PASCAL'S CALCULUS OF VARIATIONS.

Die Variationsrechnung. Von E. PASCAL. Deutsch von A. SCHEPP. B. G. Teubner, Leipzig, 1899. vi. + 146 pp.

THE German translation of the "Calcolo delle variazioni" published by Ernesto Pascal in 1897 gives to American mathematicians in convenient form the best book on the calculus of variations that has, to our knowledge, appeared up to the present time. The book consists of only 150 octavo pages and presents concisely the principal facts of the subject. A valuable feature of the work will certainly be found to be the very excellent and apparently complete bibliography given in connection with brief accounts of the development of the calculus of variations. No book with which we are acquainted could be better adapted to controvert the lay opinion that everything in mathematics is exact and beyond dispute. Again and again the author calls attention to errors made by writers in this field. He often calls attention, too, to gaps which remain still to be filled in the theory, and makes the reader sometimes feel that the results which we already have rest on a rather precarious foundation. The chief fault of the book, from our point of view, is that it sacrifices simple and natural discussion to the pursuit of the end so dear to Italian mathematicians, the greatest possible generality.

The apparent purpose of the author is to give an account, absolutely rigorous as far as it goes, of the present condition of the science. That such an end is in the calculus of variations especially difficult to attain appears from the fact that the proofs are not always precise and that the author prefers often to tell us that the work given is not rigorous rather than to attempt to make it so. A serious deficiency is the almost entire lack of reference to the work of Weierstrass. This lack is, of course, intentional and doubtless due to the author's courtesy in hesitating to refer to what is to appear in Schwarz's edition of Weierstrass's works. It seems to us, however, that Pascal, with entire justice to Weierstrass, might have absorbed a little more of Weierstrass's spirit, and distinguished more carefully among the different sorts of variations employed, laying a proper emphasis on the function of each kind. The most unsatisfactory part of the book is the treatment of Jacobi's criterium. This condition is abstruse and difficult to grasp in all its beauty and ingenuity even in the simplest case. Pascal gives a purely analytical treatment of the general case. We ven-

ture to say that few readers will care to follow him through these difficult generalities. Nevertheless, this part of the book is of value, presenting as it does results which are to be found, we believe, only in various periodicals. But of the simple geometrical meaning of Jacobi's criterium, of the fact that it is a necessary as well as a sufficient condition for a maximum or a minimum, of the fact that from this criterium we may discover how long a curve continues to satisfy the conditions of the problem—not a word is said. From this we may see then that the book is not perhaps so readable as it might have been made.

We mention some points that to us seem to require criticism. A complete list would be of considerable length and would perhaps lay an undue emphasis on the faults of the work. In the opening chapter (p. 15) the author defines a variation as something "unendlich klein" and similar to a differential. This is an unsatisfactory and confusing definition, and is amusing in juxtaposition to Pascal's comment on Lagrange (p. 5). On page 16 we find the following: "In der That, wenn die beiden unendlich nahen Curven das Bestreben haben, mit einander zusammenzufallen, so ist es selbstverständlich, dass z. B. auch ihre Tangenten zusammenzufallen suchen." Such a primeval statement is utterly out of place in a book purporting to be rigorous. In the italics preceding this quotation the "variation of the derivative" is used in place of the "derivative of the variation." This slip would seem less worthy of notice if the author considered it evident, as it really is, that the two are identical. He does not, however, but proves at length two pages afterwards that they are so. The discussion of the restrictions of the "regular variation," a common term which by the way Pascal does not use, is on the whole, gratifying. He omits, however, to state what seems to us the point of chief interest, that any special variation may with perfect freedom be used to obtain a *necessary* condition for a maximum or a minimum. In pages 17–19 the author does not always make clear his assumptions as to the nature of the variation. On page 21 he says F must be assumed continuous, while two pages before he assumed that F had finite partial derivatives, an assumption certainly implying continuity. On page 39 the reasoning is purely formal, precisely that which he condemns most severely. His statement here that his process leads to the problem of maxima and minima of ordinary analysis is not accurate. His result he justifies by a footnote which admits that the text is inconclusive.

The treatment of multiple integrals though brief and incomplete is, on the whole, very good. We are glad to see several problems mentioned which are sometimes omitted in a course on this subject, such as the general problem of the calculus of variations. This Pascal calls, without sufficient reason we think, "Mayer's problem." The book closes with a discussion of the most famous problems of the calculus, *e. g.*, Newton's problem and the brachistochrone, with shorter notices of many others. The translator has added to the book some references to articles that have appeared since 1897, and indexes.

J. K. WHITTEMORE.

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NOTES.

THE second number of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY, which has just appeared, consists of 162 pages and contains the following articles:—"On the metric geometry of the plane n-line," by F. MORLEY; "On relative motion," by ALEXANDER S. CHESSIN; "Plane cubics and irrational covariant cubics," by HENRY S. WHITE; "A purely geometric representation of all points in the projective plane," by JULIAN LOWELL COOLIDGE; "The decomposition of the general collineation of space into three skew reflections," by EDWIN B. WILSON; "A new method of determining the differential parameters and invariants of quadratic differential quantics," by HEINRICH MASCHKE; "On the extension of Delaunay's method in the lunar theory to the general problem of planetary motion," by G. W. HILL; "On the types of linear partial differential equations of the second order in three independent variables which are unaltered by the transformations of a continuous group," by J. E. CAMPBELL.

THE INTERNATIONAL MATHEMATICAL CONGRESS AT PARIS: At the two general sessions of the congress, the following addresses will be delivered:—August 6th: "On the historiography of mathematics," by Professor M. CANTOR.—"Three Italian analysts, Betti, Brioschi, Casorati, and three ways of considering the questions of analysis; their influence," by Professor V. VOLTERRA.—August 11th: "A page from the life of Weierstrass," by Professor G. MIT-