, proposed by M. Leau, in favor of the adoption of some special artificial language as the vehicle for all scientific communications. Though no particular language was named in the resolution, it was made clear that "Esperanto" was the language intended. Its advocates, MM. Leau, Padoa, Boccardi, Laisant, and others, disclaimed any wish to substitute it for natural languages, but urged its adoption as the vehicle for international intercourse ; this view they upheld with great earnestness "on behalf of humanity," as M. Laisant put it. The opposite view was upheld with equal earnestness, if less vehemence, by MM. Schroeder, Vassi-

lief, Maggi, and others, chiefly on the ground that any such language is entirely unnecessary ; as M. Maggi remarked, mathematics already has a universal language, the language of formulæ. In the end the suggestion of M. Vassilief was adopted, that the Congress should place itself on record as opposed to any unnecessary diversity in the languages employed, that is, practically, to the use of any language for scientific purposes other than English, French, German, and Italian, though these languages were not specified in the resolution adopted.

On Saturday, August 11th, the concluding general meeting was held at 9 a. m. The first business was to determine the time and place for the next meeting. At Zürich, Professor Klein, on behalf of the German Mathematical Society, had expressed their great desire that the third congress should be held in Germany ; and a definite invitation to this effect was now laid before the Congress, and unanimously accepted. The place of meeting will probably be Baden-Baden ; the date decided upon is 1904, and the time is to be either at the beginning or the end of the summer vacation. No other business was transacted, and the two general addresses appointed for the day were then delivered by MM. Mittag-Leffler and Poincaré. Immediately after the conclusion of the President's address, he dismissed the Congress with the words " La séance est levée, le congrès est clos."

M. Mittag-Leffler's address was entitled " Une page de la vie de Weierstrass"; in this he considered in some detail Weierstrass's attitude towards some of the mathematical ideas of his time, illustrating by copious extracts from his correspondence; unfortunately it is not possible to give any adequate account of it. M. Poincaré's can be given more fully.

H. POINCARE: *Du rôle de l'intuition et de la logique en mathé-matiques.*

It is obvious that there are two entirely different types of mind among mathematicians, manifesting themselves in two distinct methods of treating mathematical questions. Those of the first type are dominated by logic; those of the second are guided by intuition. They may be called analysts and geometers, though it is not really a question of the subject with which they deal ; the analyst remains an analyst even when working at geometry, and the geometer

employing himself on pure analysis is still a geometer. Nor is the distinction a mere matter of education ; a man is born a mathematician, he does not become one; and either he is born an analyst or he is born a geometer. The two types of mind are equally necessary for the progress of the science ; each has accomplished great things that would have been impossible to the other.

At first sight the ancients seem to have all been intuitionalists, but this impression disappears on closer study. Euclid, for instance, was a logician, even though every stone of his edifice is due to intuition. The natural tendencies have not changed, only their manifestation. There has been an evolution, due to the increasing recognition of the fact that intuition cannot give rigor, nor even certainty ; a proof that relies on concrete images may be very deceptive. It was soon realized that rigor cannot be expected in the demonstrations unless it is to be found in the definitions ; so long as the objects of reasoning were given simply by the bodily senses or the imagination, there was no precise idea on which reasoning could be based. Thus the efforts of the logicians were concentrated on the definitions, one result of which is that mathematics has become arithmetized.

The question arises, is this evolution ended—have we at last attained to absolute rigor, or do we deceive ourselves as our fathers did ? Philosophers tell us that it is impossible to eliminate intuition altogether from our reasonings, for no science can spring from pure logic alone. To designate this other essential, we have no name but intuition ; but this covers many different ideas. There is (1) the appeal to the bodily senses and to imagination ; (2) generalization by induction ; (3) the intuition of pure number ; on this last a veritable mathematical method is based, while from the first two no certainty can be derived. The analysis of the present day constructs its demonstrations solely from syllogisms and this intuition of pure number ; we may say that at last absolute rigor is attained.

The philosophers now object that what has been gained in rigor has been lost in actuality ; the approach toward the logical ideal has been secured by cutting the ties with reality. For the sake of the demonstration a mathematical definition is substituted for the object, and it still remains to prove that the concrete reality answers to the definition. But as this is an experimental truth, it is not the business of mathematics to establish it. It is a great step forward to have separated these two things; nevertheless there is some-

thing in the philosophic objection. In becoming rigorous,
mathematics has assumed a certain character of artificiality;
if it is clear how questions can be resolved, it is no longer
clear how and why they arise. We seek for reality ; but
this does not reside in the separate steps of the demonstra-
tion ; it must be sought rather in the something that makes
for unity. The microscopic examination of an elephant
gives no idea of the animal itself ; the fairy-like structure of
silicious needles which is all that is left of certain sponges
cannot be understood without reference to the living sponge
by which this form was imposed on the silicious particles.
Logic by itself cannot give the view of the whole which is
indispensable alike to the inventor and to him who desires
really to understand the inventor. Logic, which alone
gives certainty, is simply the instrument of demonstration ;
the instrument of discovery is intuition.

But analysts also are inventors ; hence they cannot al-
ways be proceeding from the general to the particular, as
the rules of formal logic demand, for scientific conquests are
made only by generalization. There is however a per-
fectly rigorous process, that of mathematical induction, by
which it is possible to pass from the particular to the gen-
eral.* For the profitable use of this, to recognize the analo-
gies whose presence makes it applicable, the analyst must
have the direct feeling for the unity of an argument, for its
soul and spirit ; for him the most abstract entities must be
living beings. What is this but intuition ? This however
does not invalidate the distinction already drawn, for it is
an intuition entirely different in nature from the sensible
intuition founded in imagination alone, even though psy-
chologists may finally pronounce it also to have a sensual
foundation. It is the intuition of pure logical form, which
together with the intuition of pure number makes not only
demonstration, but also discovery, possible to the analyst.
Thus among the analysts inventors do exist, but not many;
it remains true that the most usual instrument of invention
in mathematics is sensible intuition.

On the evening preceding the formal opening of the Con-
gress an informal reunion of the members, about half of
whom were present, was held at the Café Voltaire. On
Tuesday afternoon, after the rising of the sections, the
members were entertained at the École normale supérieure.
At noon on Sunday, August 12th, a very successful banquet

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* Poincaré, "Sur la nature du raisonnement mathématique," *Revue de
métaphysique et de morale*, vol. 2 (1894), pp. 371-384.

was held at the Salle de l'Athénée-Saint-Germain, when, in the absence of M. Poincaré, M. Darboux presided very pleasantly. Toasts to those present, to the hosts, to the absent, to M. Darboux, and to the next Congress, were proposed by MM. Darboux, Geiser, J. Tannery, Stephanos, and Vassilief. Invitations to receptions held by the President of the Republic and by Prince Roland Bonaparte were accepted by a number of the members.

A very large attendance had been expected, on account of the additional attractions offered by the Exhibition ; and the answers to the circulars first sent out went far to justify this anticipation, for up to last December about 1,000 mathematicians had signified their intention of being present, with 680 members of their families. The membership fee was fixed at 30 francs, with an additional 5 francs for every member of the family. As a matter of fact, the total attendance can hardly have exceeded 250 in all. There seems very little doubt that a large proportion were kept away by distaste of the crowds that were supposed to be visiting the Exhibition, and by the rumored difficulty in obtaining accommodation, a difficulty that seems to have existed mainly in the circulars of the various agencies ; but the great heat of July certainly decided many to stay away who would otherwise have been present.

The countries represented were as follows : France, 90 ; Germany, 25: United States, 17; Italy, 15; Belgium, 13; Russia, 9 ; Austria, 8 ; Switzerland, 8 ; England, 7 ; Sweden, 7; Denmark, 4; Holland, 3 ; Spain, 3 ; Roumania, 3 ; Servia, 2; Portugal, 2 ; South America, 4 ; with single representatives from Turkey, Greece, Norway, Canada, Japan, Mexico. This list is only approximate, as no revised list of members was issued. Among the members from the United States were Professors Allardice, E. W. Brown, Dickson, Ely, Hagen, Halsted, Hancock, Harkness, Keppel, Lovett, Macfarlane, Pell, Scott, Stringham, Webster.

While the four languages, English, French, German, and Italian, were admitted on equal terms, by the constitution of the congresses, the great preponderance of French was noticeable. At Zürich, this preponderance existed in friendly intercourse, but French and German were about equally used in the communications, whereas in Paris all the general addresses, and most of the sectional papers, were in French, possibly out of compliment to our hosts. Probably at Baden-Baden French and German will be used in about equal proportions as at Zürich.

The distribution of the authors of communications among

the different countries may be of interest. The four general addresses were delivered by representatives from France, Italy, Germany, Sweden ; including these, papers were read by members from France, 13 ; Italy, 9 ; United States, 6 ; Germany, 4 ; Sweden, 4 ; Austria, 2 ; England, Greece, Holland, Japan, Portugal, Russia, Servia, and Spain, one each. This list may require some slight modification, as the only programme issued needed some corrections, but it is substantially correct. In some cases two or three communications were made by one member, thus bringing up the number of sectional papers to about 50 ; some of these were presented by title only in the absence of their authors. The general meetings occupied four hours, the sectional meetings 26. About 200 members were present at each of the general meetings ; at the sectional meetings the average attendance was about 90, and as two sections were usually sitting at the same time, this accounts very fairly for the members. About 160 were present at the banquet.

The arrangements excited a good deal of criticism. The committee of organization had doubtless special difficulties to contend with, as M. Laisant, to whom the secretarial part had been assigned, was unable to undertake it owing to the pressure of other duties. The mistake was then made of entrusting a part of this responsibility to the firm of Carré and Naud, whereas in such a case personal interest and individual responsibility are indispensable for ensuring proper attention to the various details of organization. Owing to this, members arriving in Paris had very great difficulty, during the first two days, in obtaining the necessary information in time for it to be of any service. The want of a common assembly room, where members might conveniently meet one another with or without concerted arrangement, was seriously felt. The arrangements in Zürich were so admirably complete in every point, that these defects were even more conspicuous by comparison. There is no doubt that a smaller town lends itself best to such a gathering ; it is not so much that there is less division of interests, as that the members are more in evidence, and so have a better chance of *realizing* one another. The first object of an international congress of mathematicians is to enable its members to meet one another in circumstances that shall excite the mathematical faculty, and make manifestations of mathematical interest most natural, thus encouraging the exchange of ideas both between individuals and larger groups. The mathematician

who is in any degree a specialist is in general rather soli-
tary in the average college—he would have been better off
in Noah's Ark, for at the worst there would have been two
of a kind.   For his mind's health it is well that he should
occasionally be thrown with those of kindred interests ; it
is well too that he should be made to feel the unity of
mathematics.

The general addresses are of course under one aspect the
most important part of the formal proceedings, giving broad
views of the whole subject, and helping the pronounced
specialist to realize his affiliations with other regions.   Did
a congress achieve no other end than that of evoking such
addresses as those of Klein, Hurwitz, Cantor, Poincaré,
Volterra, at Zürich and Paris, and presenting them to
mathematicians of such widely differing interests, its exist-
ence would be amply justified.   Something more might be
done in a similar direction as regards the sectional meet-
ings ; an address, not necessarily by the president of the
section, dealing in a broad manner with the region, or some
part of the region, assigned to the section, might very well
be arranged for.   While the general addresses enable the
individual to appreciate the relation of his special subject to
mathematics as a whole, these sectional addresses would
assist him in the equally important and even more difficult
task of gauging his own relation to his special subject.
One or two of the communications offered, both at Paris
and at Zürich, were of this nature ; but the matter should
not be left to chance, it is well worth systematically arrang-
ing.   Such an address would deal sometimes with the gen-
eral ideas of the subject, sometimes with what remained
to be done, sometimes with what had been accomplished
during the last few years.

As to the nature of the more special papers, it hardly
seems advisable to restrict the present liberty, not even to
the extent that is found salutary in the regular meetings of
a mathematical society ; encouragement is perhaps more
needed.   But some control over the time consumed should
be exercised ; not only a theoretical control, confined to a
printed statement that it must not exceed twenty minutes,
but a real control, exerted as a regular thing.   Ten or
fifteen minutes, well employed, is quite long enough for the
ordinary type of special theorem, for an outline of the
method of proof is sufficient in an oral communication, and
another five or ten minutes ought to enable the speaker to
indicate its connections within the subject.   The strict
limitation to a time previously determined would in most

cases be beneficial to the author, obliging him to select, subordinate, and group his details. It is tolerably certain that if the author regards all details as equally important, his auditors will regard all as equally unimportant.

One thing very forcibly impressed on the listener is that the presentation of papers is usually shockingly bad. Presumably the reader desires to be heard and understood ; to compass these ends, instead of speaking to the audience, he reads his paper to himself in a monotone that is sometimes hurried, sometimes hesitating, and frequently bored. He does not even take pains to pronounce his own language clearly, but slurs or exaggerates its characteristics, so that he is often both tedious and incomprehensible. These failings are not confined to any one nationality ; on the whole the Italians, with their clear and spirited enunciation, come nearest to being free from them. It would be invidious and impertinent to mention names ; the special sinners sit in both high and low places. But it is perhaps pardonable to refer to M. Mittag-Leffler's presentation of his paper to Section II as showing in how admirable and engaging a style the thing can be done. It is not given to everyone to do it with this charm ; but there is no excuse for any normally constituted human being, sufficiently versed in mathematics, failing to interest a suitable audience for a reasonable time in that which interests himself, always provided that it be of sufficient novelty either in matter or in mode of treatment to justify him in presenting it at all.

At the Zürich Congress certain matters were energetically discussed in Section V ; extensive support was then given to resolutions in favor of constituting permanent commissions charged to consider 1° general reports on the progress of mathematics, 2° matters of bibliography and terminology, 3° the possibility of giving some permanent character to the Congress, by means of a central bureau or otherwise. Though these resolutions were not voted upon directly, it being felt that they required more deliberate discussion, yet at the concluding general meeting the members of the Zürich bureau were appointed a commission to consider the questions that seemed of most importance, and to furnish the Mathematical Society of France with such information on these points as might be useful in preparation for the Congress of 1900. At the joint sitting of Sections V and VI in Paris M. Dickstein asked a question on behalf of the members, namely, was not the Congress to hear anything of the deliberations of this commission ? No satisfactory answer was forthcoming ; M. Laisant replied simply that

the Mathematical Society of France had been so taken up
with material preparations for the Congress that it had not
been able to enter upon any of these matters.   He took the
opportunity, however, of directing attention to the *Annuaire
des Mathématiciens*, projected by Carré and Naud, as carrying
into effect one suggestion made at Zürich.   The question
then dropped, but it was felt that this left matters in a very
unsatisfactory state.   It is to be hoped that a different re-
port will be given in 1904, that the members of the Baden
congress will not simply hear papers and meet friends, but
have a chance to consider these matters of international
concern.   Some questions of business arise in every science ;
they tend to settle themselves by a kind of tentative proc-
ess, a survival of the fittest, or rather by general tacit con-
sent.   This is often the best process ; any attempt at forcing
an expression of general agreement may result in checking
development by encouraging too early a crystallization.
But there are some matters, depending on concerted action,
that are ripe for decision, and that cannot profitably be set-
tled by any one nation for itself ; matters in which for want
of a general agreement labor may be wasted.   Such matters
may naturally be decided by an international congress,
whose decisions will simply have the force of general con-
sent.   One such question is that of a classification of math-
ematical sciences.   At least two well-known systems are in
use, and there may be others.   Multiplication of systems of
classification, like the multiplication of universal languages,
practically destroys the good of any and all ; the congress
ought to pronounce in favor of some one.   As to the prepa-
ration of special reports, it seems doubtful whether this will
be done best by the congress at present.   In course of time
it may assume academic functions and responsibilities, but
it will be necessary for it to prove its continuity before it
can with any propriety expect to control mathematical ef-
forts.   Encouragement and recognition would seem to be
its appropriate province at present in these respects.   The
question as to how this continuity is to be obtained is a
rather serious one, and deserving of careful discussion.   A
central bureau with various functions was suggested when
the matter was under discussion at Zürich ; but there are
objections to this.   If the organizers of each congress will
make it a point of honor to act on the recommendations of
the preceding congress, taking into consideration the reso-
lutions passed and doing what can be done towards carrying
them into effect, possibly sufficient continuity may be at-
tained without the red tape that would coil itself about any

permanent bureau. Some questions are better left unde-
cided. International agreement is not wanted on all points ;
international rivalry and emulation still have their part to
play, helped by the international friendships that are pro-
moted by such gatherings as these international congresses.

In conclusion, I must express my thanks to many of
those whose names appear in this report for the assistance
I have received from them in its preparation.

<div align="right">CHARLOTTE ANGAS SCOTT.</div>

BRYN MAWR COLLEGE,
   October, 1900.

THE FORTY-NINTH ANNUAL MEETING OF THE AMERICAN ASSOCIATION FOR THE AD- VANCEMENT OF SCIENCE.

THE forty-ninth annual meeting of the American Asso-
ciation for the Advancement of Science, which was held at
Columbia University June 23–30, was from the point of view
of scientific work one of the most successful in the history
of the Association. Sixteen affiliated societies met with the
Association and contributed greatly to the importance and
interest of the meeting. Two of these, the American Math-
ematical Society and the Astronomical and Astrophysical
Society, held joint sessions with Section A.

The next meeting of the Association will be held in Den-
ver during the last week of August, 1901, under the
presidency of Professor Minot of the Harvard Medical
School. Professor James McMahon, Cornell University,
will be vice-president of Section A, and Professor H. C.
Lord, Ohio State University, will be secretary. Forty-one
out of the total number of forty-nine annual meetings of the
Association have been held during the month of August,
while the recent New York meeting was the first that was
held in June. The next meeting will be farther west than
hitherto, but it seemed to be the general opinion that this
was desirable in order to extend the influence of the Asso-
ciation. Pittsburg was recommended as the place of meet-
ing in 1902.

The meetings of the section of mathematics and astronomy
were well attended. The officers of this section were : vice-
president, Asaph Hall, Jr.; secretary, W. M. Strong ; coun-