

# Triadophilia: A special corner in the landscape

Philip Candelas<sup>1</sup>, Xenia de la Ossa<sup>1</sup>, Yang-Hui He<sup>1,2</sup>,  
and Balázs Szendrői<sup>1</sup>

<sup>1</sup>Mathematical Institute, Oxford University, 24-29 St. Giles',  
Oxford OX1 3LB, UK

<sup>2</sup>Merton College, Oxford OX1 4JD, UK

## Abstract

It is well known that there are a great many apparently consistent vacua of string theory. We draw attention to the fact that there appear to be very few Calabi–Yau manifolds with the Hodge numbers  $h^{1,1}$  and  $h^{2,1}$  both small. Of these, the case  $(h^{1,1}, h^{2,1}) = (3, 3)$  corresponds to a manifold on which a three-generation heterotic model has recently been constructed. We point out also that there is a very close relation between this manifold and several familiar manifolds including the “three-generation” manifolds with  $\chi = -6$  that were found by Tian and Yau, and by Schimmrigk, during early investigations. It is an intriguing possibility that we may live in a naturally defined corner of the landscape. The location of these three-generation models with respect to a corner of the landscape is so striking that we are led to consider the possibility of transitions between heterotic vacua. The possibility of these transitions, that we here refer to as transgressions, is an old idea that goes back to Witten. Here we apply this idea to connect three-generation vacua on different Calabi–Yau manifolds.

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*Triadophilia* from G. a love of three-ness, a nostalgia for a world of three generations. Less precise but also less cumbersome than *tritogeneia-philía*.

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## 1 Introduction and summary

### 1.1 Survey of constructions of Calabi–Yau manifolds

Until the interest in Calabi–Yau manifolds that derived from string theory, very few of these were known explicitly; the manifolds had, at that time, only recently been shown to exist. Owing to the interest from string theory, increasingly large classes of Calabi–Yau manifolds were constructed. Since, however, it seems to be impossible to construct classes of manifolds with desired properties, one must perforce construct large classes of manifolds that are then searched for cases that are phenomenologically interesting. Tian and Yau [1] were nevertheless able, at a very early stage, to find two manifolds with Euler number  $\chi = -6$  leading to a model with three generations of particles. Both of these have Hodge numbers  $(h^{11}, h^{21}) = (6, 9)$ . Motivated by these examples, Schimmrigk [2] found a third manifold with  $\chi = -6$ , and with the same Hodge numbers. This same manifold was rediscovered, shortly afterwards, by Gepner [3] in the process of constructing rational conformal field theories. We can denote the three families of manifolds, families because they have parameters, in the following way:

$$M'' = N''/A, \quad M' = N'/\widehat{A} \times B \quad \text{and} \quad M = N/\widehat{A} \times \widehat{B} \times C, \quad (1.1)$$

where

$$N'' = \begin{matrix} \mathbb{P}^3 \\ \mathbb{P}^3 \end{matrix} \begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}_{-18}, \quad N' = \begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^3 \end{matrix} \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}_{-54}, \quad N = \begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix}_{-162}. \quad (1.2)$$

The notation denotes that  $N''$ , for example, is realized in the product  $\mathbb{P}^3 \times \mathbb{P}^3$  by three equations whose degrees, in the variables of the two projective spaces, are given by the columns of the matrix. The subscripts appended to the matrix are the Euler numbers of the manifolds and the notation in (1.1) indicates that the manifolds are quotiented by certain groups  $A$ ,  $B$  and  $C$ . Each of these groups is abstractly a  $\mathbb{Z}_3$ . The group  $A$  acts freely but  $B$  and  $C$  each leave fixed a certain curve, in fact a torus, within the manifold and the hats indicate that these fixed tori are resolved. It was suspected, on the basis of the identity of the Hodge numbers together with the fact that they fall into a sequence, that the manifolds in fact belong to the same irreducible family, and this was shown to be the case in [4].

Despite the ease with which these early examples had been found, further examples of manifolds with  $\chi = \pm 6$  proved much more elusive. The class of all Calabi–Yau manifolds that can be realised as a complete intersection of polynomials in a product of projective spaces, known as the Complete Intersection Calabi–Yau manifolds or CICYs, generalizing the construction of the manifolds of (1.2), was constructed in [5–7]. This class, consisting

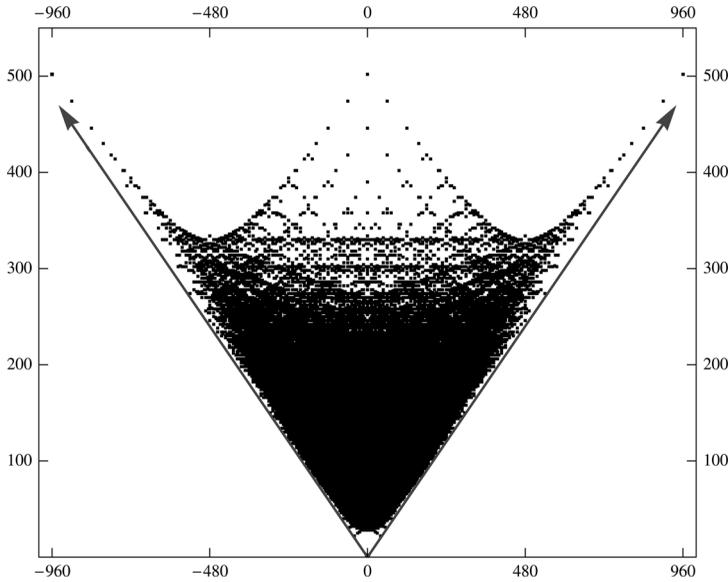


Figure 1: A plot of the Hodge numbers of the Kreuzer–Skarke list.  $\chi = 2(h^{11} - h^{21})$  is plotted horizontally and  $h^{11} + h^{21}$  is plotted vertically. The oblique axes bound the region  $h^{11} \geq 0, h^{21} \geq 0$ .

of almost 8000 manifolds, was searched for manifolds with  $\chi = -6$ , and for manifolds whose quotients by a freely acting group could have  $\chi = -6$ . None was found beyond the three above which inspired the construction.

The number of examples of Calabi–Yau manifolds was increased by the construction of manifolds [8–10] given by polynomials in weighted  $\mathbb{P}^4$  and increased again very greatly by the construction of manifolds as hypersurfaces in toric varieties following the methods introduced by Batyrev [11]. In a tour de force of computer calculation [12, 13] Kreuzer and Skarke compiled a list of all 4-dimensional reflexive polyhedra, each of which corresponds to a family of anticanonical hypersurface Calabi–Yau manifolds in the corresponding toric variety. The list runs to almost 500,000,000 polyhedra and gives rise to some 30,000 distinct pairs of Hodge numbers<sup>1</sup> which we plot in figure 1.

For comparison, we include a plot in figure 2 of the Hodge numbers of the 264 distinct pairs of Hodge numbers for the CICY’s plotted to the same

<sup>1</sup>It is not known how many of these manifolds are distinct. Manifolds with distinct Hodge numbers are certainly distinct, however, the converse is not true in general, so the number of distinct manifolds is somewhere between 30,000 and 500,000,000. For the CICY’s there are 264 pairs of Hodge numbers and roughly 8,000 manifolds. For this case it is known [14] that at least 2590 of the manifolds are distinct as classical manifolds.

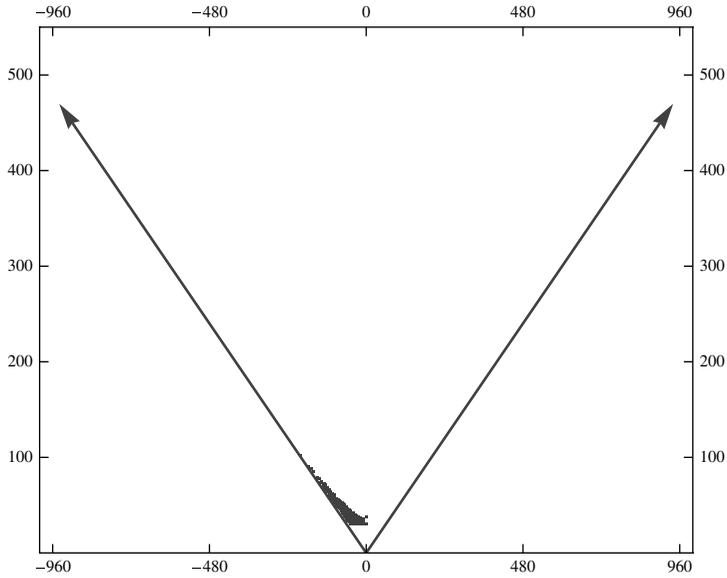


Figure 2: A plot of the 264 distinct pairs of Hodge Numbers for the CICY's.

scale. This is something of a cautionary tale showing what can happen when a seemingly large class of manifolds turns out to be rather special.

The Kreuzer–Skarke list, vast though it is, does not exhaust all possibilities. An obvious extension is to include the possibility of higher codimension corresponding to the case of more than one polynomial in a toric variety of higher dimension; these one might term toric CICY's. A correspondence with lattice polyhedra that generalizes the construction of Batyrev for the case of a single polynomial has been given by Batyrev and Borisov [15]. Two simple examples of such manifolds will appear later and it is worth writing one of them here to explain the notation and to give an idea of the immense number of possible members of this class. Consider the manifold that is denoted by

$$\mathbb{P} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 3 & 3 \\ 3 & 0 \end{bmatrix}.$$

The first matrix is the weight matrix and the second one is the degree matrix. Each column of the weight matrix corresponds to a coordinate, so in this case we have coordinates  $(z_1, z_2, \dots, z_7)$  and the two rows of the first matrix indicate that there are two independent scalings with the columns of the matrix corresponding to the weights of each coordinate. Under a scaling the coordinates transform as

$$(z_1, z_2, z_3, z_4, z_5, z_6, z_7) \rightarrow (\lambda\mu z_1, \lambda\mu z_2, \lambda z_3, \lambda z_4, \lambda z_5, \lambda z_6, \mu z_7),$$

with  $\lambda, \mu$  nonzero complex numbers. The second matrix indicates that there are two polynomials  $p^1$  and  $p^2$  and that under a scaling  $p^1 \rightarrow \lambda^3 \mu^3 p^1$  and  $p^2 \rightarrow \lambda^3 p^2$ . The fact that the manifold has vanishing first Chern class is ensured by the condition that the row sum of each row of the weight matrix is equal to the row sum of the corresponding row of the degree matrix. The dimension count, in this case, is that we have seven coordinates that are identified under two scalings and subject to two polynomial constraints yielding a manifold of dimension  $7 - 2 - 2 = 3$ . The question of when a configuration of this type gives rise to a nonsingular manifold is answered by the Batyrev–Borisov procedure. The Kreuzer–Skarke list corresponds to the special case that the degree matrix has only one column, the hypersurfaces in weighted  $\mathbb{P}^4$  correspond to the case that the weight matrix has only one row, and the CICY’s to the very special case that all the entries of the weight matrix are either 1 or 0 and moreover that each column of the weight matrix contain precisely one 1. The number of possible configurations would seem to be immense and the scale of the enterprise of examining this class would seem to preclude any complete listing, though several hundred new pairs of Hodge numbers have been found by studying the interesting region along the edges of the plots where one of the Hodge numbers is small [16, 17].

Batyrev and Kreuzer [18] have also found many new pairs of Hodge numbers by examining reflexive polyhedra for hypersurfaces in toric varieties that admit conifold singularities, blowing down the  $\mathbb{P}^1$ ’s and smoothing the resulting manifolds.

Even this of course is not everything, since there are Calabi–Yau manifolds that are not covered by these constructions; we are, moreover, also interested in heterotic vacua corresponding to vector bundles  $\mathcal{V}$  on the Calabi–Yau manifold for which  $c_1(\mathcal{V}) = 0$ . If  $\mathcal{V} = \mathcal{T}$ , the tangent bundle, then the number of generations is  $|\frac{1}{2}c_3(\mathcal{T})| = |\frac{1}{2}\chi|$  so three generations corresponds to  $\chi = \pm 6$  as is the case for  $N = 2$  compactifications. For general heterotic vacua, however, there is much greater freedom. The restrictions on  $\mathcal{V}$  are that it be stable, have  $c_1(\mathcal{V}) = 0$  and have  $c_2(\mathcal{V})$  satisfy a certain condition. The number of generations of particles is then  $|\frac{1}{2}c_3(\mathcal{V})|$ . There are presumably a great many of these heterotic vacua; while toric geometry has afforded us a considerable degree of control over Calabi–Yau manifolds, it is not yet known how to extend this degree of control to bundles over these manifolds.

Special cases can, however, be studied and the group at the University of Pennsylvania [19–22] has developed a small number of three-generation heterotic models based on quotients of a special<sup>2</sup> CICY, the split bicubic

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<sup>2</sup>The split bicubic is special in so far as it has  $(h^{11}, h^{21}) = (19, 19)$  the value for  $h^{11}$  being the largest for any CICY. All the CICY’s have Euler numbers in the range  $-200 \leq \chi \leq 0$  and the split bicubic, together with a manifold with Hodge numbers (15,15), are the only two CICY’s which have  $\chi = 0$  and can possibly be self-mirror.

family

$$X^{19,19} = \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \left[ \begin{matrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{matrix} \right]_0^{19,19} \tag{1.3}$$

where we append superscripts to record the values of  $(h^{11}, h^{21})$ . One special feature of the split bicubic is that it is a bi-elliptic fibration. To see this, consider the form of the equations [23] for this space

$$\begin{aligned} t_1 \left( \sum_{j=1}^3 \xi_j^3 - 3a \xi_1 \xi_2 \xi_3 \right) + 3ct_2 \sum_{j=1}^3 \xi_j^2 \xi_{j-1} &= 0 \\ 3ct_1 \sum_{j=1}^3 \eta_j^2 \eta_{j-1} + t_2 \left( \sum_{j=1}^3 \eta_j^3 - 3b \eta_1 \eta_2 \eta_3 \right) &= 0, \end{aligned} \tag{1.4}$$

where we have chosen coordinates  $t$  for the  $\mathbb{P}^1$  and  $\xi$  and  $\eta$  for the two  $\mathbb{P}^2$ 's. These are particularly symmetric polynomials of the given degrees of a form to which we will return later; the most general polynomials would contain 19 parameters. However this simple choice is sufficient to illustrate the following point. Consider the equations for fixed  $(t_1, t_2) \in \mathbb{P}^1$ ; each equation is then a cubic in a  $\mathbb{P}^2$ , generically an elliptic curve (i.e., a 2-torus). Thus the split bicubic is a fibration over  $\mathbb{P}^1$  with fibre  $E_1(t) \times E_2(t)$ , where for generic  $t$ , both  $E_i(t)$  are elliptic curves, which degenerate for certain special values of  $t$ .

Holomorphic vector bundles on elliptic curves were classified by Atiyah [24] and the extension to spaces that are fibred by elliptic curves was considered by Donagi [25] and by Friedman, Morgan and Witten [26, 27]. Further work investigated the problem of constructing stable  $SU(n)$  bundles, for  $n = 3, 4, 5$ , on the large class of Calabi–Yau 3-folds that are elliptically fibred, that is for which there is a map to a  $\mathbb{P}^1$  for which the generic fibre is an elliptic curve [21, 28]. An explicit construction of a heterotic model whose low energy effective theory has the particle content of the minimum supersymmetric standard model (MSSM) nevertheless proved elusive until such a model based on a stable  $SU(4)$  vector bundle, corresponding to a gauge group  $SO(10)$  in spacetime, was presented in [20]. The manifold of this model is a  $\mathbb{Z}_3 \times \mathbb{Z}_3$  quotient of the split bicubic. A breaking of the  $SO(10)$  symmetry via the Hosotani mechanism, that takes advantage of the fundamental group  $\mathbb{Z}_3 \times \mathbb{Z}_3$ , yields the particle spectrum of the MSSM, without exotics, and with no anti-generations. A version with  $SU(5)$  vector bundle, and hence also  $SU(5)$  in spacetime, was also found [22].

Returning to figure 1 and the Kreuzer–Skarke list, it is apparent that the central part of the plot is very dense with essentially every site occupied. The main point that we wish to make here is that the tip of the diagram where  $(h^{11}, h^{21})$  are both small is thinly populated and this remains true

even if we include the CICY's, the Klemm–Kreuzer toric CICY's, the toric conifolds and other examples of which we are aware. One way of attempting to populate the tip is to seek Calabi–Yau manifolds that are free quotients, with a nontrivial fundamental group. Such manifolds seem, however, to be genuinely rare and especially so for larger fundamental groups that would produce small Hodge numbers. This was apparent for the CICY's from the first investigations [6, 7].

Recently Batyrev and Kreuzer [29] have searched the Kreuzer–Skarke list for manifolds with a nontrivial fundamental group and find just 16 examples; moreover the fundamental groups that they find are: one occurrence of  $\mathbb{Z}_5$  and two occurrences of  $\mathbb{Z}_3$ , with the remaining 13 instances corresponding to  $\mathbb{Z}_2$ 's. This is not everything: a quotient manifold will only appear in the Kreuzer–Skarke list if the quotient is realized torically. Thus the occurrence of the  $\mathbb{Z}_5$  corresponds to a quotient of the quintic 3-fold

$$\mathbb{P}^4[5]/\mathbb{Z}_5, \quad (h^{11}, h^{21}) = (1, 21)$$

with the generator of the group corresponding to the action  $x_j \rightarrow \zeta^j x_j$  on the coordinates of the embedding space, for  $\zeta$  a nontrivial fifth root of unity. The further quotient

$$\mathbb{P}^4[5]/\mathbb{Z}_5 \times \mathbb{Z}_5, \quad (h^{11}, h^{21}) = (1, 5)$$

is not present in the list owing to the fact that the generator of the second symmetry group acts by cyclic permutation of the coordinates, not by multiplying the coordinates by roots of unity.<sup>3</sup> One of the two occurrences of  $\mathbb{Z}_3$  also involves a familiar manifold

$$\mathbb{P}^2 \begin{bmatrix} 3 \\ 3 \end{bmatrix} / \mathbb{Z}_3, \quad (h^{11}, h^{21}) = (2, 29),$$

where the  $\mathbb{Z}_3$  can be chosen to be either the symmetry  $A$  of (1.1) or a certain diagonal subgroup of  $B \times C$ . The other  $\mathbb{Z}_3$  quotient is

$$\mathbb{P}(1\ 1\ 1\ 3\ 3)[9]/\mathbb{Z}_3, \quad (h^{11}, h^{21}) = (2, 38).$$

The Batyrev–Kreuzer search does not find

$$\mathbb{P}^3 \begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix} / \mathbb{Z}_3, \quad (h^{11}, h^{21}) = (6, 9)$$

because this space is described by three polynomials while the Kreuzer–Skarke list corresponds to spaces that are defined by a single polynomial.

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<sup>3</sup>It is possible to choose coordinates so that the first generator acts cyclically on the coordinates and the second acts by multiplication by fifth roots of unity. It is, however, not possible to arrange for both generators to act torically.

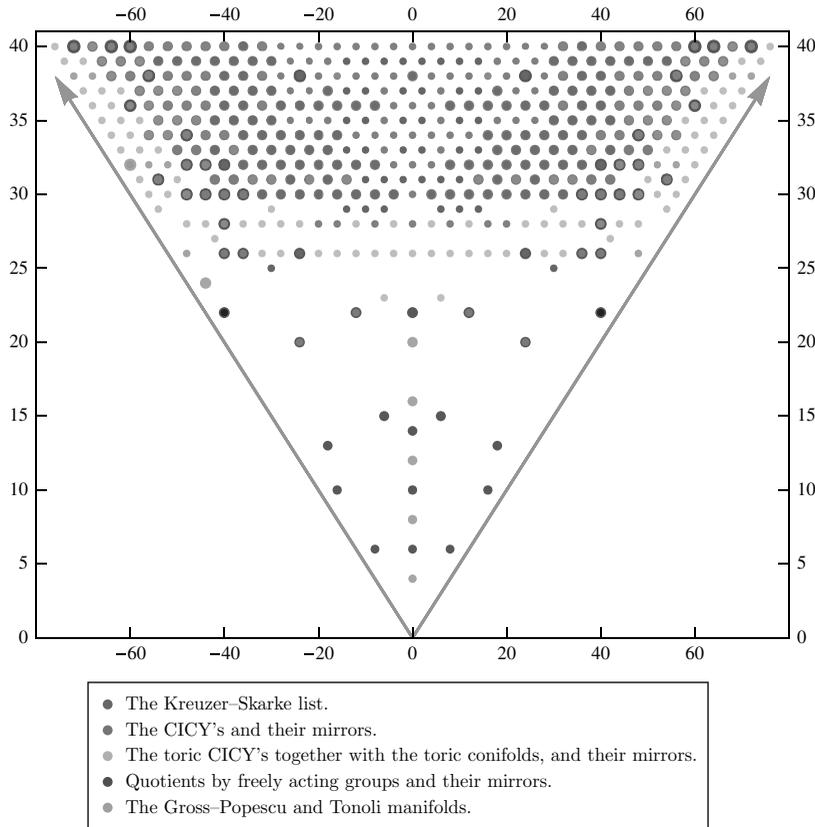


Figure 3: The underpopulated corner of the landscape.  $\chi = 2(h^{11} - h^{21})$  is plotted horizontally,  $h^{11} + h^{21}$  is plotted vertically and the oblique axes bound the region  $h^{11} \geq 0, h^{21} \geq 0$ . In the electronic version of this figure the points are coloured according to provenance and have partial transparency in order to show overlays. The manifolds with  $h^{11} + h^{21} \leq 22$  are identified in table 1.

In a recent article, Bouchard and Donagi [32] make a detailed classification of quotients of the split bicubic and find quotients with  $(h^{11}, h^{21}) = (11, 11), (7, 7), (5, 5), (3, 3)$  that are included in table 1.

To emphasize the paucity of manifolds with both Hodge numbers small we present in figure 3 and table 1 the tip of the plot of Hodge numbers for  $h^{11} + h^{21} \leq 40$  including the CICY's together with their mirrors, the toric CICY's, the toric conifolds, the quotient manifolds of which we are aware, a special class of manifolds fibred by abelian surfaces due to Gross and Popescu [34], manifolds constructed by Borisov and Hua [35] and by Hua [36], and certain interesting examples due to Rødland [31], and Tonoli [39]. For two of these manifolds, those with  $(h^{11}, h^{21}) = (1, 23), (1, 31)$ , we do not show the points corresponding to the mirror manifolds since the constructions are such that the mirror manifolds are not known to exist. Our observation is

Table 1: The manifolds of the Tip. The manifolds with  $y = h^{11} + h^{21} \leq 22$  from figure 3. In the ‘Manifold’ column  $X^{19,19}$  denotes the split bicubic and multiple quotient groups indicates different quotients with the same Hodge numbers. The  $\mathbb{H}$  in the quotient of  $\mathbb{P}^7[2, 2, 2, 2]$  denotes the quaternion group and the notation  $\mathbb{P}^7[2, 2, 2, 2]^\sharp$  and  $\mathbb{P}^7[2, 2, 2, 2]^\#$  denotes two different singularizations. The vectors appended to the symmetries of the two weighted CICY’s indicate how the generators act. The generator  $(\mathbb{Z}_3 : 1, 2, 1, 2, 0, 0, 0)$ , for example, acts by multiplying the first coordinate by  $\omega$ , the second by  $\omega^2$ , etc., with  $\omega$  a nontrivial cube root of unity. In the entry corresponding to  $(h^{11}, h^{21}) = (6, 6)$  we write  $\mathbb{P}^5[3, 3]^\sharp$  to denote a resolution of a conifold of  $\mathbb{P}^5[3, 3]$  and  $\mathbf{1}$  denotes the trivial group. For each manifold with  $\chi < 0$  there is a mirror which we do not list explicitly.

$(\chi, y)$	$(h^{11}, h^{21})$	Manifold	Reference
$(-40, 22)$	$(1, 21)$	$\mathbb{P}^4[5]/\mathbb{Z}_5$	–
$(-12, 22)$	$(8, 14)$	$\mathbb{P} \begin{pmatrix} 4 & 2 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} [8 & 4] \\ [0 & 4] \\ [2 & 0] \end{matrix}$	[30]
$(0, 22)$	$(11, 11)$	$X^{19,19}/\mathbb{Z}_2$	[32]
$(-24, 20)$	$(4, 16)$	$\mathbb{P} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} [3 & 3] \\ [3 & 0] \end{matrix} / (\mathbb{Z}_3 : 1, 2, 1, 2, 0, 0, 0)$	[30]
$(-20, 20)$	$(5, 15)$	$\mathbb{P}^1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} / \mathbb{Z}_2 \times \mathbb{Z}_2$	$(x_{\alpha,1}, x_{\alpha,2}) \rightarrow (x_{\alpha,1}, -x_{\alpha,2})$ $(x_{\alpha,1}, x_{\alpha,2}) \rightarrow (x_{\alpha,2}, x_{\alpha,1})$ $\alpha = 1, \dots, 5, j = 1, 2$
$(0, 20)$	$(10, 10)$	$\mathbb{P}^7[\widehat{2, 2, 2, 2}]^\sharp$	[34, 6.17], [36, Rem. 7.8], §1.1
$(0, 16)$	$(8, 8)$	$\mathbb{P}(11114)[8]^\sharp$	[34, 2.2]
$(-6, 15)$	$(6, 9)$	$\mathbb{P}^3 \begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix} / A$	§1.1
$(0, 14)$	$(7, 7)$	$X^{19,19} / \{\mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2\}$	[32]
$(-18, 13)$	$(2, 11)$	$\mathbb{P}^2 \begin{bmatrix} 3 \\ 3 \end{bmatrix} / A \times D$	§1.1

(0,12)	(6,6)	$\widehat{\mathbb{P}^5[3,3]^\sharp}/G, G \subset \mathbb{Z}_6 \times \mathbb{Z}_6$	[34, 4.10], §1.1 below
(-16,10)	(1,9)	$\frac{\mathbb{P}^1 \begin{bmatrix} 1 & 1 \\ \mathbb{P}^1 & 1 \\ \mathbb{P}^1 & 1 \\ \mathbb{P}^1 & 1 \\ \mathbb{P}^1 & 1 \end{bmatrix}}{\mathbb{Z}_5}$	$x_{\alpha,j} \rightarrow x_{\alpha+1,j}$ $\alpha = 1, \dots, 5, j = 1, 2$ $\alpha$ understood mod 5.
(-16,10)	(1,9)	$\mathbb{P}_5^1[3,3]/\mathbb{Z}_3 \times \mathbb{Z}_3$	–
(-16,10)	(1,9)	$\mathbb{P}^7[2,2,2,2]/\{\mathbb{H}, \mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\}$	[34, 4.10], §1.1 below
(0,10)	(5,5)	$X^{19,19}/\mathbb{Z}_4$	[32]
(-14,9)	(1,8)	{Resolution of a Pfaffian CY manifold}/ $\mathbb{Z}_7$	[31]
(0,8)	(4,4)	Resolution of a Horrocks-Mumford quintic	[34, 3.2]
(-8,6)	(1,5)	$\mathbb{P}^4[5]/\mathbb{Z}_5 \times \mathbb{Z}_5$	–
(-8,6)	(1,5)	$\frac{\mathbb{P}^1 \begin{bmatrix} 1 & 1 \\ \mathbb{P}^1 & 1 \\ \mathbb{P}^1 & 1 \\ \mathbb{P}^1 & 1 \\ \mathbb{P}^1 & 1 \end{bmatrix}}{\mathbb{Z}_5 \times \mathbb{Z}_2}$	$x_{\alpha,j} \rightarrow x_{\alpha+1,j}$ $(x_{\alpha,1}, x_{\alpha,2}) \rightarrow (x_{\alpha,1}, -x_{\alpha,2})$
(-8,6)	(1,5)	$\mathbb{P}^7[2,2,2,2]/G,  G  = 16$	[33, 35, 36], §1.1 below
(0,6)	(3,3)	$X^{19,19}/\{\mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_2, \mathbb{Z}_5\}$	[32]
(-4,4)	(1,3)	$\frac{\mathbb{P}^1 \begin{bmatrix} 1 & 1 \\ \mathbb{P}^1 & 1 \\ \mathbb{P}^1 & 1 \\ \mathbb{P}^1 & 1 \\ \mathbb{P}^1 & 1 \end{bmatrix}}{\mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2}$	$x_{\alpha,j} \rightarrow x_{\alpha+1,j}$ $(x_{\alpha,1}, x_{\alpha,2}) \rightarrow (x_{\alpha,1}, -x_{\alpha,2})$ $(x_{\alpha,1}, x_{\alpha,2}) \rightarrow (x_{\alpha,2}, x_{\alpha,1})$
(-4,4)	(1,3)	$\mathbb{P}^7[2,2,2,2]/G,  G  = 32$	[33, 35, 36], §1.1 below
(0,4)	(2,2)	$\widehat{\mathbb{P}^7[2,2,2,2]^\sharp}/G,  G $ divides 64	[34, 5.8, 6.9, 7.5], [35, 36], §1.1
(0,4)	(2,2)	Resolution of Pfaffian CY with 49 nodes	[32]

that the tip is sparsely populated with some of the lowest points corresponding to manifolds we have discussed explicitly above or their quotients. The Kreuzer–Skarke list contains many pairs of points with  $\chi = \pm 6$ . These have Hodge numbers  $(h, h + 3)$  and  $(h + 3, h)$  for certain values of  $h$  in the range  $13 \leq h \leq 128$  and it is easier to state the values of  $h$  that are not found. These excluded values are  $h = 102, 103, 115, 117, 119\text{--}126$ . It is interesting that the Tian–Yau manifold, with  $(h^{11}, h^{21}) = (6, 9)$  has Hodge numbers that are smaller than these other manifolds, which are simply connected.

Conifold resolutions of certain symmetric members of the  $\mathbb{P}^7[2, 2, 2, 2]$  family and their quotients play an interesting role in the tip and we pause to comment on these. In a very early paper, Strominger and Witten [33] note that there is a system of four quadrics for  $\mathbb{P}^7$  that are invariant under  $\mathbb{Z}_8 \times \mathbb{Z}_8$  and are fixed point free. These quadrics are equivalent to

$$p^\alpha = z_\alpha^2 + z_{\alpha+4}^2 + a z_{\alpha+2}z_{\alpha+6} + b(z_{\alpha+1}z_{\alpha-1} + z_{\alpha+3}z_{\alpha-3}) \tag{1.5}$$

where the affix  $\alpha$  is understood mod 8 and we see from the form of the equation that  $p^{\alpha+4} = p^\alpha$ . The locus  $p^\alpha = 0$  is invariant under the symmetries

$$A : z_\alpha \rightarrow z_{\alpha+1} \text{ and } B : z_\alpha \rightarrow \zeta^\alpha z_\alpha$$

with  $\zeta$  a nontrivial eighth root of unity. These quadrics are however not transverse since one can check that the form  $dp^1 \wedge dp^2 \wedge dp^3 \wedge dp^4$  vanishes at 64 points which correspond to a single group orbit. Thus the variety defined by the quadrics is singular. Owing to this Strominger and Witten deform the quadrics to

$$p^\alpha = z_\alpha^2 + z_{\alpha+4}^2 + a z_{\alpha+2}z_{\alpha+6} + b(z_{\alpha+1}z_{\alpha-1} + z_{\alpha+3}z_{\alpha-3}) + c(z_{\alpha+1}z_{\alpha+3} + z_{\alpha+5}z_{\alpha-1})$$

With this extra term the quadrics are invariant under the subgroup  $\mathbb{Z}_8 \times \mathbb{Z}_4$  generated by  $A$  and  $B^2 : z_\alpha \rightarrow i^\alpha z_\alpha$  and the equations are moreover transverse. The quotient is a manifold with  $(h^{11}, h^{21}) = (1, 3)$  and fundamental group  $\mathbb{Z}_8 \times \mathbb{Z}_4$ . It is noted also that there are several symmetry groups  $G$  of order 32, both abelian and nonabelian, for which the quadrics are fixed point free and transverse. Curiously, among the possible symmetries noted in this paper is the quaternion group,  $\mathbb{H}$ , a fact that was rediscovered later by Beauville [37].

The question of finding all possible quotients of  $\mathbb{P}^7[2, 2, 2, 2]$  has been investigated by Borisov and Hua [35] and by Hua [36]. On returning to the  $\mathbb{Z}_8 \times \mathbb{Z}_8$ -invariant quadrics (1.5), it is noted that the 64 singular points of the variety are nodes, that is conifold points, and that, somewhat remarkably, they can be simultaneously resolved to give a simply connected Calabi–Yau manifold with  $(h^{11}, h^{21}) = (2, 2)$ . This resolution is one of the Gross–Popescu manifolds that was constructed by completely different reasoning. It is noted also that there are 3 groups of order 64 that preserve the

quadrics (1.5) and a further two groups of order 64 that preserve another two-parameter family of quadrics. This second family of quadrics also have the property that they are fixed point free but not transverse and have 64 nodes that can be simultaneously resolved to give a Calabi–Yau manifold which, in this case, have  $(h^{11}, h^{21}) = (10, 10)$ . For the family (1.5) the groups lift to the resolutions and remain fixed point free. Taking the quotient by the group does not affect the Picard group or the number of parameters, since the locus  $p^\alpha = 0$  is invariant. Thus these quotients also have  $(h^{11}, h^{21}) = (2, 2)$  and have fundamental groups of order 64, which are the largest known. It follows that it is possible to take the quotient by any subgroup of the relevant group of order 64 and this yields further manifolds with the same Hodge numbers. The groups of order 64 have the property that all the subgroups of order 32 preserve 3-parameter families of quadrics that are fixed point free and transverse. Quotients by these groups yield manifolds with  $(h^{11}, h^{21}) = (1, 3)$ . Taking quotients by smaller subgroups yields manifolds with  $(h^{11}, h^{21}) = (1, 2^n + 1)$  for  $n = 2, \dots, 5$ . For the second family of quadrics only a two-dimensional subspace of the ten-dimensional family of resolutions admit a group action and it is conjectured that the quotients of the manifolds of the subspace have  $(h^{11}, h^{21}) = (2, 2)$ . The Gross-Popescu manifolds merit further study but we shall mention just one further fact: which is that the manifold with  $(h^{11}, h^{21}) = (6, 6)$  is the resolution of a conifold of  $\mathbb{P}^5[3, 3]$  that is symmetric under  $\mathbb{Z}_6 \times \mathbb{Z}_6$ . This symmetry lifts to the resolution where it is fixed point free. Taking quotients by subgroups yields nonsimply connected manifolds with the same Hodge numbers.

We have made the observation that in order to find manifolds with low values of  $h^{11} + h^{12}$  it is good to seek manifolds with a nontrivial fundamental group. The fundamental group, however, cannot be quite the right attribute since it is not respected by mirror symmetry. In figure 3, for example,  $\mathbb{P}^4[5]/\mathbb{Z}_5$  has fundamental group  $\mathbb{Z}_5$  while its mirror has trivial fundamental group. The attribute that we are after, for a manifold  $Y$ , is torsion in the integer cohomology ring  $H^\bullet(Y, \mathbb{Z})$ ; for a clear discussion see [29]. The torsion is a finite component of the cohomology ring that is absent if we work over  $\mathbb{C}$  or  $\mathbb{R}$ . For the case that  $H^1(Y, \mathbb{Z}) \cong 0$  the cohomology groups take the general form

$$\begin{aligned} H^0(Y, \mathbb{Z}) &\cong \mathbb{Z} & H^6(Y, \mathbb{Z}) &\cong \mathbb{Z} \\ H^1(Y, \mathbb{Z}) &\cong 0 & H^5(Y, \mathbb{Z}) &\cong \mathfrak{A}(Y)^* \\ H^2(Y, \mathbb{Z}) &\cong \mathfrak{A}(Y) \oplus \mathbb{Z}^{h^{11}} & H^4(Y, \mathbb{Z}) &\cong \mathfrak{B}(Y)^* \oplus \mathbb{Z}^{h^{11}} \\ & & H^3(Y, \mathbb{Z}) &\cong \mathfrak{B}(Y) \oplus \mathbb{Z}^{2h^{21}+2} \end{aligned}$$

where  $\mathfrak{A}(Y)$  and  $\mathfrak{B}(Y)$  are finite groups and  $\mathfrak{A}(Y)^*$  and  $\mathfrak{B}(Y)^*$  are the corresponding dual groups. The group  $\mathfrak{B}(Y)$  is the torsion of  $H^3(Y, \mathbb{Z})$  and is known as the Brauer group. The group  $\mathfrak{A}(Y)$  is closely related to the

fundamental group through the isomorphism

$$\mathfrak{A}(Y) \cong \text{Hom}(\pi_1(Y), \mathbb{Q}/\mathbb{Z}).$$

On the other hand, it is conjectured that under mirror symmetry, there is a relation

$$\mathfrak{A}(Y^*) \oplus \mathfrak{B}(Y^*)^* \cong \mathfrak{A}(Y)^* \oplus \mathfrak{B}(Y),$$

where  $Y^*$  is the mirror of  $Y$ .

For the 16 examples of toric free group actions found by Batyrev and Kreuzer, it is the case that if a manifold  $Y$  has a nontrivial fundamental group then its Brauer group is trivial and the mirror  $Y^*$  has trivial fundamental group but nontrivial Brauer group. For manifolds defined by more than one polynomial, however, there are manifolds for which both  $\mathfrak{A}$  and  $\mathfrak{B}$  are simultaneously nontrivial. Thus the attribute that we are seeking is nontrivial torsion in the homology ring. Indeed, one of the manifolds [34, Theorem 6.9] at the current tip of the cone with Hodge numbers (2,2), a resolution of a very special nodal  $\mathbb{P}^7[2, 2, 2, 2]$ , is simply connected, but has recently been shown [40] to have Brauer group  $\mathfrak{B}(Y) \cong \mathbb{Z}_8 \times \mathbb{Z}_8$ , the largest known; its mirror [40, Remark 1.5] is conjectured to have torsion in both its fundamental group and Brauer group.

Let us remark finally that string compactifications are often asymmetrical with respect to mirror symmetry. For both the models based on the Tian–Yau manifold and the quotient of the split bicubic, the Hosotani mechanism is used to reduce the spacetime gauge group, which requires a nontrivial fundamental group. It is compelling that there should be a mirror description of these models with the role of the fundamental group reflected onto the Brauer group. It would be of considerable interest to understand this relation.

## 1.2 Key points

This has been a long introduction, perhaps overly beset by detail, so let us summarize our three main points.

- The geography of Calabi–Yau manifolds has a “tip” which appears to be sparsely populated. The sparse population seems to be a reflection of the fact that Calabi–Yau manifolds whose homology has nontrivial torsion seem to be genuinely rare. It is striking that the tip contains the manifold, that we shall call  $X^{3,3}$ , a quotient of  $X^{19,19}$  introduced in (1.3), for which there is a heterotic model that has the particle content of the MSSM. The tip also contains the Tian–Yau manifold

for which there is also a three-generation model. The fact that the tip is sparsely populated makes the fact that we find two three-generation models here more surprising. The fact that  $X^{3,3}$  is almost at the very end of the tip is a fact that would still be true even if new constructions increase the population.

- It is natural to ask if the two three-generation models are related and an answer is that the manifolds on which they are based are indeed closely related by conifold transitions. The most direct relation is that the Tian–Yau manifold is related via a conifold transition to a manifold  $X^{7,7}$  which is a 3-fold covering space of  $X^{3,3}$ .
- It is also natural to ask, in relation to transitions between heterotic models, if it is possible to transfer bundles across a conifold transition. We refer to this process here as a transgression of bundles. A necessary condition for a bundle to arise as a transgression on the split manifold is that the bundle should be trivial on the lines that arise as the blow ups of the nodes. Remarkably this is the case for the heterotic bundle on  $X^{3,3}$ , suggesting that the heterotic bundle can be thought of as arising in this way.

We have explained the first point in this introduction. In the remainder of the paper we elaborate on the second and third points. In §2 we discuss the relations between the bicubic and the split bicubic and their quotients. An interesting fact that is not obvious at the outset is that we may pass from the covering space of the Tian–Yau manifold to the split bicubic via a conifold transition

$$\begin{matrix} \mathbb{P}^3 \\ \mathbb{P}^3 \end{matrix} \begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix} \longleftrightarrow \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

It follows, upon taking the quotient by  $A \cong \mathbb{Z}_3$ , that the Tian–Yau manifold is related via a conifold transition to the quotient  $X^{7,7} = X^{19,19}/A$  of the split bicubic, a manifold also in the tip with  $(h^{11}, h^{21}) = (7, 7)$  and which is a 3-fold cover of  $X^{3,3}$ .

Given that the three-generation manifolds that we are considering are related by conifold transitions, we turn in §3 to the question of whether the vector bundles of their heterotic models are related. It is an old suggestion that it should, in certain circumstances, be possible to transfer bundles across a conifold transition. We examine this process. Although we have not been able to relate the tangent bundle of the Tian–Yau manifold to the vector bundle of  $X^{3,3}$  directly, we note that it should be possible to transgress the tangent bundle of the Tian–Yau manifold to a, hitherto unknown, bundle on the  $X^{7,7}$ -manifold with  $\chi = -6$ . The fact that the vector bundle on  $X^{3,3}$

is trivial on the “conifold lines”, that is on the  $\mathbb{P}^1$ s that arise from the split

$$\mathbb{P}^2 \begin{bmatrix} 3 \\ 3 \end{bmatrix}^{2,11} / A \times D \quad \longleftrightarrow \quad \begin{matrix} \mathbb{P}^1 \begin{bmatrix} 1 & 1 \end{bmatrix}^{3,3} \\ \mathbb{P}^2 \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} / A \times D \end{matrix}$$

suggests that the bundle can be thought of as arising from the manifold on the left. We examine this process and find, via a monad construction, some candidate bundles with the right Chern classes although we are not yet able to answer the question in the affirmative.

Seeing the tip of the landscape in figure 3 it is hard not to speculate on the possibility of a dynamical mechanism that would allow the universe to drift towards the tip. This of course is what makes the possibility of transgression so interesting. The burden of our discussion of transgression in §3 is that although the manifolds that we discuss seem to be discretely different, and the plots reinforce that impression, nevertheless the parameter spaces of different heterotic models meet in certain mildly singular manifolds and it is natural to ask if it is possible for the universe to move among these models. We conclude with a brief speculation along these lines in §4.

### 1.3 Dramatis personæ

Here we list the principal actors of the story for reference and to fix notation for the rest of the paper.

- In the Tian–Yau sequence, we have the three families of spaces

$$(N'')^{14,23} = \mathbb{P}^3 \begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \quad (N')^{8,35} = \mathbb{P}^2 \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}, \quad N^{2,83} = \mathbb{P}^2 \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

together with (resolutions of) respective quotients

$$(M'')^{6,9} = (N'')^{14,23} / A ,$$

$$(M')^{6,9} = \widehat{(N')^{8,35}} / A \times B ,$$

$$M^{6,9} = \widehat{N^{2,83}} / A \times B \times C .$$

The actions of the groups  $A$ ,  $B$  and  $C$  are explained in §2.1; the hats indicate resolutions of singularities of the quotients. We will show that  $M$ ,  $M'$  and  $M''$  all belong to the same irreducible family and so following §2.1, they will all be denoted by  $M^{6,9}$ .

- The bicubic  $N^{2,83}$  admit further free quotients

$$N^{2,29} = \mathbb{P}^2 \begin{bmatrix} 3 \\ 3 \end{bmatrix} / A \quad \text{and} \quad N^{2,11} = \mathbb{P}^2 \begin{bmatrix} 3 \\ 3 \end{bmatrix} / A \times D$$

that are studied in §2.2.

- In the split bicubic sequence, we have the family

$$X^{19,19} = \mathbb{P}^2 \begin{bmatrix} \mathbb{P}^1 & 1 & 1 \\ 3 & 0 & \\ \mathbb{P}^2 & 0 & 3 \end{bmatrix}$$

as well as free quotients

$$X^{7,7} = X^{19,19} / A \quad \text{and} \quad X^{3,3} = X^{19,19} / A \times D,$$

where the actions of the groups  $A$  and  $D$  are explained in §2.3.

## 2 Relations between three-generation manifolds

In this section, we first explain the relation between the three equivalent presentations of the Tian–Yau manifold. Then we will examine the various conifold transitions between the bicubic and the split bicubic and their quotients.

### 2.1 Three families of three-generation manifolds

Consider the three families of manifolds  $N''$ ,  $N'$  and  $N$  from §1.3 as well as resolutions of certain quotients  $M''$ ,  $M'$  and  $M$ . The groups  $A$ ,  $B$  and  $C$  are all abstractly isomorphic to  $\mathbb{Z}_3$ . The group  $A$  acts freely, while  $B$  and  $C$  have fixed curves that are tori. The upshot is that the Euler numbers  $\chi = 2(h^{11} - h^{21})$  of the (resolved) quotient manifolds are obtained from that of the covering manifold by dividing by the orders of the groups, with the result that  $M$ ,  $M'$  and  $M''$  all have  $\chi = -6$ . We will see also that the three manifolds also all have Hodge numbers  $(h^{11}, h^{21}) = (6, 9)$ . As stated previously, these families of manifolds have been shown to belong to the same irreducible component of the moduli space in [4], but it is useful to review some aspects of this correspondence in order to establish conventions and notation. A detailed study of the Tian–Yau manifold  $(M'')^{6,9}$ , and of a three-generation model based on this manifold, is to be found in [41].

We begin by discussing the Tian–Yau manifold. The covering space  $(N'')^{14,23}$  is described by three equations of the indicated bidegrees. One

may take, for example, the equations

$$\begin{aligned}
 F &= f_0 x_0 y_0 + f_1 \sum_j x_j y_j + f_2 \sum_j x_j y_{j+1} + f_3 \sum_j x_{j+1} y_j \\
 &\quad + f_4 x_0 \sum_j y_j + f_5 \left( \sum_j x_j \right) y_0, \\
 G &= x_0^3 - x_1 x_2 x_3 + g_1 \sum_j x_j^3 + g_2 x_0 \sum_j x_j x_{j+1}, \\
 H &= y_0^3 - y_1 y_2 y_3 + h_1 \sum_j y_j^3 + h_2 y_0 \sum_j y_j y_{j+1}. \tag{2.1}
 \end{aligned}$$

where the  $(x_0, x_j)$  and  $(y_0, y_j)$ ,  $j = 1, 2, 3$ , are projective coordinates for the two  $\mathbb{P}^3$ 's and  $f_a, g_a$  and  $h_a$  are coefficients. The separate treatment of the zeroth coordinate anticipates the action of the symmetry group  $A \cong \mathbb{Z}_3$  with generator

$$A : (x_0, x_j) \times (y_0, y_k) \longrightarrow (x_0, x_{j+1}) \times (y_0, y_{k+1}), \tag{2.2}$$

with the  $j$  and  $k$  indices understood as being reduced mod 3. The freedom to make changes of coordinates has been used to ensure that, in  $G$ , the term  $x_1 x_2 x_3$  appears with coefficient  $-1$  and the terms of the form  $x_\mu^2 x_\nu$ , with  $\mu \neq \nu$ , appear with coefficient zero, and similarly for  $H$ . The polynomial  $F$  may be redefined by an overall scale, but apart from this all the coefficients are significant. Apart from this freedom to make coordinate changes and scale  $F$ , these are the most general polynomials invariant under  $A$ , yielding a total of  $10-1 = 9$  parameters. To check that  $A$  acts on  $(M'')^{6,9}$  without fixed points, note that the fixed point set of  $A$  in the first  $\mathbb{P}^3$  consists of the curve  $(x_0, x_1, x_1, x_1)$  and two points of the form  $(0, 1, \omega, \omega^2)$  with  $\omega$  a nontrivial cube root of unity. The two isolated points do not satisfy the equation  $G = 0$ , for general values of the parameters, and so do not lie on  $(M'')^{6,9}$ . It remains to check the points of the form

$$(x_0, x_1, x_1, x_1) \times (y_0, y_1, y_1, y_1).$$

and it is easy to see that these points do not satisfy the equations for general values of the parameters. Thus the quotient is smooth and has Euler number  $-6$ . The Hodge numbers may be calculated individually by applying the Lefschetz hyperplane theorem; however, a counting of the number of ways in which the equations may be deformed works in this case and gives  $h^{2,1} = 9$ , so  $h^{1,1} = 6$ .

Next we turn to

$$(N')^{8,35} = \mathbb{P}^2 \left[ \begin{array}{cc} 3 & 0 \\ 1 & 3 \end{array} \right]_{-54}$$

and write  $(\xi_1, \xi_2, \xi_3)$  for the projective coordinates of the  $\mathbb{P}^2$  and, as before,  $(y_0, y_j)$  for the coordinates of the  $\mathbb{P}^3$ . We write the equations for this space

in the form

$$\begin{aligned}
 F &= f_0 x_0 y_0 + f_1 \sum_j x_j y_j + f_2 \sum_j x_j y_{j+1} + f_3 \sum_j x_{j+1} y_j \\
 &\quad + f_4 x_0 \sum_j y_j + f_5 \left( \sum_j x_j \right) y_0, \\
 H &= y_0^3 - y_1 y_2 y_3 + h_1 \sum_j y_j^3 + h_2 y_0 \sum_j y_j y_{j+1}.
 \end{aligned}$$

The polynomials  $F$  and  $H$  are as before but we take now

$$x_0 = \xi_1 \xi_2 \xi_3, \quad x_j = \xi_j^3, \quad j = 1, 2, 3. \tag{2.3}$$

Note that with this identification the  $x$ 's satisfy the singular cubic

$$G_0 = x_0^3 - x_1 x_2 x_3 = 0.$$

We have a symmetry  $A$  that acts cyclically as before and also a new symmetry,  $B$ , that acts on the  $\xi_j$  leaving the  $x$ 's and  $y$ 's unchanged:

$$\begin{aligned}
 A : \xi_j &\longrightarrow \xi_{j+1}, \quad (y_0, y_j) \longrightarrow (y_0, y_{j+1}), \\
 B : \xi_j &\longrightarrow \omega^j \xi_j.
 \end{aligned}$$

Note that one consequence of identifying the  $\xi$ 's under  $B$  is to render the map  $\xi \rightarrow x$  injective.

The symmetry  $A$  acts without fixed points, as before, but  $B$  has fixed points in its action on  $\xi$  which are  $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ . These lead, via  $F$ , to a linear constraint on the  $y$ 's which, in conjunction with the equation  $H = 0$ , leads to three fixed tori which are identified under the action of  $A$ . Since the fixed point set has Euler number zero the Euler number of the singular quotient is  $-6$  and the Euler number of the resolved quotient is also  $-6$  since the resolution replaces the singular torus by a torus multiplied pointwise by the  $A_2$  Eguchi–Hanson surface. The parameter count proceeds by noting that we now have  $2 + 5 = 7$  parameters in the equations and the  $A_2$  Eguchi–Hanson surface has two  $(1,1)$ -forms from which one may form two  $(2,1)$ -forms on the 3-fold by multiplying these by the holomorphic form of the torus. Thus we have  $h^{2,1} = 7 + 2 = 9$  and hence also  $h^{1,1} = 6$  as before.

In fact, it is easy to see from our discussion that a resolved quotient  $(M')^{6,9}$  deforms back smoothly to a Tian–Yau manifold  $(M'')^{6,9}$ . After taking the quotient by  $A \times B$ , the space is given by the equations  $F, G_0$  and  $H$ , and contains a curve of  $A_2$  singularities along the torus discussed above. To obtain  $(M')^{6,9}$ , we resolve this singularity; to obtain  $(M'')^{6,9}$ , we smooth the defining equation  $G_0$  to a smooth  $G$ . However, it is well known that the resolution and the deformation of a surface  $A_2$  singularity

are smooth deformation equivalent, and this also works for the family of singularities parametrized by the torus. Hence indeed,  $(M')^{6,9}$  can be deformed to  $(M'')^{6,9}$ .

For the third manifold

$$N^{2,83} = \mathbb{P}^2 \left[ \begin{matrix} 3 \\ 3 \end{matrix} \right]_{-162}$$

the construction is performed twice taking the equation to be

$$F = f_0 x_0 y_0 + f_1 \sum_j x_j y_j + f_2 \sum_j x_j y_{j+1} + f_3 \sum_j x_{j+1} y_j + f_4 x_0 \sum_j y_j + f_5 \left( \sum_j x_j \right) y_0,$$

where now

$$x_0 = \xi_1 \xi_2 \xi_3, \quad x_j = \xi_j^3, \quad j = 1, 2, 3 \quad \text{and} \quad y_0 = \eta_1 \eta_2 \eta_3, \quad y_j = \eta_j^3, \quad j = 1, 2, 3.$$

We now have symmetries  $A$  and  $B$  as before and a new symmetry  $C$ , for which it is convenient to take the generator to be

$$C : \eta_j \longrightarrow \omega^{-j} \eta_j.$$

The symmetry  $A$  acts without fixed points while  $B$  and  $C$  each have a fixed torus. The equation  $F$  is transverse as a function of the  $\xi$ 's and  $\eta$ 's for general values of the coefficients. A transverse  $F$  is obtained by taking, for example,  $f_0 = f_1 = 1$ ,  $f_2 = f_3 = c$  and  $f_4 = f_5 = 0$  providing  $c \neq 0, 1$ . The parameter count is again  $h^{2,1} = 5 + 2 + 2 = 9$  with 5 parameters coming from  $F$  and two each from the resolution of the two fixed tori. The same argument as before shows that  $M^{6,9}$  also deforms smoothly to  $(M')^{6,9}$ , and hence to  $(M'')^{6,9}$ .

We have shown that  $M$ ,  $M'$  and  $M''$  all have Hodge numbers  $(6, 9)$ , and that these three manifolds belong to the same irreducible family. From now on we will omit the primes since no confusion can arise and will choose at various times whichever presentation is convenient.

### 2.2 Some quotients of the bicubic

We now examine briefly two related quotients of the manifold  $N^{2,83}$ . First, there is

$$N^{2,29} = \mathbb{P}^2 \left[ \begin{matrix} 3 \\ 3 \end{matrix} \right] / A,$$

a free quotient with Euler number  $\chi = -54$ . Next, we also have

$$N^{2,11} = \frac{\mathbb{P}^2 \begin{bmatrix} 3 \\ \mathbb{P}^2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix}}{A \times D}$$

that we will need in the following, where we denote by  $D$  the diagonal subgroup of  $B \times C$  with generator

$$D : \xi_j \rightarrow \omega^j \xi_j, \quad \eta_k \rightarrow \omega^{-k} \eta_k.$$

There are a total of 12 independent polynomials invariant under  $A \times D$  that are given in the following table:

Number	Polynomial
6	$\sum_j \xi_j^2 \xi_{j\pm 1} \eta_{j+k}^2 \eta_{j+k\pm 1}$
3	$\sum_j \xi_j^3 \eta_{j+k}^3$
1	$\xi_1 \xi_2 \xi_3 \sum_j \eta_j^3$
1	$\left(\sum_j \xi_j^3\right) \eta_1 \eta_2 \eta_3$
1	$\xi_1 \xi_2 \xi_3 \eta_1 \eta_2 \eta_3$

It is easy to see that  $A \times D$  acts without fixed points for generic coefficients of the defining polynomial. Since there are 12 coefficients and an overall scale is irrelevant we have  $h^{21} = 11$  and since the group action is fixed point free we have  $\chi = -18$  so  $h^{11} = 2$ .

### 2.3 Quotients of the split bicubic

Having addressed the three avatars of the Tian–Yau manifold that were central to the discussion of three generation models in early investigations, we turn now to the split bicubic  $X^{19,19}$  and its quotients

$$X^{7,7} = \frac{\mathbb{P}^1 \begin{bmatrix} 1 & 1 \\ \mathbb{P}^2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}}{A}{}^{7,7} \quad \text{and} \quad X^{3,3} = \frac{\mathbb{P}^1 \begin{bmatrix} 1 & 1 \\ \mathbb{P}^2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}}{A \times D}{}^{3,3}.$$

For these manifolds take coordinates  $t_r$ ,  $r = 1, 2$  on the  $\mathbb{P}^1$ , and coordinates  $\xi_j$  and  $\eta_j$  on the two  $\mathbb{P}^2$ 's as previously. The generator of  $A$  can be chosen

to act cyclically as before leaving the  $t$ 's invariant [23]:

$$A : \xi_j \rightarrow \xi_{j+1}, \quad \eta_j \rightarrow \eta_{j+1}, \quad t_r \rightarrow t_r.$$

The group  $D$  is the diagonal subgroup of  $B \times C$  considered above and which we also take to leave the coordinates of the  $\mathbb{P}^1$  invariant:

$$D : \xi_j \rightarrow \omega^j \xi_j, \quad \eta_j \rightarrow \omega^{-j} \eta_j, \quad t_r \rightarrow t_r.$$

Under  $A \times D$  the only cubic monomials in the  $\xi_j$  that are invariant are  $\sum_j \xi_j^3$  and  $\xi_1 \xi_2 \xi_3$  and similarly for the  $\eta_k$ . With a certain choice of  $t$  coordinate we may write the polynomials for the  $A \times D$  quotient in the form

$$\begin{aligned} F^1 &= t_1 \left( \sum_{j=1}^3 \xi_j^3 - 3a \xi_1 \xi_2 \xi_3 \right) + 3ct_2 \xi_1 \xi_2 \xi_3 \\ F^2 &= 3ct_1 \eta_1 \eta_2 \eta_3 + t_2 \left( \sum_{j=1}^3 \eta_j^3 - 3b \eta_1 \eta_2 \eta_3 \right) \end{aligned} \tag{2.4}$$

and the coefficients  $a, b, c$  give the correct count for  $h^{2,1}(X^{3,3}) = 3$ . Under  $A$  the fixed point set, in the embedding space, is of the form

$$(t_1, t_2) \times (1, \alpha, \alpha^2) \times (1, \beta, \beta^2); \quad \alpha^3 = \beta^3 = 1.$$

and these points do not satisfy the equations unless  $c^2 = (a - 1)(b - 1)$ . The fixed point set under  $D$  is

$$(t_1, t_2) \times \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \times \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

and these points do not satisfy the equations for  $(t_1, t_2)$  a point in  $\mathbb{P}^1$ .

The equations that we have introduced in (2.4) are different from the original defining polynomials (1.4). The action of  $D$  is also different since the action on the original polynomials extends to a nontrivial action on the coordinates of the  $\mathbb{P}^1$ :

$$D : \xi_j \rightarrow \omega^j \xi_j, \quad \eta_j \rightarrow \omega^{-j} \eta_j, \quad (t_1, t_2) \rightarrow (t_1, \omega t_2).$$

Let us denote by  $Y^{3,3}$  the quotient of (1.4) by this action of  $A \times D$ . We assume, for the purposes of this article, that  $X^{3,3}$  and  $Y^{3,3}$  are holomorphically equivalent manifolds. We follow [23] in adopting the representation (2.4). It is known that the Hodge numbers are the same for both quotients as are the Yukawa couplings over  $\mathbb{R}$  and the action of the second Chern class on  $H^2$ . That is, for both  $X^{3,3}$  and  $Y^{3,3}$ , it is known that there is a basis

$\{\tau_1, \tau_2, \phi\}$  for  $H^2$  over  $\mathbb{R}$  such that

$$\begin{aligned} \tau_i^3 &= 0 & \phi \tau_i &= 3 \tau_i^2 ; & i &= 1, 2 ; \\ \tau_1^2 \tau_2 &= \tau_1 \tau_2^2 = 3 , & \phi^2 &= 0 ; \\ c_2 &= 12(\tau_1^2 + \tau_2^2) . \end{aligned} \tag{2.5}$$

Rather than repeat our analysis for the two cases we shall focus on the representation given in (2.4). The discussion of the transgression that we will give in §3.4 requires only certain features of the heterotic bundle and does not require the precise form of the polynomials or of the  $D$ -action. Our argument therefore applies to the manifold  $Y^{3,3}$  in the event that the manifolds are different.

If the equations for  $X^{19,19}$  are merely to be invariant under  $A$ , then we are allowed the terms  $\sum_j \xi_j^2 \xi_{j\pm 1}$  and  $\sum_j \eta_j^2 \eta_{j\pm 1}$ . We are also now free to make coordinate redefinitions  $\xi_j \rightarrow \xi_j + r \xi_{j+1} + s \xi_{j-1}$  and similarly for  $\eta$ . We may use this freedom to bring the equations to the form

$$\begin{aligned} F^1 &= t_1 \left( \sum_{j=1}^3 \xi_j^3 - 3a \xi_1 \xi_2 \xi_3 + d_+ \sum_{j=1}^3 \xi_j^2 \xi_{j+1} + d_- \sum_{j=1}^3 \xi_j^2 \xi_{j-1} \right) + 3ct_2 \xi_1 \xi_2 \xi_3 \\ F^2 &= 3ct_1 \eta_1 \eta_2 \eta_3 + t_2 \left( \sum_{j=1}^3 \eta_j^3 - 3b \eta_1 \eta_2 \eta_3 + e_+ \sum_{j=1}^3 \eta_j^2 \eta_{j+1} + e_- \sum_{j=1}^3 \eta_j^2 \eta_{j-1} \right) \end{aligned} \tag{2.8}$$

which exhibit the seven complex structure parameters corresponding to  $h^{2,1}(X^{7,7}) = 7$ .

### 2.4 Conifold transitions between the manifolds

We have introduced two sequences of manifolds: (i) the Tian–Yau sequence

$$N^{2,83} \rightarrow N^{2,29} \rightarrow N^{2,11},$$

of interest since the early days of string phenomenology and (ii) the split-bicubic sequence

$$X^{19,19} \rightarrow X^{7,7} \rightarrow X^{3,3},$$

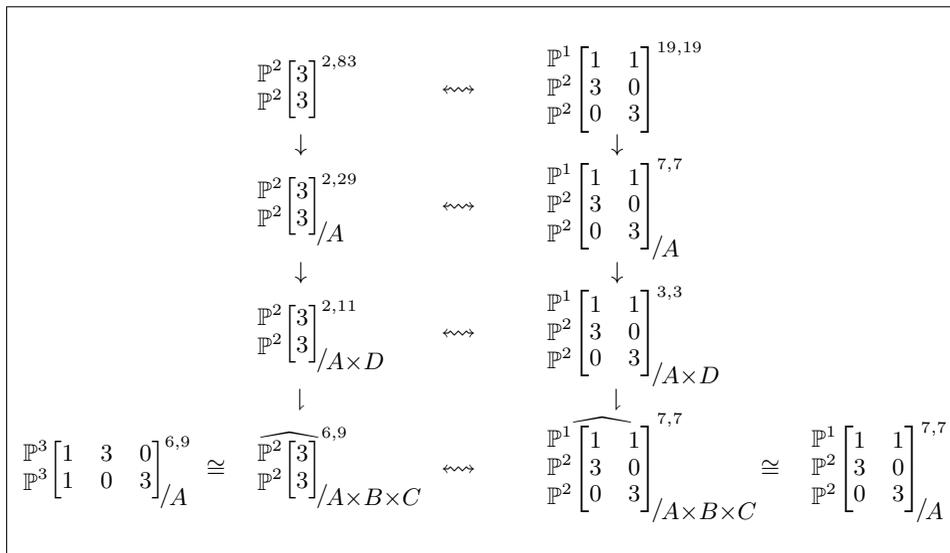
recently of interest in constructing the MSSM. In this section, we show the remarkable relation that the two are related by a conifold transition, and moreover, in the next section, how bundles on one may be transferred to another by the process of transgression. Table 2 below shows the intricate web of relations between the various manifolds we have been examining. These show the birational relations between the various quotients of the bicubic, split bicubic and the Tian–Yau manifold. The vertical arrows denote quotients by freely acting groups while the harpoons above the last row denote the process of taking a quotient by  $C$  and then resolving the singularities. The relation expressed by the  $\rightsquigarrow$  is that of a conifold transition [42].

Let us first briefly recall how a conifold transition works in the context of CICY's. This is the “splitting” process of which a good account may be found in [42]. Let us take as an example the top row of table 2. To the right is the split bicubic  $X^{19,19}$ . As in §2.3, we can write its defining equations as

$$t_1U(\xi) + t_2W(\xi) = 0, \quad t_1Z(\eta) + t_2V(\eta) = 0, \quad (2.9)$$

where as before  $t_{1,2}$  are the coordinates of the  $\mathbb{P}^1$ ,  $\xi_j, \eta_j$  are, respectively, the coordinates of the two  $\mathbb{P}^2$ 's, and  $U, V, W, Z$  are cubic polynomials. We can regard (2.9) as a matrix equation in  $t_{1,2}$ . Now, because  $t_{1,2}$  are projective coordinates on  $\mathbb{P}^1$  they cannot vanish simultaneously, and hence (2.9) can

Table 2: Birational relations between quotients of the bicubic, split bicubic and the Tian–Yau manifold; hats denote resolutions of quotients.



only hold if

$$F_0 = \det \begin{vmatrix} U(\xi) & W(\xi) \\ Z(\eta) & V(\eta) \end{vmatrix} = U(\xi)V(\eta) - W(\xi)Z(\eta) = 0. \tag{2.10}$$

This is, a bicubic equation in the  $\xi$  and  $\eta$  coordinates. In fact it is a singular limit of the manifold  $N^{2,83}$ . The limit is singular owing to the fact that all the derivatives of  $F_0$  vanish precisely at the points where  $U, V, W, Z$  are all zero and this happens, in this case, in  $3^4$  points. We have seen that a point  $(t, \xi, \eta)$  that satisfies (2.9) must be such that  $(\xi, \eta)$  satisfy (2.10). Conversely suppose that  $(\xi, \eta)$  satisfy (2.10) and that for these values the cubics  $U, V, W, Z$  are not all zero. The equations (2.9) will then determine a unique ratio  $t_1/t_2$  hence a unique point  $t \in \mathbb{P}^1$ . If, however, all four of the cubics vanish then the equations (2.9) are satisfied for all values of  $t \in \mathbb{P}^1$ . For suitable cubics the split manifold defined by (2.9) is smooth while the conifold defined by  $F_0 = 0$  is singular at a certain number of nodes. The split manifold projects down onto the conifold such that a unique point projects to every nonsingular point of the conifold but an entire  $\mathbb{P}^1$  of the split manifold projects down onto each node. Alternatively we pass from the conifold to the split manifold by blowing up each node to a  $\mathbb{P}^1$ .

The first three rows of table 2 are simply conifold transitions between  $N^{2,83}$  and  $X^{19,19}$  as well as their quotients. The relation between the extreme ends of the last row is less obvious and corresponds to the relation

$$\begin{matrix} \mathbb{P}^3 [1 & 3 & 0]^{6,9} \\ \mathbb{P}^3 [1 & 0 & 3]_{/A} \end{matrix} \rightsquigarrow \begin{matrix} \mathbb{P}^1 [1 & 1 & 0 & 0]^{7,7} \\ \mathbb{P}^3 [1 & 0 & 3 & 0] \\ \mathbb{P}^3 [0 & 1 & 0 & 3]_{/A} \end{matrix} \cong \begin{matrix} \mathbb{P}^1 [1 & 1]^{7,7} \\ \mathbb{P}^2 [3 & 0] \\ \mathbb{P}^2 [0 & 3]_{/A} \end{matrix} \tag{2.11}$$

which we explain in the following. The relation is true whether or not we take the  $A$ -quotient, so we shall first explain the relation before taking the quotient. The isomorphism between the second and third families in the above relation may be surprising; it is this that we will now explain.

### 2.4.1 Three avatars of the split bicubic

Consider the family of complete intersection Calabi–Yau manifolds

$$\begin{matrix} & & \tilde{G}^0 & \tilde{H}^0 & G^1 & G^2 & G^3 & H^1 & H^2 & H^3 \\ t & \mathbb{P}^1 & [1 & 1 & 0 & 0 & 0 & 0 & 0 & 0] \\ \xi & \mathbb{P}^2 & [0 & 0 & 1 & 1 & 1 & 0 & 0 & 0] \\ \eta & \mathbb{P}^2 & [0 & 0 & 0 & 0 & 0 & 1 & 1 & 1] \\ x & \mathbb{P}^3 & [1 & 0 & 1 & 1 & 1 & 0 & 0 & 0] \\ y & \mathbb{P}^3 & [0 & 1 & 0 & 0 & 0 & 1 & 1 & 1] \end{matrix} = X, \tag{2.12}$$

where we have indicated the coordinates of the projective spaces and have also named the polynomials whose degrees correspond to the columns of the

matrix. We will soon see that calling this manifold  $X$  is justified since we shall show that  $X/A$  is isomorphic to both the second and third CICY's in (2.11). This is a matter of showing, first, that we obtain maps between the various manifolds by eliminating variables from the defining equations and, secondly, that these maps are isomorphisms rather than being merely birational equivalences.

Consider the three equations  $G^j$ . These are bilinear in the coordinates  $\xi$  and  $x$  of the first  $\mathbb{P}^2$  and first  $\mathbb{P}^3$

$$G^j = G^j_{k\alpha} \xi^k x^\alpha = G(x)^j_k \xi^k ; G(x)^j_k = G^j_{k\alpha} x^\alpha ,$$

where we are using a summation convention and  $j, k = 1, 2, 3$  while  $\alpha = 0, \dots, 3$ . Now, the equations  $G(x)^j_k \xi^k = 0$  are three equations in the three  $\xi^k$  and these coordinates are not all zero. So we have

$$G(x) = \det \left( G(x)^j_k \right) = 0.$$

We see that  $G(x)$  is the determinant of a  $3 \times 3$  matrix of linear forms and so is a cubic. It is also a classical fact that any cubic in variables  $x^\alpha$ ,  $\alpha = 0, \dots, 3$  can be written as the determinant of a matrix of linear forms (for a review of this and related facts see [43]; the original reference is [44]). Analogous considerations apply also to the equations  $H^j$  leading to a determinant  $H(y)$ .

We have now reduced the triple  $G^j$  (and, respectively,  $H^j$ ) into a single equation  $G$  (and, respectively,  $H$ ). It thus remains to show that

$$\begin{array}{c} \tilde{G}^0 \tilde{H}^0 G H \\ t \quad \mathbb{P}^1 \left[ \begin{array}{cccc} 1 & 1 & 0 & 0 \\ x & \mathbb{P}^3 \left[ \begin{array}{ccc} 1 & 0 & 3 \\ y & \mathbb{P}^3 \left[ \begin{array}{ccc} 0 & 1 & 0 \end{array} \right] \end{array} \right] \end{array} \right] \end{array} \quad (2.13)$$

is another avatar of  $X$ . This is done by simply remarking that the rank of the matrix  $G(x)^j_k$  is at most 2 owing to the fact that it has zero determinant. For a general point  $x$ , the rank is 2 and in fact this is true for all  $x$  since to have rank 1 would impose 4 independent linear relations on  $x \in \mathbb{P}^3$  which cannot be satisfied in a 3-dimensional space. This being so there is a one-to-one relation between allowed points  $\xi$  and allowed points  $x$  and the manifold that results from eliminating  $\xi$  and  $\eta$  in this way is smooth. This establishes that the CICY (2.13) is isomorphic to  $X$ .

Finally, we show that  $X$  is indeed the familiar split bicubic,  $X^{19,19}$ . This is done by eliminating variables in another way. The four equations  $\tilde{G}^0 = 0$

and  $G^j = 0$  can be written

$$\tilde{G}(t, \xi)^{\alpha\beta} x^\beta = 0 \quad \text{where} \quad \tilde{G}(\xi)^j{}_\beta = G^j{}_{k\beta} \xi^k.$$

Thus we have a set of four equations for the four coordinates  $x^\beta$ , which cannot all be zero. It follows that

$$F^1 = \det(\tilde{G}(t, \xi)^{\alpha\beta}) = 0.$$

Since  $\tilde{G}(t, \xi)^0{}_\beta$  is independent of  $\xi$  and the  $\tilde{G}(t, \xi)^j{}_\beta$  are independent of  $t$  we see that  $F^1$  has bidegree  $(1, 3)$  in  $t$  and  $\xi$ . Analogous considerations apply also to the matrix  $\tilde{H}(t, \eta)^{\alpha\beta}$  and to

$$F^2 = \det(\tilde{H}(t, \eta)^{\alpha\beta}).$$

One checks, analogously to the previous case, that the matrices  $\tilde{G}(t, \xi)^{\alpha\beta}$  and  $\tilde{H}(t, \eta)^{\alpha\beta}$  always have rank 3. We have now shown that indeed the manifold

$$\begin{array}{c} F^1 \ F^2 \\ t \ \mathbb{P}^1 \left[ \begin{array}{cc} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{array} \right] \\ \xi \ \mathbb{P}^2 \\ \eta \ \mathbb{P}^2 \end{array} = X.$$

### 2.4.2 The conifold transition between $M$ and $X^{7,7}$

We have now seen three equivalent representations of  $X^{19,19}$ . Next, we proceed to show the relation proposed in (2.11). This is seen easily since the determinants  $G$  and  $H$  become the cubics of the covering space of the Tian–Yau manifold

$$\begin{array}{c} F \ G \ H \\ x \ \mathbb{P}^3 \left[ \begin{array}{ccc} 1 & 3 & 0 \\ 1 & 0 & 3 \end{array} \right] \\ y \ \mathbb{P}^3 \end{array}$$

On making the conifold transition it is possible to approach the conifold from the Tian–Yau side by singularizing the bilinear equation  $F$  while leaving  $G$  and  $H$  fixed. The coordinates  $t$  can be chosen such that the equations  $\tilde{G}^0$  and  $\tilde{H}^0$  take the form

$$\begin{aligned} \tilde{G}^0 &= t_1 \left( \sum_j x_j - 3ax_0 \right) + 3ct_2x_0 \\ \tilde{H}^0 &= 3ct_1y_0 + t_2 \left( \sum_j y_j - 3by_0 \right). \end{aligned}$$

Again we have two equations in the coordinates  $(t_1, t_2)$ , which cannot both be zero, so the determinant must vanish

$$F_0 = \left( \sum_j x_j - 3ax_0 \right) \left( \sum_j y_j - 3by_0 \right) - 9c^2x_0y_0 = 0 \tag{2.14}$$

and the polynomial  $F$  is a deformation of  $F_0$ . Comparing with (2.4) we see that the parameter count is that there are two parameters in each of  $G$  and  $H$  and three in  $F_0$ . This accounts for the seven complex structure parameters of the split bicubic.

Consider now the manifold

$$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \left[ \begin{matrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{matrix} \right]^{3,3} / A \times D$$

One might at first seek to relate this to the Tian–Yau manifold by taking a quotient of (2.11) by  $D$  but this cannot be done, owing to the fact that  $D$  does not act on the Tian–Yau manifold. Another reason why this fails is that it is not possible to find a nonsingular CICY of the form (2.12) that is invariant under  $A \times D$ .

We have, instead, the splitting

$$\begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \left[ \begin{matrix} 3 \\ 3 \end{matrix} \right]^{2,11} / A \times D \rightsquigarrow \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \left[ \begin{matrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{matrix} \right]^{3,3} / A \times D \tag{2.15}$$

as in table 2 and we have seen previously that the group  $A \times D$  acts freely on both manifolds. We can take the quotient by  $C$  and then resolve. This gives us the Tian–Yau manifold on the left but the relation that we find is

$$\begin{matrix} \mathbb{P}^3 \\ \mathbb{P}^3 \end{matrix} \left[ \begin{matrix} 1 & 3 & 0 \\ 1 & 0 & 3 \end{matrix} \right]^{6,9} / A \rightsquigarrow \begin{matrix} \widehat{\mathbb{P}^1} \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \left[ \begin{matrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{matrix} \right]^{7,7} / A \times B \times C = \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \left[ \begin{matrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{matrix} \right]^{7,7} / A$$

where it should be noted that  $B$  and  $C$  have fixed points whose resolution is implicit in this identity. We also know from the previous subsection that

$$\begin{matrix} \mathbb{P}^3 \\ \mathbb{P}^3 \end{matrix} \left[ \begin{matrix} 1 & 3 & 0 \\ 1 & 0 & 3 \end{matrix} \right]^{6,9} / A \rightsquigarrow \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \left[ \begin{matrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{matrix} \right]^{7,7} / A .$$

The upshot is that we again relate the Tian–Yau manifold to  $X^{7,7}$  rather than to  $X^{3,3}$ .

### 3 Transgression of vector bundles

Given the close relation between the Tian–Yau manifold and the quotient of the split bicubic, it is natural to ask how the tangent bundle of the

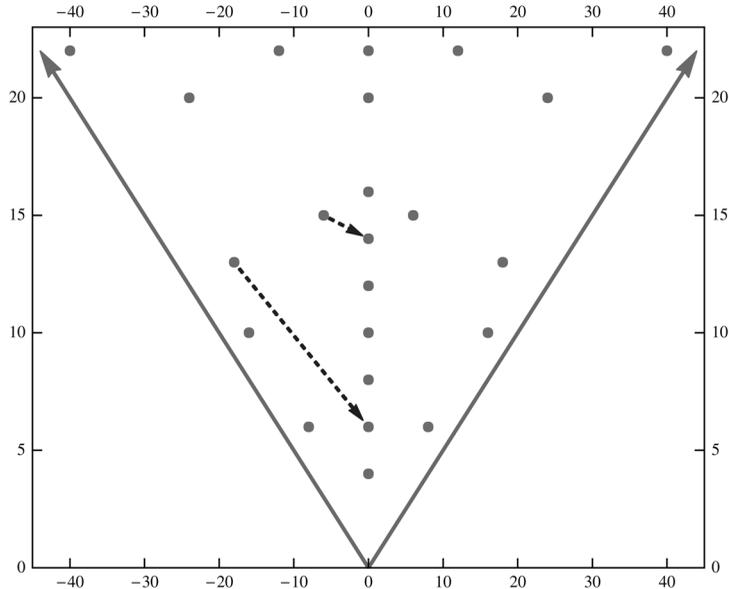


Figure 4: The manifolds with  $h^{11} + h^{21} \leq 22$  showing the two transgressions that we propose here between the Tian–Yau manifold and  $X^{7,7}$ , and between  $N^{2,11}$  and  $X^{3,3}$ .

Tian–Yau manifold is related to the vector bundles of the heterotic models based on the split bicubic. We will describe here a process of transferring a bundle between two manifolds that are related by a conifold transition. Every manifold comes equipped with a tangent bundle,  $\mathcal{T}$ , and a conifold transition induces a change in the tangent bundles of the manifolds. This change is rather drastic since  $c_3(\mathcal{T})$  is the Euler number of the manifold and, as the result of a conifold transition, this changes by twice the number of nodes of the conifold. This jump is the inevitable consequence of the fact that the tangent bundle is singular where the manifold fails to have a well-defined tangent space. It was pointed out to one of the present authors, by E. Witten, many years ago, that other bundles are better behaved and can be expected to deform smoothly across the transition. This fits in well with heterotic models since there is no reason, in general, to prefer the tangent bundle over others. We will here apply this idea to the three-generation models that we have been discussing. As a matter of language, we prefer to refer to the process of transferring bundles across a conifold transition as a transgression<sup>4</sup> of bundles. In figure 4 we illustrate two transgressions that we shall propose here.

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<sup>4</sup>transgress from L. *transgredior* = *trans* + *egredior*, to go across, to cross over.

### 3.1 Local analysis of transgressions

In coordinates  $w^j = (u, v, w, z)$ , a conifold singularity at the origin in  $\mathbb{C}^4$  is described locally by the vanishing of the equation

$$f_0 = uv - wz.$$

The conifold is smoothed to a manifold, that we denote by  $M_s$ , by deforming  $f_0$  to  $f_s = f_0 + sg$ , where  $g$  is any function such that the derivatives of  $f_s$  can no longer all vanish simultaneously. The split manifold,  $\check{M}$ , is realized in  $\mathbb{C}^4 \times \mathbb{P}^1$  by the equations

$$\begin{aligned} t_1u + t_2w &= 0, \\ t_1z + t_2v &= 0. \end{aligned} \tag{3.1}$$

The sections of the tangent bundle  $\mathcal{T}_s$  of  $M_s$  consist of the set of vector fields  $V^j$  that are tangent to  $M_s$ ; this condition can be written as

$$V^j \frac{\partial f_s}{\partial w^j} = 0. \tag{3.2}$$

For  $s \neq 0$ , this condition ensures that  $\mathcal{T}_s$  has rank 3. However, at the conifold point  $w^j = 0$  all derivatives of  $f_0$  vanish, and the dimension of the space of solutions to (3.2) jumps from 3 to 4. It is this behaviour that we understand as characterizing the singularity of the tangent bundle at the conifold.

We can however first deform  $\mathcal{T}_s$  by replacing (3.2) by the condition

$$V^j \left( \frac{\partial f_s}{\partial w^j} + h_j \right) = 0, \tag{3.3}$$

where the  $h_j$  are functions that are not all zero at  $w^j = 0$  and are not the derivatives of a function  $h$ , the latter condition ensuring that we are deforming  $\mathcal{T}_s$  while keeping the underlying space  $M_s$  fixed. Then, with the deformed bundle, we may proceed to the limit  $f_s \rightarrow f_0$  and the space of allowed sections remains 3-dimensional even at the conifold point. The key point is that we have divorced the bundle from being the tangent bundle. We would expect “most” bundles to remain nonsingular at the conifold in the same way as the deformation of the tangent bundle.

We may also think of Equations (3.3) as defining sections of a rank 3 vector bundle on  $\check{M}$ . The sections thus defined are independent of position on the  $\mathbb{P}^1$  of the resolution, and hence the obtained bundle is trivial as a bundle over the  $\mathbb{P}^1$ . This bundle is therefore not a deformation of the tangent bundle of  $\check{M}$ . It is clear that the bundle we have defined on  $\check{M}$  deforms back to a bundle on  $M_s$ .

### 3.2 Deformation of the tangent bundle for the quintic 3-fold

We wish to pass from a local to a global description of transgression for compact Calabi–Yau manifolds. Our interest is primarily with the deformations of the tangent bundle of Tian–Yau manifold and its close relatives and the relation of these to bundles on the quotients of the split bicubic. These exhibit certain interesting complications so before examining this case, we recall the corresponding situation for the quintic 3-fold  $\mathbb{P}^4[5]$ . We will first give a standard “explicit” construction [45] and then a more general account of a type that is more favoured by mathematicians. This has the advantage of being applicable in situations where the explicit construction fails and of generalizing in a straightforward way to other bundles. We will repeat this analysis in the following subsection for the Tian–Yau manifold which is a case for which the naive construction fails.

To save writing, we denote  $\mathbb{P}^4[5]$  by  $Q$  and take the manifold to be defined by a quintic  $p(x)$  in the homogeneous coordinates  $x_j$ ,  $j = 1, \dots, 5$  of the  $\mathbb{P}^4$ . The tangent bundle,  $\mathcal{T}_Q$ , is the set of pairs  $(x, V)$  with  $x$  a point of  $Q$  and  $V = V^j \frac{\partial}{\partial x^j}$  a vector that now satisfies

$$V^j \frac{\partial p}{\partial x^j} = \mu p \tag{3.4}$$

for some  $\mu$ , and is, additionally, subject to the identification

$$V^j + \rho x^j \simeq V^j \tag{3.5}$$

for all  $\rho$ . These conditions are consistent owing to the fact that  $p$  is a homogeneous quintic and therefore satisfies the Euler relation  $x^j \frac{\partial p}{\partial x^j} = 5p$ . The two conditions reduce the dimension of the space of allowed vectors, at a given point, from 5 to the correct value of 3.

The tangent bundle can be deformed by deforming (3.4) by quartic polynomials  $r_j$  such that

$$V^j \left( \frac{\partial p}{\partial x^j} + r_j \right) = \mu p. \tag{3.6}$$

In order to maintain the identification (3.5), we need to require that

$$x^j r_j = 0. \tag{3.7}$$

We could have imposed the weaker condition  $x^j r_j = \mu' p$ ; however, an adjustment

$$r_j \rightarrow r_j - \frac{\mu'}{5} \frac{\partial p}{\partial x^j}$$

can be made so as to enforce (3.7). This constraint also prevents the  $r_j$  from being the derivatives  $\frac{\partial r}{\partial x^j}$  of a quintic  $r$ ; such a deformation would represent a deformation of  $Q$ , rather than that of  $\mathcal{T}_Q$  on a fixed  $Q$ .

This construction can be used to estimate the dimension of  $H^1(Q, \text{End}Q) = H^1(Q, \mathcal{T}_Q \otimes \mathcal{T}_Q^*)$ , the space of first order deformations of  $\mathcal{T}_Q$ . Having come this far we cannot resist completing the calculation. Each  $r_j$  is a homogeneous polynomial of degree 4 in five variables. In these, there are a total of  $5 \binom{4+5-1}{4} = 350$  parameters. The constraint (3.7), which, being of degree 5 in five variables, imposes  $\binom{5+5-1}{5} = 126$  conditions. We conclude that the number of deformations to  $\mathcal{T}_Q$  is at least  $350 - 126 = 224$ . In this case, in fact  $h^1(Q, \mathcal{T}_Q \otimes \mathcal{T}_Q^*) = 224$ ; for a recent discussion of the deformation of the tangent bundle for the quintic, see [46, 47]. The quintic is somewhat exceptional; it can happen that the tangent bundle of a manifold has deformations that are not accounted for in this way.

We now take a more abstract approach, in order to consider deformations of bundles other than the tangent bundle, in particular, the direct sum of the tangent bundle and trivial bundles. The adjunction formula [48] states that for a projective variety  $Y$  embedded in  $\mathbb{P}^n$ , the tangent bundle  $\mathcal{T}_Y$  and the normal bundle  $\mathcal{N}_Y$  fit into the short-exact sequence

$$0 \rightarrow \mathcal{T}_Y \rightarrow \mathcal{T}_{\mathbb{P}^n}|_Y \rightarrow \mathcal{N}_Y \rightarrow 0, \tag{3.8}$$

where  $\mathcal{T}_{\mathbb{P}^n}|_Y$  is the tangent bundle of  $\mathbb{P}^n$  restricted to  $Y$ . Next, the Euler sequence on  $\mathbb{P}^n$ , restricted to  $Y$ , reads

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_Y(1)^{\oplus n+1} \rightarrow \mathcal{T}_{\mathbb{P}^n}|_Y \rightarrow 0,$$

and thus, composing the right-hand maps in the two sequences, there is a third sequence [49]

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{O}_Y(1)^{\oplus n+1} \xrightarrow{f} \mathcal{N}_Y \rightarrow 0,$$

where  $\mathcal{F}$  is defined by  $\mathcal{F} = \ker(f)$ , in order to make the sequence exact. The map  $f$  is the analogue of (3.7). For the quintic  $Q$ , we get

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{O}_Q(1)^{\oplus 5} \xrightarrow{f} \mathcal{O}_Q(5) \rightarrow 0, \quad f = \left( \frac{\partial p}{\partial x^j} \right), \tag{3.9}$$

with  $f$  now explicitly written as a specific  $5 \times 1$  matrix of degree 4 homogeneous polynomials, the partial derivatives of  $p$ , mapping from  $\mathcal{O}_Q(1)^{\oplus 5}$  to  $\mathcal{O}_Q(5)$ .

Fitting (3.8) and (3.9) together yields the following intertwined diagram of short exact sequences:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \rightarrow & \mathcal{T}_Q & \rightarrow & \mathcal{T}_{\mathbb{P}^4}|_Q & \longrightarrow & \mathcal{O}_Q(5) \rightarrow 0 \\
 & & \uparrow & & \uparrow & & \parallel \\
 0 & \rightarrow & \mathcal{F} & \rightarrow & \mathcal{O}_Q(1)^{\oplus 5} & \xrightarrow{f} & \mathcal{O}_Q(5) \rightarrow 0 \\
 & & \uparrow & & \uparrow & & \\
 & & \mathcal{O}_Q & = & \mathcal{O}_Q & & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 
 \end{array} \tag{3.10}$$

The important fact is that the lowest term in the two vertical sequences is the trivial bundle  $\mathcal{O}_Q$  on the quintic.

From (3.10) we see that the bundle  $\mathcal{F}$  is an extension of  $\mathcal{T}_Q$  by  $\mathcal{O}_Q$ , and hence is a deformation of the direct sum  $\mathcal{T}_Q \oplus \mathcal{O}_Q$  (as explained in [49],  $\mathcal{F}$  is a nonsplit extension). Deformations of  $\mathcal{F}$  are obtained by deforming the map  $f$ , given by the partial derivatives  $(\frac{\partial p}{\partial x^j})$ , to a map  $(\frac{\partial p}{\partial x^j} + r_j)$ , for some quartics  $r_j$  as in (3.6). Imposing (3.7) corresponds, in this language, to the statement that the deformed  $f$  maps the distinguished sub-bundle  $\mathcal{O}_Q$  of  $\mathcal{O}_Q(1)^{\oplus 5}$  to zero, and thus we obtain a deformation of  $\mathcal{T}_Q$ . If (3.7) fails, then we obtain a deformation of  $\mathcal{F}$ , and hence of  $\mathcal{T}_Q \oplus \mathcal{O}_Q$ .

### 3.3 Deforming the tangent bundle of the Tian–Yau manifold

Let us move on to the manifold of our principal interest here. It is instructive to first run through the explicit construction of the deformation of the tangent bundle to see how this fails.

To avoid possible complications related to the resolution of the fixed point sets of  $B \times C$  let us focus on  $M$ , presented as

$$\begin{array}{c}
 F \quad G \quad H \\
 x \quad \mathbb{P}^3 \begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix} \\
 y \quad \mathbb{P}^3 \begin{bmatrix} 1 & 0 & 3 \end{bmatrix} /_A
 \end{array} = M$$

There are now three defining equations, which we will also denote by  $p^\alpha = (F, G, H)$ . We can think of the tangent vector as a differential operator

$$V = X + Y; \quad X = X^a \frac{\partial}{\partial x^a}, \quad Y = Y^b \frac{\partial}{\partial y^b},$$

where the indices  $a, b = 0, \dots, 3$  run through the two  $\mathbb{P}^3$ 's. We will also, as previously, have need of indices  $j, k$  that run over the range  $1, 2, 3$ , and which are understood to take values modulo 3.

The analogue to (3.6), requiring  $V$  to be tangent to  $M$ , is the condition

$$V(p^\alpha) = m^\alpha_\beta p^\beta, \tag{3.11}$$

where now  $m^\alpha_\beta$  is, in general, a matrix of polynomials but which in this case, owing to the degrees of the polynomials, is a diagonal matrix of constants. The Euler relations now read

$$x^a \frac{\partial p^\alpha}{\partial x^a} = \text{deg}_x(p^\alpha) p^\alpha, \quad y^b \frac{\partial p^\alpha}{\partial y^b} = \text{deg}_y(p^\alpha) p^\alpha,$$

where here  $\text{deg}_x(p^\alpha)$  denotes the degree of  $p^\alpha$  as a function of  $x$  and similarly for  $\text{deg}_y(p^\alpha)$ . We also identify

$$(X^a + \rho x^a, Y^b + \sigma y^b) \simeq (X^a, Y^b). \tag{3.12}$$

the tangent bundle  $\mathcal{T}_M$  is now deformed by requiring that  $V = X + Y$  satisfy

$$X^a \left( \frac{\partial p^\alpha}{\partial x^a} + r^\alpha_a \right) + Y^b \left( \frac{\partial p^\alpha}{\partial y^b} + s^\alpha_b \right) = m^\alpha_\beta p^\beta, \tag{3.13}$$

where the quantities  $r^\alpha_a$  and  $s^\alpha_b$  are matrices of polynomials of the same degrees as  $\frac{\partial p^\alpha}{\partial x^a}$  and  $\frac{\partial p^\alpha}{\partial y^b}$ , respectively, while  $m^\alpha_\beta$  is a matrix of constants. In order to maintain the identification (3.12), we write

$$R^\alpha_a = \frac{\partial p^\alpha}{\partial x^a} + r^\alpha_a \quad \text{and} \quad S^\alpha_b = \frac{\partial p^\alpha}{\partial y^b} + s^\alpha_b$$

and require

$$x^a R^\alpha_a = n^\alpha_\beta p^\beta, \quad y^b S^\alpha_b = \tilde{n}^\alpha_\beta p^\beta \tag{3.14}$$

for some constant matrices  $n^\alpha_\beta, \tilde{n}^\alpha_\beta$ .

Note first that, since the second defining equation,  $p^2 = G$ , has zero degree in  $y^a$  and similarly,  $p^3 = H$  has zero degree in  $x^a$ , the quantities  $S^2_b$  and  $R^3_a$  both vanish. A little thought then shows that the most general form for  $r^2_a$  that is invariant under  $A$  and satisfies (3.14) has the form

$$\begin{aligned} r^2_a dx^a &= \sum_j \left( \kappa x_0 + \lambda x_j + \sum_{\pm} \mu_{\pm} x_{j\pm 1} \right) (x_j dx_0 - x_0 dx_j) \\ &\quad + \sum_j \sum_{\pm} \nu_{\pm} (x_{j\pm 1}^2 - x_j x_{j\mp 1}) dx_j. \end{aligned}$$

and contains six parameters. A similar expression holds for  $s^3_b$ .

A problem, however, arises in relation to the deformation corresponding to  $p^1 = F$ . This polynomial is bilinear so the conditions (3.14), applied to this case, force the relations

$$R^1_a = u \frac{\partial F}{\partial x^a} \quad \text{and} \quad S^1_b = v \frac{\partial F}{\partial y^b}$$

for some constants  $u$  and  $v$ . Our previous observation concerning transgression of deformation of tangent bundles, as in (3.3), was that once the tangent bundle has been deformed sufficiently generally, then the analogues of our quantities  $R^\alpha_a$  and  $S^\alpha_b$  are all nonzero at the conifold and prevent the bundle from becoming singular. Here this fails since at the conifold (as in (2.14))  $F$  is necessarily of the form  $F_0 = UV - WZ$ , for some polynomials  $U, V, W, Z$ , that vanish at the nodes. Thus all the derivatives of  $F$  vanish at the nodes and hence also the quantities  $R^1_a$  and  $S^1_b$ . So in this case, the deformation does not help to avoid the singularity associated with the tangent bundle at the conifold point.

We can however save the situation by abandoning the identification in (3.12). Our bundle is then no longer a deformation of the tangent bundle of  $M$ , but a deformation of  $\mathcal{T}_M \oplus \mathcal{O} \oplus \mathcal{O}$ . Alternatively we could abandon just one of the identifications of (3.12), in which case the bundle becomes a deformation of  $\mathcal{T}_M \oplus \mathcal{O}$ .

We can also study deformations in the language of intertwined short exact sequences. As we have seen, the key equation is  $F$ ; we take  $M^F$  as the (5-dimensional) hypersurface given by the vanishing of  $F$  in the ambient space  $\mathbb{P}^3 \times \mathbb{P}^3$  (we can impose the other equations later). In complete analogy with (3.10), we have

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \rightarrow & \mathcal{T}_{M^F} & \rightarrow & \mathcal{T}_{\mathbb{P}^3 \times \mathbb{P}^3}|_{M^F} & \rightarrow & \mathcal{O}_{M^F}(1, 1) \rightarrow 0 \\
 & & \uparrow & & \uparrow \varphi & & \parallel \\
 0 & \rightarrow & \mathcal{F} & \rightarrow & \mathcal{O}_{M^F}(1, 0)^{\oplus 4} \oplus \mathcal{O}_{M^F}(0, 1)^{\oplus 4} & \xrightarrow{f} & \mathcal{O}_{M^F}(1, 1) \rightarrow 0 \\
 & & \uparrow & & \uparrow & & \\
 & & \mathcal{O}_{M^F}^{\oplus 2} & = & \mathcal{O}_{M^F}^{\oplus 2} & & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 
 \end{array} \tag{3.15}$$

The important facts to notice are the following. The map  $f$  is now given by an  $8 \times 1$  matrix with four entries of bidegree (1,0) and four entries of bidegree (0,1), corresponding to the operators  $\frac{\partial}{\partial x^a}$  and  $\frac{\partial}{\partial y^a}$ . The kernel of the map  $\varphi$  now is actually  $\mathcal{O}_{M^F}^{\oplus 2}$ , two copies of the trivial bundle on  $M^F$ . Therefore the bundle  $\mathcal{F}$ , which is the kernel of  $f$ , and hence whose

deformations are measured by (3.13), is a deformation of  $\mathcal{T}_{MF} \oplus \mathcal{O}_{MF}^{\oplus 2}$ , a rank 5 bundle. Varying the map  $f$  results in further deformations of this bundle. If either  $u$  or  $v$  vanished, then we would be deforming the rank 4 bundle  $\mathcal{T}_{MF} \oplus \mathcal{O}_{MF}$ ; in case the both vanish, we are back to deformations of  $\mathcal{T}_{MF}$ .

It is interesting that, on the split bicubic side, the motivation for studying bundles of higher rank derives from the fact that the commutant of the background gauge group in  $E_8$  is the gauge group of the low energy effective theory. For the “standard embedding”, the bundle associated with the background gauge group is identified with the tangent bundle, the background gauge group is  $SU(3)$  and the gauge group of the low energy effective theory is  $E_6$ , the commutant of  $SU(3)$  in  $E_8$ . A bundle of higher rank can have a larger background gauge group leading to a gauge group for the low energy theory that is a subgroup of  $E_6$ . In particular a rank 4 bundle on the Calabi–Yau manifold can give rise to a gauge group  $SO(10)$  in space-time and a rank 5 bundle can give rise to a gauge group  $SU(5)$ . Bundles of higher rank are therefore of interest because they can lead to appealing phenomenology. It is curious that we are here driven to rank 5 bundles by a wish to transgress.

### 3.4 On the geometry of $X^{3,3}$ and the heterotic vector bundle it carries

The manifold  $X^{3,3}$  is a  $\mathbb{Z}_3 \times \mathbb{Z}_3$  quotient of the split bicubic  $X^{19,19}$ . The latter is the fibre product of two ninth del Pezzo ( $dP_9$ ) surfaces [20],

$$\begin{matrix} t & \mathbb{P}^1 \\ \xi & \mathbb{P}^2 \end{matrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = B_1 \quad \text{and} \quad \begin{matrix} t & \mathbb{P}^1 \\ \eta & \mathbb{P}^2 \end{matrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = B_2.$$

The equations for the split bicubic  $X^{19,19}$  are of the form

$$\begin{aligned} F^1 &= t_1 U(\xi) + t_2 W(\xi) \\ F^2 &= t_1 Z(\eta) + t_2 V(\eta) \end{aligned}$$

with  $U$  and  $W$  cubics in  $\xi$ , and  $V$  and  $Z$  cubics in  $\eta$ . Particular equations invariant under  $A \times D$  are given by (2.4). The surfaces  $B_1$  and  $B_2$  correspond to the vanishing of  $F^1$  and  $F^2$ , respectively. Consider the surface  $B_1$ . The two cubics  $U(\xi)$  and  $V(\xi)$  vanish simultaneously in nine points and for each  $t$  the locus  $F^1 = 0$  is a cubic curve,  $E_1(t)$ , in  $\mathbb{P}^2$  which passes through these nine points. It is easy to see that for each  $\xi \in \mathbb{P}^2$ , that is not one of these nine special points, there is a unique  $t \in \mathbb{P}^1$  such that  $E_1(t)$  contains  $\xi$ . As  $t$  varies,  $E_1(t)$  sweeps out the surface  $B_1$ . Apart from the nine special points, each point of  $B_1$  derives from a unique point of  $\mathbb{P}^2$ . Consider the map that projects the points of  $B_1$  back onto the corresponding points of  $\mathbb{P}^2$  in this

way. For  $\xi_*$  one of the special points, the equation  $F^1 = 0$  is satisfied for every  $t$ . Thus an entire  $\mathbb{P}^1$  of  $B_1$  projects down to  $\xi_*$ . We shall refer to the  $\mathbb{P}^1$ 's that project down to the nine special points as exceptional lines. We have just worked through the classical fact that the surfaces  $B_i$  above are  $dP_9$ 's, surfaces obtained by blowing up  $\mathbb{P}^2$  in nine points. Along with the birational morphisms  $B_i \rightarrow \mathbb{P}^2$  just discussed, there are also the projections  $B_i \rightarrow \mathbb{P}^1$ , which exhibit the  $dP_9$ 's as elliptic fibrations.

We turn now to a description of the homology [19] of the quotient manifold  $X^{3,3}$ . The space  $H^2(X^{3,3}, \mathbb{C}) = H^2(X^{19,19}, \mathbb{C})^{\mathbb{Z}_3 \times \mathbb{Z}_3}$  is 3-dimensional, with basis  $(\tau_1, \tau_2, \phi)$ . Here the  $\tau_i$  are the pull-backs of the  $\mathbb{Z}_3 \times \mathbb{Z}_3$ -invariant divisors in  $B_i$ , the hyperplane classes pulled back from  $\mathbb{P}^2$ . The class  $\phi$  is the pull-back of the common fibre class  $f$ , the fibre of the projection of the 3-fold to  $\mathbb{P}^1$ . The intersection numbers are given by (2.5) in particular,  $H^4(X^{3,3}, \mathbb{C}) = H^4(X^{19,19}, \mathbb{C})^{\mathbb{Z}_3 \times \mathbb{Z}_3}$  is spanned by the classes  $(\tau_1^2, \tau_2^2, \tau_1\tau_2)$ .

The stable  $\mathbb{Z}_3 \times \mathbb{Z}_3$ -equivariant  $SU(4)$  bundle<sup>5</sup>  $\mathcal{V}$  on  $X^{3,3}$  which gives the MSSM spectrum [20] descends from an  $\mathbb{Z}_3 \times \mathbb{Z}_3$ -equivariant bundle  $\tilde{\mathcal{V}}$  on  $X^{19,19}$ . The bundle  $\tilde{\mathcal{V}}$  in turn is given by the short-exact sequence

$$0 \rightarrow \mathcal{V}_1 \rightarrow \tilde{\mathcal{V}} \rightarrow \mathcal{V}_2 \rightarrow 0 \tag{3.16}$$

of two rank 2 bundles,  $\mathcal{V}_1$  and  $\mathcal{V}_2$ . Each of these is itself the tensor product of a line bundle with a rank 2 bundle pulled back from a  $dP_9$  factor of  $X^{19,19}$ :

$$\mathcal{V}_1 = \mathcal{O}_X(-\tau_1 + \tau_2) \otimes \pi_1^*(\mathcal{W}_1), \quad \mathcal{V}_2 = \mathcal{O}_X(\tau_1 - \tau_2) \otimes \pi_2^*(\mathcal{W}_2),$$

where

$$\begin{aligned} 0 &\rightarrow \alpha_1 \mathcal{O}_{B_1}(-f) \rightarrow \mathcal{W}_1 \rightarrow \alpha_1^2 \mathcal{O}_{B_1}(f) \otimes I_3^{B_1} \rightarrow 0, \\ 0 &\rightarrow \alpha_2 \mathcal{O}_{B_2}(-f) \rightarrow \mathcal{W}_2 \rightarrow \alpha_2^2 \mathcal{O}_{B_2}(f) \otimes I_6^{B_2} \rightarrow 0; \end{aligned}$$

here  $I_3^{B_1}$  and  $I_6^{B_2}$  denote the ideal sheaf of 3 and 6 points in  $B_1$  and  $B_2$ , respectively, and  $\alpha_1$  and  $\alpha_2$  are third roots of unity. The Chern classes of  $\tilde{\mathcal{V}}$  can be computed from these sequences to be

$$c_1(\tilde{\mathcal{V}}) = 0, \quad c_2(\tilde{\mathcal{V}}) = \tau_1^2 + 4\tau_2^2 + 4\tau_1\tau_2, \quad c_3(\tilde{\mathcal{V}}) = -54. \tag{3.17}$$

### 3.5 Transgression between $N^{2,11}$ and $X^{3,3}$

Recall from §2.4 (compare table 2) that there is a conifold transition between the quotient  $X^{3,3}$  of the split bicubic, and the manifold  $N^{2,11}$ . We wish to

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<sup>5</sup>The following construction applies to the manifold  $Y^{3,3}$  discussed in §2.3. We remind the reader that we are assuming that  $X^{3,3}$  and  $Y^{3,3}$  are holomorphically equivalent. In the event that they are not then the following discussion should be taken as applying to  $Y^{3,3}$ .

investigate the possibility that the heterotic bundle discussed above can arise as a transgression under this transition. Before we do that, we make one further remark concerning the relation between exceptional lines on the  $dP_9$ 's to the conifold lines of the split bicubic  $X^{19,19}$ , the  $\mathbb{P}^1$ 's of the split bicubic that project down to the nodes of the conifold. We have seen that the conifold lines correspond to the solutions of

$$U(\xi) = W(\xi) = 0 \quad \text{and} \quad V(\eta) = Z(\eta) = 0.$$

These equations are solved precisely where  $(\xi, \eta) = (\xi_*, \eta_*)$ . In this way we see that the 81 conifold lines of the split bicubic are in correspondence to the 81 pairs of exceptional lines on  $B_1 \times B_2$ .

### 3.5.1 Triviality of the heterotic bundle on conifold lines

A necessary condition for a bundle to be a transgression through a conifold is that the bundle be trivial on the conifold lines. For the present case, we denote the homology class of the conifold lines of the quotient  $X^{3,3}$  by  $L \in H^4(X^{3,3}, \mathbb{C})$ .

The nonsingular manifold  $N^{2,11}$  has a 2-dimensional cohomology group  $H^2(N^{2,11}, \mathbb{C})$  spanned by classes  $H_1$  and  $H_2$ , the two hyperplane classes restricted from  $\mathbb{P}^2 \times \mathbb{P}^2$ . The hyperplane classes also give cohomology classes in  $H^2(N_0^{2,11})$ , the conifold degeneration. On the other hand, we have an inclusion  $H^2(N_0^{2,11}, \mathbb{C}) \hookrightarrow H^2(X^{3,3}, \mathbb{C})$ , where  $X^{3,3}$  is the split manifold after the conifold transition; under this map, the classes  $H_i$  map to the classes  $\tau_i$  for  $i = 1, 2$ , the latter being hyperplane sections from the two individual  $\mathbb{P}^2$ 's. Indeed, for the manifold  $N^{2,11}$  we have the intersection numbers

$$H_1^3 = H_2^3 = 0, \quad H_1^2 H_2 = H_2^2 H_1 = 3.$$

which agrees with (3.15) under the identification  $H_i \mapsto \tau_i$ .

Since the classes  $\tau_i$  arise as pullbacks from  $N_0^{2,11}$  under the resolution map  $X^{3,3} \rightarrow N_0^{2,11}$ , we know by the projection formula that the curve class  $L \in H^4(X^{3,3}, \mathbb{C})$  must be perpendicular to  $\tau_1$  and  $\tau_2$ . Hence we can write  $L$  in terms of our homology basis as

$$L = -\tau_1^2 - \tau_2^2 + \tau_1 \tau_2, \tag{3.18}$$

where we also used the condition that since  $\phi$  is an effective class on  $X^{3,3}$ , we must have  $\phi \cdot L \geq 0$ .

We learn about the restriction of  $\tilde{\mathcal{V}}$  to a conifold curve  $L \cong \mathbb{P}^1 \subset X^{19,19}$  in the cover by restricting the exact sequence (3.16) to get

$$0 \rightarrow (\mathcal{O}_X(-\tau_1 + \tau_2) \otimes \pi_1^*(\mathcal{W}_1))|_L \rightarrow \tilde{\mathcal{V}}|_L \rightarrow (\mathcal{O}_X(\tau_1 - \tau_2) \otimes \pi_2^*(\mathcal{W}_2))|_L \rightarrow 0.$$

Now, from (3.15) and (3.18), we see that

$$\mathcal{O}_X(-\tau_1 + \tau_2)|_L \cong \mathcal{O}_L(\tau_1^2\tau_2 + \tau_1\tau_2^2) \cong \mathcal{O}_L$$

and similarly for  $\mathcal{O}_X(\tau_1 - \tau_2)|_L$ . Hence,

$$0 \rightarrow \pi_1^*(\mathcal{W}_1)|_L \rightarrow \tilde{\mathcal{V}}|_L \rightarrow \pi_2^*(\mathcal{W}_2)|_L \rightarrow 0. \tag{3.19}$$

On the other hand, we claim that both restrictions  $\pi_i^*(\mathcal{W}_i)|_L$  are trivial. From the definition of  $\mathcal{W}_i$ ,

$$0 \rightarrow \mathcal{O}_{B_i}(-f)|_L \rightarrow \mathcal{W}_i|_L \rightarrow \mathcal{O}_{B_i}(f)|_L \rightarrow 0,$$

where we used the fact that the points which define the ideal sheaves  $I_3$  and  $I_6$  can be moved away from  $L$ . Thus the extension class of  $\mathcal{W}_i|_L$  lives in

$$\text{Ext}^1(\mathcal{O}_{B_i}(f)|_L, \mathcal{O}_{B_i}(-f)|_L \rightarrow 0) \cong H^1(L, \mathcal{O}_L(2f|_L)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6)) \neq 0$$

since  $f \cdot \pi_i(\tau_i) = 3$ . The generic element of this Ext-group corresponds to the extension  $\mathcal{W}_i|_L \cong \mathcal{O}_{\mathbb{P}^1}^2$ ; since the extensions defining the  $\mathcal{W}_i$  can be chosen to be generic, we are dealing here with a generic extension as well. Substituting into (3.19), we conclude that

$$\tilde{\mathcal{V}}|_L \cong \mathcal{O}_X^{\oplus 4}$$

on the conifold curves  $L \subset X$ .

### 3.5.2 Candidate bundles

We now approach the transition from the other side, attempting to construct some bundles on the manifold  $N^{2,11}$  with the right Chern classes. The first salient feature of the bundle  $\tilde{\mathcal{V}}$  is that  $c_3(\tilde{\mathcal{V}}) = -6$  on  $X^{3,3}$ . The bundle descends from an equivariant bundle on the covering space  $X^{19,19}$  with  $c_3 = -54$ . For the transgression we should seek a bundle with  $c_3 = -6$  on  $N^{2,11}$  or an equivariant bundle also with  $c_3 = -54$  on the bicubic  $N^{2,83}$ .

There has been some recent revival in interest in CICY's, especially in constructing bundles over them [50]. A large class of bundles can be constructed over algebraic manifolds via the monad construction. Over projective spaces, for example, all bundles arise this way. Let us try, therefore, to construct a monad-type bundle, as the kernel of bundle maps between

sums of line-bundles, over the bicubic. Such bundles  $\mathcal{V}$  are defined by the short-exact sequence

$$0 \rightarrow \mathcal{V} \rightarrow \mathcal{B} \xrightarrow{f} \mathcal{C} \rightarrow 0,$$

with

$$\mathcal{B} = \bigoplus_{i=1}^{r_{\mathcal{B}}} \mathcal{O}_X(b_1^i, b_2^i), \quad \mathcal{C} = \bigoplus_{j=1}^{r_{\mathcal{C}}} \mathcal{O}_X(c_1^j, c_2^j). \tag{3.20}$$

where  $\mathcal{O}(a, b)$  is the line bundle of bidegree  $(a, b)$  on the bicubic. In terms of the hyperplane classes, this line bundle is to be thought of as  $\mathcal{O}(aH_1, bH_2)$ . The  $b_s^i$  and  $c_s^j$  are integers such that for  $s = 1, 2$  and all  $i, j$ ,  $c_s^j \geq b_s^i \geq 0$  in order that  $f$  be a well-defined polynomial map. In particular,  $\mathcal{V}$  has the following topological properties ( $r, s, t = 1, 2$ ):

$$\begin{aligned} \text{rk}(\mathcal{V}) &= r_{\mathcal{B}} - r_{\mathcal{C}}, \\ [c_1(\mathcal{V})]_s &= \sum_{i=1}^{r_{\mathcal{B}}} b_s^i - \sum_{j=1}^{r_{\mathcal{C}}} c_s^j, \\ [c_2(\mathcal{V})]_r &= \frac{1}{2} \sum_{s,t=1}^k y_{rst}^{\text{top}} \left( \sum_{j=1}^{r_{\mathcal{C}}} c_s^j c_t^j - \sum_{i=1}^{r_{\mathcal{B}}} b_s^i b_t^i \right), \\ c_3(\mathcal{V}) &= \frac{1}{3} \sum_{r,s,t=1}^k y_{rst}^{\text{top}} \left( \sum_{i=1}^{r_{\mathcal{B}}} b_r^i b_s^i b_t^i - \sum_{j=1}^{r_{\mathcal{C}}} c_r^j c_s^j c_t^j \right), \end{aligned} \tag{3.21}$$

where  $y_{rst}^{\text{top}}$  is the intersection form on the bicubic. Finally, we must descend this bundle onto the  $A \times D$  quotient  $X^{3,3}$ . This can be done by finding an  $A \times D$ -equivariant map.

It is now a matter of solving for non-negative integer values for  $b_s^i$  and  $c_s^j$  such that first of all  $c_1(\mathcal{V}) = 0$  and  $c_3(\mathcal{V}) = -54$ . Then, in the  $(H_1, H_2)$  basis, we can form the integer 2-vector  $c_2(\mathcal{V}) \cdot (H_1, H_2)$ . This should then be compared, with  $c_2(\tilde{\mathcal{V}})$  expressed in the  $(\tau_1, \tau_2)$  basis. Examining (3.17), we see that in this basis

$$c_2(\tilde{\mathcal{V}}) = (\tau_1^2 + 4\tau_2^2 + 4\tau_1\tau_2) \cdot (\tau_1, \tau_2) = (24, 15).$$

After performing a computer scan for possible solutions, we find the following rank 4 candidate monad bundles with all the above requisite properties.

$\mathcal{B}$	$\mathcal{C}$
$\mathcal{O}(0, 1)^{\oplus 3} \oplus \mathcal{O}(0, 2) \oplus \mathcal{O}(1, 0)^{\oplus 2}$	$\mathcal{O}(1, 2) \oplus \mathcal{O}(1, 3)$
$\mathcal{O}(0, 1)^{\oplus 3} \oplus \mathcal{O}(1, 0) \oplus \mathcal{O}(1, 1)^{\oplus 3}$	$\mathcal{O}(1, 2) \oplus \mathcal{O}(1, 3) \oplus \mathcal{O}(2, 1)$
$\mathcal{O}(0, 1)^{\oplus 3} \oplus \mathcal{O}(1, 1)^{\oplus 6}$	$\mathcal{O}(1, 2)^{\oplus 4} \oplus \mathcal{O}(2, 1)$
$\mathcal{O}(0, 1)^{\oplus 3} \oplus \mathcal{O}(0, 2) \oplus \mathcal{O}(1, 0) \oplus \mathcal{O}(1, 1)^{\oplus 3}$	$\mathcal{O}(1, 2)^{\oplus 4}$

In this section we have, inspired by the intimate relation between the Tian–Yau and the bicubic sequences of manifolds, discussed in detail how one may transgress a bundle from a manifold to another through a conifold transition. Though the transgressions of bundles are old ideas in principle, here we have considered an explicit example. Our construction predicts a bundle on  $N^{2,11}$ , which is a transgression of the heterotic MSSM bundle on  $X^{3,3}$  and which may afford here a simpler description. This is an intriguing investigation to which we intend to return in a future publication.

#### 4 Concluding speculations

We began by plotting discretely different Calabi–Yau manifolds and were naturally drawn to the fact that each manifold has a parameter space and these spaces meet in loci corresponding to certain singular manifolds. For the cases we have been considering the relationship between the manifolds is close and the singular manifolds in which the parameter spaces meet are the conifolds which are only mildly singular. It is an old speculation [51] that the space of all Calabi–Yau manifolds may be connected in this way.

At a technical level, it is known that the parameter spaces of a great many Calabi–Yau manifolds form a connected web. For example, it is known that all CICY’s are connected by a series of conifold transitions [42, 52]. It is known also that the parameter spaces of all the manifolds of the Kreuzer–Skarke list form a connected web [53], which is moreover connected to the web of CICY’s, though the singular manifolds can be more singular than conifolds. Much work has been done on string theory compactified on conifolds and related spaces subsequent to [54] and in many cases conifold transitions seem to be physically acceptable. Nevertheless it is not known to what extent it is possible for the universe to move from one Calabi–Yau manifold to another. However if it is possible in heterotic string theory then transgression will be an important part of the story.

Looking at the tip of the landscape, it is hard not to speculate that there may be a physical mechanism allowing transitions between what appear classically to be different vacua thereby permitting the universe to trickle down to a very special corner of the landscape, an oasis where only very few Calabi–Yau manifolds reside.

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