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D-Brane superpotentials in Calabi–Yau orientifolds

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Abstract

We develop computational tools for the tree-level superpotential of B-branes in Calabi–Yau orientifolds. Our method is based on a systematic implementation of the orientifold projection in the geometric approach of Aspinwall and Katz. In the process, we lay down some ground rules for orientifold projections in the derived category.

Contents

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1 Introduction

D-branes in Type IIB orientifolds are an important ingredient in constructions of string vacua. A frequent problem arising in this context is the computation of the tree-level superpotential for holomorphic D-brane configurations. This is an important question for both realistic model building as well as dynamical supersymmetry breaking.

Various computational methods for the tree-level superpotential have been proposed in the literature. A geometric approach which identifies the superpotential with a three-chain period of the holomorphic $(3, 0)$-form has been investigated in [1–5]. A related method, based on two-dimensional holomorphic Chern–Simons theory, has been developed in [6–9]. The tree-level superpotential for fractional brane configurations at toric Calabi–Yau singularities has been computed in [10–15]. Using exceptional collections, one can also compute the superpotential for non-toric del Pezzo singularities [16–19]. Perturbative disc computations for superpotential interactions have been performed in [20–22]. Finally, a mathematical approach based on versal deformations has been developed in [23] and extended to matrix-valued fields in [24].

A systematic approach encompassing all these cases follows from the algebraic structure of B-branes on Calabi–Yau manifolds. Adopting the point of view that B-branes form a triangulated differential graded category [25–30], the computation of the superpotential is equivalent to the computation of a minimal A_∞ structure for the D-brane category [31–35].

This approach has been employed in the Landau–Ginzburg D-brane category [36–38], and in the derived category of coherent sheaves [39, 40]. These are two of the various phases that appear in the moduli space of a generic $\mathcal{N} = 2$ Type II compactification. In particular, Aspinwall and Katz [39] developed a general computational approach for the superpotential, in which the A_∞ products are computed using a Čech cochain model for the off-shell open string fields.

The purpose of the present paper is to apply a similar strategy for D-branes wrapping holomorphic curves in Type II orientifolds. This requires a basic understanding of the orientifold projection in the derived category, which is the subject of Section 2. In Section 3, we propose a computational scheme for the superpotential in orientifold models. This relies on a systematic implementation of the orientifold projection in the calculation of the A_∞ structure.

We show that the natural algebraic framework for deformation problems in orientifold models relies on L_∞ rather than A_∞ structures. This observation leads to a simple prescription for the D-brane superpotential in the presence of an orientifold projection: one has to evaluate the superpotential of the underlying unprojected theory on invariant on-shell field configurations. This is the main conceptual result of the paper, and its proof necessitates the introduction of a lengthy abstract machinery.

Applying our prescription in practice requires some extra work. The difficulty stems from the fact that while the orientifold action is geometric on the Calabi–Yau, it is *not* naturally geometric at the level of the derived category. Therefore, knowing the superpotential in the original theory does not trivially lead to the superpotential of the orientifolded theory. To illustrate this point, we compute the superpotential in two different cases. Both will involve D-branes wrapping rational curves, the difference will be in the way these curves are obstructed to move in the ambient space.

The organization of the paper is as follows. Section 2 reviews the construction of the categorical framework in which we wish to impose the orientifold projection, as well as how to do the latter. Section 3 describes the calculation of the D-brane superpotential in the presence of the projection.

Finally, Section 4 offers concrete computations of the D-brane superpotential for obstructed curves in Calabi–Yau orientifolds.

2 D-brane categories and orientifold projection

This section will be concerned with general aspects of topological B-branes in the presence of an orientifold projection. Our goal is to find a natural formulation for the orientifold projection in D-brane categories.

For concreteness, we will restrict ourselves to the category of topological B-branes on a Calabi–Yau threefold X, but our techniques extend to higher dimensions. In this case, the D-brane category is the derived category of coherent sheaves on X [25, 27]. In fact, a systematic off-shell construction of the D-brane category [28, 29] shows that the category in question is actually larger than the derived category. In addition to complexes, one has to also include twisted complexes as defined in [41]. We will show below that the off-shell approach is the most convenient starting point for a systematic understanding of the orientifold projection.

2.1 Review of D-brane categories

Let us begin with a brief review of the off-shell construction of D-brane categories [28, 29, 41]. It should be noted at the offset that there are several different models for the D-brane category, depending on the choice of a fine resolution of the structure sheaf $\mathcal{O}_X$. In this section, we will work with the Dolbeault resolution, which is closer to the original formulation of the boundary topological B-model [42]. This model is very convenient for the conceptual understanding of the orientifold projection, but it is unsuitable for explicit computations. In Section 4, we will employ a Čech cochain model for computational purposes, following the path pioneered in [39].

Given the threefold X, one first defines a differential graded category $\mathcal{C}$ as follows

$$\mathrm{Ob}(\mathcal{C})\colon \text{holomorphic vector bundles } (E,\overline{\partial}_E) \text{ on } X$$

$$\mathrm{Mor}_{\mathcal{C}}\left((E,\overline{\partial}_E),(F,\overline{\partial}_F)\right) = \left(\oplus_p A_X^{0,p}(\mathscr{H}om_X(E,F)),\overline{\partial}_{EF}\right),$$

where we have denoted by $\overline{\partial}_{EF}$ the induced Dolbeault operator on $\mathscr{H}om_X$ (E,F)-valued $(0,p)$ forms.[1] The space of morphisms is a $\mathbb{Z}$-graded differential complex. In order to simplify the notation we will denote the objects

[1] $\mathscr{H}om_X(E,F)$ is the sheaf Hom of E and F, viewed as sheaves.

of $\mathcal{C}$ by E, the data of an integrable Dolbeault operator $\overline{\partial}_E$ being implicitly understood.

The composition of morphisms in $\mathcal{C}$ is defined by exterior multiplication of bundle-valued differential forms. For any object E, composition of morphisms determines an associative algebra structure on the endomorphism space $\mathrm{Mor}_{\mathcal{C}}(E,E)$. This product is compatible with the differential, therefore we obtain a differential graded associative algebra (DGA) structure on $\mathrm{Mor}_{\mathcal{C}}(E,E)$.

At the next step, we construct the *shift completion* $\widetilde{\mathcal{C}}$ of $\mathcal{C}$, which is a category of holomorphic vector bundles on X equipped with an integral grading.

$$\begin{aligned}&\mathrm{Ob}(\widetilde{\mathcal{C}})\colon \text{pairs } (E,n),\ \text{with } E \text{ an object of } \mathcal{C} \text{ and } n\in\mathbb{Z}\\ &\mathrm{Mor}_{\widetilde{\mathcal{C}}}((E,n),(F,m)) = \mathrm{Mor}_{\mathcal{C}}(E,F)[n-m].\end{aligned}$$

The integer n is the boundary ghost number introduced in [27]. Note that for a homogeneous element

$$f\in \mathrm{Mor}^k_{\widetilde{\mathcal{C}}}((E,n),(F,m)),$$

we have

$$k = p + (m-n),$$

where p is the differential form degree of f. The degree k represents the total ghost number of the field f with respect to the bulk-boundary BRST operator. In the following, we will use the notations

$$|f| = k,\quad c(f) = p,\quad h(f) = m-n.$$

The composition of morphisms in $\widetilde{\mathcal{C}}$ differs from the composition of morphisms in $\mathcal{C}$ by a sign, which will play an important role in our construction. Given two homogeneous elements

$$f\in \mathrm{Mor}_{\widetilde{\mathcal{C}}}((E,n),(E',n'))\quad g\in \mathrm{Mor}_{\widetilde{\mathcal{C}}}((E',n'),(E'',n'')),$$

one defines the composition

$$(g\circ f)_{\widetilde{\mathcal{C}}} = (-1)^{h(g)c(f)}(g\circ f)_{\mathcal{C}}. \tag{2.1}$$

This choice of sign leads to the graded Leibniz rule

$$\overline{\partial}_{EE''}(g\circ f)_{\widetilde{\mathcal{C}}} = \left(\overline{\partial}_{E'E''}(g)\circ f\right)_{\widetilde{\mathcal{C}}} + (-1)^{h(g)}\left(g\circ \overline{\partial}_{EE'}(f)\right)_{\widetilde{\mathcal{C}}}.$$

Now we construct a pre-triangulated DG category $\text{Pre-Tr}(\widetilde{\mathcal{C}})$ of twisted complexes as follows:

$$\text{Ob}\left(\text{Pre-Tr}(\widetilde{\mathcal{C}})\right): \quad \begin{array}{l}\text{finite collections of the form}\\ \left\{(E_i, n_i, q_{ji})\middle|\ q_{ji} \in \text{Mor}^1_{\widetilde{\mathcal{C}}}((E_i, n_i), (E_j, n_j))\right\}\\ \text{where the } q_{ji} \text{ satisfy the Maurer–Cartan equation}\\ \overline{\partial}_{E_iE_j}(q_{ji}) + \sum_k (q_{jk} \circ q_{ki})_{\widetilde{\mathcal{C}}} = 0.\end{array}$$

$$\text{Mor}_{\text{Pre-Tr}(\widetilde{\mathcal{C}})}((E_i, n_i, q_{ji}), (F_i, m_i, r_{ji})) = \left(\bigoplus_{i,j} \text{Mor}_{\widetilde{\mathcal{C}}}((E_i, n_i), (F_j, m_j)), Q\right),$$

where the differential Q is defined as

$$Q(f) = \overline{\partial}_{E_iF_j}(f) + \sum_k (r_{kj} \circ f)_{\widetilde{\mathcal{C}}} - (-1)^{|f|}(f \circ q_{ik})_{\widetilde{\mathcal{C}}},$$
$$f \in \text{Mor}_{\widetilde{\mathcal{C}}}((E_i, n_i), (F_j, m_j)).$$

$|f|$ is the degree of f in $\text{Mor}_{\widetilde{\mathcal{C}}}((E_i, n_i), (F_j, m_j))$ from above. For each object, the index i takes finitely many values between 0 and some maximal value which depends on the object. Note that $Q^2 = 0$ because $\{q_{ji}\}, \{r_{ji}\}$ satisfy the Maurer–Cartan equation. Composition of morphisms in $\text{Pre-Tr}(\widetilde{\mathcal{C}})$ reduces to composition of morphisms in $\widetilde{\mathcal{C}}$.

Finally, the triangulated D-brane category $\mathcal{D}$ has by definition the same objects as $\text{Pre-Tr}(\widetilde{\mathcal{C}})$, while its morphisms are given by the zeroth cohomology under Q of the morphisms of $\text{Pre-Tr}(\widetilde{\mathcal{C}})$:

$$\text{Ob}(\mathcal{D}) = \text{Ob}\left(\text{Pre-Tr}(\widetilde{\mathcal{C}})\right)$$
$$\text{Mor}_{\mathcal{D}}((E_i, n_i, q_{ji}), (F_i, m_i, r_{ji}))$$
$$= H^0\left(Q, \text{Mor}_{\text{Pre-Tr}(\widetilde{\mathcal{C}})}((E_i, n_i, q_{ji}), (F_i, m_i, r_{ji}))\right). \tag{2.2}$$

The bounded derived category of coherent sheaves $D^b(X)$ is a full subcategory of $\mathcal{D}$. To see this, consider the objects of the form (E_i, n_i, q_{ji}) such that

$$n_i = -i, \quad q_{ji} \neq 0 \Leftrightarrow j = i - 1. \tag{2.3}$$

Since $q_{ji} \in \text{Mor}^1_{\widetilde{\mathcal{C}}}((E_i, n_i), (E_j, n_j))$, the second condition in (2.3) implies that their differential form degree must be 0. The Maurer–Cartan equation for such objects reduces to

$$\overline{\partial}_{E_iE_{i-1}} q_{i-1,i} = 0, \quad (q_{i-1,i} \circ q_{i,i+1})_{\widetilde{\mathcal{C}}} = 0.$$

Therefore the twisted complex (E_i, n_i, q_{ji}) is in fact a complex of holomorphic vector bundles

$$\cdots \longrightarrow E_{i+1} \xrightarrow{q_{i,i+1}} E_i \xrightarrow{q_{i-1,i}} E_{i-1} \longrightarrow \cdots \tag{2.4}$$

We will use the alternative notation

$$\cdots \longrightarrow E_{i+1} \xrightarrow{d_{i+1}} E_i \xrightarrow{d_i} E_{i-1} \longrightarrow \cdots \tag{2.5}$$

for complexes of vector bundles, and also denote them by the corresponding Gothic letter, here $\mathfrak{E}$.

One can easily check that the morphism space (2.2) between two twisted complexes of the form (2.3) reduces to the hypercohomology group of the local Hom complex $\mathscr{H}om(\mathfrak{E}, \mathfrak{F})$

$$\mathrm{Mor}_{\mathcal{D}}((E_i, n_i, q_{ji}), (F_i, m_i, r_{ji})) \simeq \mathbb{H}^0(X, \mathscr{H}om(\mathfrak{E}, \mathfrak{F})). \tag{2.6}$$

As explained in [30], this hypercohomology group is isomorphic to the derived morphism space $\mathrm{Hom}_{D^b(X)}(\mathfrak{E}, \mathfrak{F})$. Assuming that X is smooth and projective, any derived object has a locally free resolution, hence $D^b(X)$ is a full subcategory of $\mathcal{D}$.

2.2 Orientifold projection

Now we consider orientifold projections from the D-brane category point of view. A similar discussion of orientifold projections in matrix factorization categories has been outlined in [43].

Consider a four-dimensional $N = 1$ IIB orientifold obtained from an $N = 2$ Calabi–Yau compactification by gauging a discrete symmetry of the form

$$(-1)^{\epsilon F_{\mathrm{L}}} \Omega \, \sigma$$

with $\epsilon = 0, 1$. Employing common notation, Ω denotes world-sheet parity, F_{L} is the left-moving fermion number and $\sigma \colon X \to X$ is a holomorphic involution of X satisfying

$$\sigma^* \Omega_X = (-1)^{\epsilon} \, \Omega_X, \tag{2.7}$$

where Ω_X is the holomorphic $(3, 0)$-form of the Calabi–Yau. Depending on the value of ϵ, there are two classes of models to consider [44]:

1. $\epsilon = 0$: theories with $O5/O9$ orientifolds planes, in which the fixed-point set of σ is either one or three complex dimensional;

2. $\epsilon = 1$: theories with $O3/O7$ planes, with σ leaving invariant zero or two complex dimensional submanifolds of X.

Following the same logical steps as in the previous subsection, we should first find the action of the orientifold projection on the category $\mathcal{C}$, which is the starting point of the construction. The action of parity on the K-theory class of a D-brane has been determined in [45]. The world-sheet parity Ω maps E to the dual vector bundle $E^\vee$. If Ω acts simultaneously with a holomorphic involution $\sigma\colon X \to X$, the bundle E will be mapped to $\sigma^*(E^\vee)$. If the projection also involves a $(-1)^{F_L}$ factor, a brane with Chan–Paton bundle E should be mapped to an anti-brane with Chan–Paton bundle $P(E)$.

Based on this data, we define the action of parity on $\mathcal{C}$ to be

$$\begin{aligned} P\colon E &\mapsto P(E) = \sigma^*(E^\vee) \\ P\colon f \in \mathrm{Mor}_{\mathcal{C}}(E,F) &\mapsto \sigma^*(f^\vee) \in \mathrm{Mor}_{\mathcal{C}}(P(F),P(E)). \end{aligned} \tag{2.8}$$

It is immediate that P satisfies the following compatibility condition with respect to composition of morphisms in $\mathcal{C}$:

$$P((g\circ f)_{\mathcal{C}}) = (-1)^{c(f)c(g)}\,(P(f)\circ P(g))_{\mathcal{C}} \tag{2.9}$$

for any homogeneous elements f and g. It is also easy to check that P preserves the differential graded structure, i.e.,

$$P(\overline{\partial}_{EF}(f)) = \overline{\partial}_{P(F)P(E)}(P(f)). \tag{2.10}$$

Equation (2.9) shows that P is not a functor in the usual sense. Since it is compatible with the differential graded structure, it should be interpreted as a functor of A_∞ categories [46]. Note however that P is “almost a functor”: it fails to satisfy the compatibility condition with composition of morphisms only by a sign. For future reference, we will refer to A_∞ functors satisfying a graded compatibility condition of the form (2.9) as *graded functors.*

The category $\mathcal{C}$ does not contain enough information to make a distinction between branes and anti-branes. In order to make this distinction, we have to assign each bundle a grading, that is, we have to work in the category $\widetilde{\mathcal{C}}$ rather than $\mathcal{C}$. By convention, the objects (E,n) with n even are called branes, while those with n odd are called anti-branes.

We will take the action of the orientifold projection on the objects of $\widetilde{\mathcal{C}}$ to be

$$\widetilde{P}\colon (E,n) \mapsto (P(E), m-n) \tag{2.11}$$

where we have introduced an integer shift m which is correlated with ϵ from (2.7):

$$m \equiv \epsilon \bmod 2. \tag{2.12}$$

This allows us to treat both cases $\epsilon = 0$ and $\epsilon = 1$ in a unified framework.

We define the action of P on a morphisms $f \in \mathrm{Mor}_{\widetilde{\mathcal{C}}}((E,n),(E',n'))$ as the following graded dual:

$$\widetilde{P}(f) = -(-1)^{n'h(f)} P(f), \tag{2.13}$$

where $P(f)$ was defined in (2.8).[2] Note that the graded dual has been used in a similar context in [43], where the orientifold projection is implemented in matrix factorization categories.

With this definition, we have the following:

Proposition 2.1. *$\widetilde{P}$ is a graded functor on $\widetilde{\mathcal{C}}$ satisfying*

$$\widetilde{P}((g\circ f)_{\widetilde{\mathcal{C}}}) = -(-1)^{|f||g|}(\widetilde{P}(f)\circ \widetilde{P}(g))_{\widetilde{\mathcal{C}}} \tag{2.14}$$

for any homogeneous elements

$$f \in \mathrm{Mor}_{\widetilde{\mathcal{C}}}((E,n),(E',n')), \quad g \in \mathrm{Mor}_{\widetilde{\mathcal{C}}}((E',n'),(E'',n'')).$$

Proof. It is clear that $\widetilde{P}$ is compatible with the differential graded structure of $\widetilde{\mathcal{C}}$ since the latter is inherited from $\mathcal{C}$.

Next we prove (2.14). First we have:

$$\begin{aligned}\widetilde{P}((g\circ f)_{\widetilde{\mathcal{C}}}) &= -(-1)^{n''h(g\circ f)} P((g\circ f)_{\widetilde{\mathcal{C}}}) \text{ by (2.13)}\\ &= -(-1)^{n''h(g\circ f)+h(g)c(f)} P((g\circ f)_{\mathcal{C}}) \text{ by (2.1)}\\ &= -(-1)^{n''h(g\circ f)+h(g)c(f)+c(f)c(g)} (P(f)\circ P(g))_{\mathcal{C}} \text{ by (2.9)}\end{aligned}$$

On the other hand

$$\begin{aligned}(\widetilde{P}(f)\circ \widetilde{P}(g))_{\widetilde{\mathcal{C}}} &= (-1)^{n'h(f)+n''h(g)}(P(f)\circ P(g))_{\widetilde{\mathcal{C}}} \text{ by (2.13)}\\ &= (-1)^{n'h(f)+n''h(g)}(-1)^{h(P(f))c(P(g))}(P(f)\circ P(g))_{\mathcal{C}} \text{ by (2.1)}\end{aligned}$$

But

$$h(g\circ f) = h(f)+h(g), \quad h(P(f)) = h(f), \quad c(P(g)) = c(g).$$

Now (2.14) follows from

$$n''(h(f)+h(g)) - n'h(f) - n''h(g) = (n''-n')h(f) = h(g)h(f)$$

and

$$|f||g| = (h(f)+c(f))(h(g)+c(g)). \qquad \square$$

[2]There is no a priori justification for the particular sign we chose, but as we will see shortly, it leads to a graded functor. A naive generalization of (2.8) ignoring this sign would not yield a graded functor.

The next step is to determine the action of P on the pre-triangulated category Pre-Tr($\widetilde{\mathcal{C}}$). We denote this action by $\widehat{P}$. The action of $\widehat{P}$ on objects is defined simply by

$$(E_i, n_i, q_{ji}) \mapsto (P(E_i), m - n_i, \widetilde{P}(q_{ji})). \tag{2.15}$$

Using equations (2.10) and (2.14), it is straightforward to show that the action of $\widehat{P}$ preserves the Maurer–Cartan equation, that is

$$\begin{aligned} &\overline{\partial}_{E_iE_j}(q_{ji}) + \sum_k (q_{jk} \circ q_{ki})_{\widetilde{\mathcal{C}}} = 0 \\ &\quad \Rightarrow\ \overline{\partial}_{P(E_j)P(E_i)}\widetilde{P}(q_{ji}) + \sum_k (\widetilde{P}(q_{ki}) \circ \widetilde{P}(q_{jk}))_{\widetilde{\mathcal{C}}} = 0, \end{aligned}$$

since all q_{ji} have total degree 1. Therefore, this transformation is well defined on objects. The action on morphisms is also straightforward

$$\begin{aligned} &f \in \bigoplus_{i,j} \mathrm{Mor}_{\widetilde{\mathcal{C}}}((E_i, n_i), (F_j, m_j)) \\ &\quad \mapsto \widehat{P}(f) = \widetilde{P}(f) \in \bigoplus_{i,j} \mathrm{Mor}_{\widetilde{\mathcal{C}}}((P(F_j), m - m_j), (P(E_i), m - n_i)). \end{aligned} \tag{2.16}$$

Again, equations (2.10), (2.14) imply that this action preserves the differential

$$Q(f) = \overline{\partial}_{E_iF_j}(f) + \sum_k (r_{kj} \circ f)_{\widetilde{\mathcal{C}}} - (-1)^{|f|}(f \circ q_{ik})_{\widetilde{\mathcal{C}}}$$

since $\{q_{ji}\}$, $\{r_{ji}\}$ have degree 1. This means we have

$$\begin{aligned} \widehat{P}(Q(f)) = \overline{\partial}_{P(F_j)P(E_i)}\widetilde{P}(f) &+ \sum_k (\widetilde{P}(q_{ik}) \circ \widetilde{P}(f))_{\widetilde{\mathcal{C}}} \\ &- (-1)^{|\widetilde{P}(f)|}(\widetilde{P}(f) \circ \widetilde{P}(r_{kj}))_{\widetilde{\mathcal{C}}}. \end{aligned} \tag{2.17}$$

For future reference, let us record some explicit formulas for complexes of vector bundles. A complex

$$\mathfrak{E}\colon\ \cdots \longrightarrow E_{i+1} \xrightarrow{d_{i+1}} E_i \xrightarrow{d_i} E_{i-1} \longrightarrow \cdots$$

in which E_i has degree $-i$ is mapped to the complex

$$\widehat{P}(\mathfrak{E}) :\ \cdots \longrightarrow P(E_{i-1}) \xrightarrow{\widetilde{P}(d_i)} P(E_i) \xrightarrow{\widetilde{P}(d_{i+1})} P(E_{i+1}) \longrightarrow \cdots \tag{2.18}$$

where $\widetilde{P}(d_i)$ is determined by (2.13)

$$\widetilde{P}(d_i) = (-1)^i\, \sigma^*(d_i^\vee)$$

and $P(E_i)$ has degree $i - m$. Applying $\widehat{P}$ twice yields the complex

$$\widehat{P}^2(\mathfrak{E})\colon \ \cdots \longrightarrow E_{i+1} \xrightarrow{\widetilde{P}^2(d_{i+1})} E_i \xrightarrow{\widetilde{P}^2(d_i)} E_{i-1} \longrightarrow \cdots, \quad (2.19)$$

where

$$\widetilde{P}^2(d_i) = (-1)^{m+1} d_i.$$

Therefore $\widehat{P}^2$ is not equal to the identity functor, but there is an isomorphism of complexes $J\colon \widehat{P}^2(\mathfrak{E}) \to \mathfrak{E}$:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & E_{i+1} & \xrightarrow{\widetilde{P}^2(d_{i+1})} & E_i & \xrightarrow{\widetilde{P}^2(d_i)} & E_{i-1} & \longrightarrow \cdots \\ & \downarrow J_{i+1} & & \downarrow J_i & & \downarrow J_{i-1} & \\ \cdots \longrightarrow & E_{i+1} & \xrightarrow{d_{i+1}} & E_i & \xrightarrow{d_i} & E_{i-1} & \longrightarrow \cdots \end{array} \quad (2.20)$$

where

$$J_i = (-1)^{(m+1)i} \chi \, \mathrm{Id}_{E_i}$$

and χ is a constant. Notice that $J^{-1}\colon \mathfrak{E} \to \widehat{P}^2(\mathfrak{E})$ and that $\widehat{P}^4 = \mathrm{Id}_{D^b(X)}$ implies that also $J\colon \widehat{P}^2(\widehat{P}^2(\mathfrak{E})) = \mathfrak{E} \to \widehat{P}^2(\mathfrak{E})$. Requiring J and J^{-1} to be equal constrains χ to be $(-1)^\omega$ with $\omega = 0, 1$. This sign cannot be fixed using purely algebraic considerations, and we will show in Section 4 how it encodes the difference between SO/Sp projections. In functorial language, this means that there is an isomorphism of functors $J : \widehat{P}^2 \to \mathrm{Id}_{D^b(X)}$.

We conclude this section with a brief summary of the above discussion and a short remark on possible generalizations. To simplify notation, in the rest of the paper we drop the decorations of the various P's. In other words, both $\widehat{P}$ and $\widetilde{P}$ will be denoted by P. Which P is meant will always be clear from the context.

1. The orientifold projection in the derived category is a graded contravariant functor $P\colon D^b(X) \to D^b(X)^{op}$ which acts on locally free complexes as in equation (2.18). Note that this transformation is closely related to the derived functor
$$\mathbf{L}\sigma^* \circ \mathbf{R}\, \mathscr{H}om(-, \mathcal{O}_X)[m].$$
The difference resides in the alternating signs $(-1)^i$ in the action of P on differentials, according to (2.18). From now on, we will refer to P as a graded derived functor.
2. There is an obvious generalization of this construction which has potential physical applications. One can further compose P with an autoequivalence $\mathcal{A}$ of the derived category so that the resulting graded

functor $P \circ \mathcal{A}$ has its square isomorphic to the identity. This would yield a new class of orientifold models, possibly without a direct geometric interpretation. The physical implications of this construction will be explored in a separate publication.

In the remaining part of this section, we will consider the case of D5-branes wrapping holomorphic curves in more detail.

2.3 $O5$ models

In this case, we consider holomorphic involutions $\sigma\colon X \to X$ whose fixed-point set consists of a finite collection of holomorphic curves in X. We will be interested in D5-brane configurations supported on a smooth component $C \simeq \mathbb{P}^1$ of the fixed locus, which are preserved by the orientifold projection. We describe such a configuration by a one-term complex

$$i_* V, \tag{2.21}$$

where $V \to C$ is the Chan–Paton vector bundle on C, and $i\colon C \hookrightarrow X$ is the embedding of C into X.

Since $C \simeq \mathbb{P}^1$, by Grothendieck's theorem, any holomorphic bundle V decomposes in a direct sum of line bundles. Therefore, for the time being, we take

$$V \simeq \mathcal{O}_C(a) \tag{2.22}$$

for some $a \in \mathbb{Z}$. We will also make the simplifying assumption that V is the restriction of a bundle V' on X to C, i.e.,

$$V = i^* V'. \tag{2.23}$$

This is easily satisfied if X is a complete intersection in a toric variety Z, in which case V can be chosen to be the restriction of bundle on Z.

In order to write down the parity action on this D5-brane configuration, we need a locally free resolution $\mathfrak{E}$ for $i_* V = i_* \mathcal{O}_C(a)$. Let

$$\mathfrak{V} : 0 \longrightarrow \mathcal{V}_n \xrightarrow{d_n} \mathcal{V}_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \mathcal{V}_1 \xrightarrow{d_1} \underline{\mathcal{V}_0} \longrightarrow 0 \tag{2.24}$$

be a locally free resolution of $i^*\mathcal{O}_C$[3] , where the degree of the term $\mathcal{V}(-k)$ is $(-k)$ for $k = 0, \ldots, n$. Then the complex $\mathfrak{E}$

$$\mathfrak{E} : 0 \longrightarrow \mathcal{V}_n(a) \xrightarrow{d_n} \mathcal{V}_{n-1}(a) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \mathcal{V}_1(a) \xrightarrow{d_1} \underline{\mathcal{V}_0(a)} \longrightarrow 0 \tag{2.25}$$

is a locally free resolution of $i_*\mathcal{O}_C(a)$.

The image of (2.25) under the orientifold projection is the complex $\mathcal{P}(\mathfrak{E})$:

$$0 \longrightarrow \sigma^*\mathcal{V}_0^\vee(-a) \xrightarrow{-\sigma^* d_1^\vee} \sigma^*\mathcal{V}_1^\vee(-a) \xrightarrow{\sigma^* d_2^\vee} \cdots$$
$$\cdots \xrightarrow{(-1)^{n-1}\sigma^* d_{n-1}^\vee} \sigma^*\mathcal{V}_{n-1}^\vee(-a) \xrightarrow{(-1)^n \sigma^* d_n^\vee} \sigma^*\mathcal{V}_n^\vee(-a) \longrightarrow 0. \tag{2.26}$$

The term $\sigma^*\mathcal{V}_k^\vee(-a)$ has degree $k - m$.

Lemma 2.2. *The complex* (2.26) *is quasi-isomorphic to*

$$i_*\left(V^\vee \otimes K_C\right)[m-2], \tag{2.27}$$

where $K_C \simeq \mathcal{O}_C(-2)$ *is the canonical bundle of* C.[4]

Proof. As noted below (2.18), (2.26) is isomorphic to $\sigma^*(\mathfrak{E}^\vee)[m]$. Since C is pointwise fixed by σ, it suffices to show that the dual of the locally free resolution (2.24) is quasi-isomorphic to $i_*K_C[-2]$. The claim then follows from the adjunction formula:

$$i_*V = i_*(V \otimes \mathcal{O}_C) = i_*(i^*V' \otimes \mathcal{O}_C) = V' \otimes i_*\mathcal{O}_C \tag{2.28}$$

and the simple fact that $i^*(V'^\vee) = V^\vee$.

Let us compute $(i_*\mathcal{O}_C)^\vee$ using the locally free resolution (2.24). The cohomology in degree k of the complex

$$\mathfrak{V}^\vee : 0 \to (\mathcal{V}_0)^\vee \to (\mathcal{V}_1)^\vee \to \cdots \to (\mathcal{V}_n)^\vee \to 0 \tag{2.29}$$

is isomorphic to the local Ext sheaves $\mathscr{E}xt^k_X(i_*\mathcal{O}_C, \mathcal{O}_X)$. According to [47, Chapter 5.3, page 690], these are trivial except for $k = 2$, in which case

$$\mathscr{E}xt^2_X(\mathcal{O}_C, \mathcal{O}_X) \simeq i_*\mathcal{L},$$

for some line bundle $\mathcal{L}$ on C.

[3] We usually underlined the 0th position in a complex.

[4] We give an alternative derivation of this result in Appendix A.1. That proof is very abstract and hides all the details behind the powerful machinery of Grothendieck duality. On the other hand, we will be using the details of this lengthier derivation in our explicit computations in Section 4.

To determine $\mathcal{L}$, it suffices to compute its degree on C, which is an easy application of the Grothendieck–Riemann–Roch theorem. We have

$$i_!(\mathrm{ch}(\mathcal{L})\mathrm{Td}(C)) = \mathrm{ch}(i_*\mathcal{L})\mathrm{Td}(X).$$

On the other hand, by construction

$$\mathrm{ch}_m(i_*\mathcal{L}) = \mathrm{ch}_m(\mathfrak{V}^\vee) = (-1)^m \mathrm{ch}_m(\mathfrak{V}) = (-1)^m \mathrm{ch}_m(i_*\mathcal{O}_C).$$

Using these two equations, we find

$$\deg(\mathcal{L}) = -2 \Rightarrow \mathcal{L} \simeq K_C.$$

This shows that $\mathfrak{V}^\vee$ has nontrivial cohomology i_*K_C only in degree 2.

Now we establish that the complex (2.29) is quasi-isomorphic to $i_*K_C[-2]$ by constructing such a map of complexes. Consider the restriction of the complex (2.29) to C. Since all terms are locally free, we obtain a complex of holomorphic bundles on C whose cohomology is isomorphic to K_C in degree 2 and trivial in all other degrees. Note that the kernel $\mathcal{K}$ of the map

$${\mathcal{V}_2}^\vee|_C \to {\mathcal{V}_3}^\vee|_C$$

is a torsion-free sheaf on C, therefore it must be locally free. Hence $\mathcal{K}$ is a subbundle of ${\mathcal{V}_2}^\vee|_C$. Since $C \simeq \mathbb{P}^1$, by Grothendieck's theorem both ${\mathcal{V}_2}^\vee|_C$ and $\mathcal{K}$ are isomorphic to direct sums of line bundles. This implies that $\mathcal{K}$ is in fact a direct summand of ${\mathcal{V}_2}^\vee|_C$. In particular, there is a surjective map

$$\rho\colon {\mathcal{V}_2}^\vee|_C \to \mathcal{K}.$$

Since $H^2(\mathfrak{V}^\vee|_C) = K_C$, we also have a surjective map $\tau\colon \mathcal{K} \to K_C$. By construction, then $\tau \circ \rho\colon \mathfrak{V}^\vee|_C \to K_C[-2]$ is a quasi-isomorphism. Extending this quasi-isomorphism by zero outside C, we obtain a quasi-isomorphism $\mathfrak{V}^\vee \to i_*K_C[-2]$, which proves the lemma. □

Let us now discuss parity invariant D-brane configurations. Given the parity action (2.27) one can obviously construct such configurations by taking direct sums of the form

$$i_*V \oplus i_*(V^\vee \otimes K_C)[m-2] \tag{2.30}$$

with V an arbitrary Chan–Paton bundle. Note that in this case we have two stacks of D5-branes in the covering space, which are interchanged under the orientifold projection.

However, on physical grounds, we should also be able to construct a single stack of D5-branes wrapping C which is preserved by the orientifold action.

This is possible only if

$$m = 2 \quad \text{and} \quad V \simeq V^\vee \otimes K_C. \tag{2.31}$$

The first condition in (2.31) fixes the value of m for this class of models. The second condition constrains the Chan–Paton bundle V to

$$V = \mathcal{O}_C(-1).$$

Let us now consider rank N Chan–Paton bundles V. We will focus on invariant D5-brane configurations given by

$$V = \mathcal{O}_C(-1)^{\oplus N}.$$

In this case, the orientifold image $P(i_*V) = i_*(V^\vee \otimes K_C)$ is isomorphic to i_*V, and the choice of an isomorphism corresponds to the choice of a section

$$M \in \mathrm{Hom}_C(V, V^\vee \otimes K_C) \simeq \mathcal{M}_N(\mathbb{C}). \tag{2.32}$$

where $\mathcal{M}_N(\mathbb{C})$ is the space of $N \times N$ complex matrices. We have

$$\begin{aligned} \mathrm{Hom}_C(V, V^\vee \otimes K_C) &\simeq H^0(C, S^2(V^\vee) \otimes K_C) \oplus H^0(C, \Lambda^2(V^\vee) \otimes K_C) \\ &\simeq \mathcal{M}_N^+(\mathbb{C}) \oplus \mathcal{M}_N^-(\mathbb{C}), \end{aligned}$$

where $\mathcal{M}_N^\pm(\mathbb{C})$ denotes the space of symmetric and anti-symmetric $N \times N$ matrices, respectively. The choice of this isomorphism (up to conjugation) encodes the difference between SO and Sp projections. For any value of N, we can choose the isomorphism to be

$$M = \mathrm{I}_N \in \mathcal{M}_N^+(\mathbb{C}), \tag{2.33}$$

obtaining $SO(N)$ gauge group. If N is even, we also have the option of choosing the anti-symmetric matrix

$$M = i \begin{bmatrix} 0 & \mathrm{I}_{N/2} \\ -\mathrm{I}_{N/2} & 0 \end{bmatrix} \in \mathcal{M}_N^-(\mathbb{C}) \tag{2.34}$$

obtaining $Sp(N/2)$ gauge group. This is a slightly more abstract reformulation of [48]. We will explain how the SO/Sp projections are encoded in the derived formalism in Sections 3 and 4.

2.4 $O3/O7$ models

In this case, we have $\epsilon = 1$, and the fixed-point set of the holomorphic involution can have both zero and two-dimensional components. We will consider the magnetized D5-brane configurations introduced in [49]. Suppose

$$i\colon C \hookrightarrow X, \quad i' : C' \hookrightarrow X$$

is a pair of smooth rational curves mapped isomorphically into each other by the holomorphic involution. The brane configuration consists of a stack of D5-branes wrapping C, which is related by the orientifold projection to a stack of anti-D5-branes wrapping C'. We describe the stack of D5-branes wrapping C by a one-term complex i_*V, with V a bundle on C.

In order to find the action of the orientifold group on the stack of D5-branes wrapping C, we pick a locally free resolution $\mathfrak{E}$ for i_*V. Once again, the orientifold image is obtained by applying the graded derived functor P to $\mathfrak{E}$.

Applying Proposition A.1, we have

Lemma 2.3. *$P(\mathfrak{E})$ is quasi-isomorphic to the one-term complex*

$$i'_*(\sigma^*(V^\vee)\otimes K_{C'})[m-2]. \tag{2.35}$$

It follows that a D5-brane configuration preserved by the orientifold projection is a direct sum

$$i_*V \oplus i'_*(\sigma^*(V^\vee)\otimes K_{C'})[m-2]. \tag{2.36}$$

The value of m can be determined from physical arguments by analogy with the previous case. We have to impose the condition that the orientifold projection preserves a D3-brane supported on a fixed point $p \in X$ as well as a D7-brane supported on a pointwise-fixed surface $S \subset X$.

A D3-brane supported at $p \in X$ is described by a one-term complex $\mathcal{O}_{p,X}$, where $\mathcal{O}_{p,X}$ is a skyscraper sheaf supported at p. Again, using Proposition A.1, one shows that $P(\mathfrak{V})$ is quasi-isomorphic to $\mathcal{O}_{p,X}[m-3]$. Therefore, the D3-brane is preserved if and only if $m = 3$.

If the model also includes a codimension 1 pointwise-fixed locus $S \subset X$, then we have an extra condition. Let V be the Chan–Paton bundle on S. We describe the invariant D7-brane wrapping S by $\mathfrak{L} \simeq i_*(V)[k]$ for some integer k, where $i\colon S \to X$ is the embedding.

Since S is codimension 1 in X, Proposition A.1 tells us that

$$P(\mathfrak{L}) \simeq i_*(V^\vee \otimes K_S)[m-k-1]. \tag{2.37}$$

Therefore, invariance under P requires

$$2k = m-1, \quad V\otimes V \simeq K_S. \tag{2.38}$$

Since we have found $m = 3$ above, it follows that $k = 1$. Furthermore, V has to be a square root of K_S. In particular, this implies that K_S must be even or, in other words, that S must be spin. This is in agreement with the

Freed–Witten anomaly cancellation condition [50]. If S is not spin, one has to turn on a half-integral B-field in order to cancel anomalies.

Returning to the magnetized D5-brane configuration, note that an interesting situation from the physical point of view is the case when the curves C and C' coincide. Then C is preserved by the holomorphic involution, but not pointwise fixed as in the previous subsection. We will discuss examples of such configurations in Section 4. In the next section we will focus on general aspects of the superpotential in orientifold models.

3 The superpotential

The framework of D-brane categories offers a systematic approach to the computation of the tree-level superpotential. In the absence of the orientifold projection, the tree-level D-brane superpotential is encoded in the A_∞ structure of the D-brane category [31–35].

Given an object of the D-brane category $\mathcal{D}$, the space of off-shell open string states is its space of endomorphisms in the pre-triangulated category Pre-Tr$(\widetilde{\mathcal{C}})$. This carries the structure of a $\mathbb{Z}$-graded differential cochain complex. In this section, we will continue to work with Dolbeault cochains and also specialize our discussion to locally free complexes $\mathfrak{E}$ of the form (2.5). Then the space of off-shell open string states is given by

$$\mathrm{Mor}_{\text{Pre-Tr}(\widetilde{\mathcal{C}})}(\mathfrak{E}, \mathfrak{E}) = \bigoplus_p A^{0,p}(\mathscr{H}\!om_X(\mathfrak{E}, \mathfrak{E})),$$

where

$$\mathscr{H}\!om^q_X(\mathfrak{E}, \mathfrak{E}) = \bigoplus_i \mathscr{H}\!om_X(E_i, E_{i-q}).$$

Composition of morphisms defines a natural superalgebra structure on this endomorphism space [51], and the differential Q satisfies the graded Leibniz rule. We will denote the resulting DGA by $\mathcal{C}(\mathfrak{E}, \mathfrak{E})$.

The computation of the superpotential is equivalent to the construction of an A_∞ minimal model for the DGA $\mathcal{C}(\mathfrak{E}, \mathfrak{E})$. Since this formalism has been explained in detail in the physics literature [33, 39], we will not provide a comprehensive review here. Rather we will recall some basic elements needed for our construction.

In order to extend this computational framework to orientifold models, we have to find an off-shell cochain model equipped with an orientifold projection and a compatible differential algebraic structure. We made a first step in this direction in the previous section by giving a categorical formulation of the orientifold projection. In Section 3.1, we will refine this construction, obtaining the desired cochain model.

Having constructed a suitable cochain model, the computation of the superpotential follows the same pattern as in the absence of the orientifold projection. A notable distinction resides in the occurrence of L_∞ instead of A_∞ structures, since the latter are not compatible with the involution. The final result obtained in section 3.2 is that the orientifold superpotential can be obtained by evaluating the superpotential of the underlying unprojected theory on invariant field configurations.

3.1 Cochain model and orientifold projection

Suppose $\mathfrak{E}$ is a locally free complex on X and that it is left invariant by the parity functor. This means that $\mathfrak{E}$ and $P(\mathfrak{E})$ are isomorphic in the derived category, and we choose such an isomorphism

$$\psi\colon \mathfrak{E} \to P(\mathfrak{E}). \tag{3.1}$$

Although in general ψ is not a map of complexes, it can be chosen so in most practical situations, including all cases studied in this paper. Therefore, we will assume from now on that ψ is a quasi-isomorphism of complexes:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & E_{m-i+1} & \xrightarrow{d_{m-i+1}} & E_{m-i} & \xrightarrow{d_{m-i}} & E_{m-i-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \psi_{m-i+1}} & & \downarrow{\scriptstyle \psi_{m-i}} & & \downarrow{\scriptstyle \psi_{m-i-1}} & & \\ \cdots & \longrightarrow & P(E_{i-1}) & \xrightarrow{P(d_i)} & P(E_i) & \xrightarrow{P(d_{i+1})} & P(E_{i+1}) & \longrightarrow & \cdots \end{array} \tag{3.2}$$

We have written (3.2) so that the terms in the same column have the same degree since ψ is a degree zero morphism. The degrees of the three columns from left to right are $i-m-1,\ i-m$ and $i-m+1$. For future reference, note that the quasi-isomorphism ψ induces a quasi-isomorphism of cochain complexes

$$\psi_*\colon \mathcal{C}(P(\mathfrak{E}), \mathfrak{E}) \to \mathcal{C}(P(\mathfrak{E}), P(\mathfrak{E})), \quad f \mapsto \psi \circ f. \tag{3.3}$$

The problem we are facing in the construction of a viable cochain model resides in the absence of a natural orientifold projection on the cochain space $\mathcal{C}(\mathfrak{E}, \mathfrak{E})$. P maps $\mathcal{C}(\mathfrak{E}, \mathfrak{E})$ to $\mathcal{C}(P(\mathfrak{E}), P(\mathfrak{E}))$, which is not identical to $\mathcal{C}(\mathfrak{E}, \mathfrak{E})$. How can we find a natural orientifold projection on a given off-shell cochain model?

Since $\mathfrak{E}$ and $P(\mathfrak{E})$ are quasi-isomorphic, one can equally well adopt the morphism space

$$\mathcal{C}(P(\mathfrak{E}), \mathfrak{E}) = \mathrm{Mor}_{\text{Pre-Tr}(\widetilde{\mathcal{C}})}(P(\mathfrak{E}), \mathfrak{E})$$

as an off-shell cochain model. As opposed to $\mathcal{C}(\mathfrak{E}, \mathfrak{E})$, this morphism space has a natural induced involution defined by the composition

$$\mathcal{C}(P(\mathfrak{E}), \mathfrak{E}) \xrightarrow{P} \mathcal{C}(P(\mathfrak{E}), P^2(\mathfrak{E})) \xrightarrow{J_*} \mathcal{C}(P(\mathfrak{E}), \mathfrak{E}) \ , \qquad (3.4)$$

where J is the isomorphism in (2.20). Therefore, we will do our superpotential computation in the cochain model $\mathcal{C}(P(\mathfrak{E}), \mathfrak{E})$, as opposed to $\mathcal{C}(\mathfrak{E}, \mathfrak{E})$, which is used in [39].

This seems to lead us to another puzzle, since a priori there is no natural associative algebra structure on $\mathcal{C}(P(\mathfrak{E}), \mathfrak{E})$. One can however define one using the quasi-isomorphism (3.1). Given

$$f^p_{q,k} \in A^{0,p}(\mathscr{H}om_X(P(E_k), E_{m-k-q})), \quad g^r_{s,l} \in A^{0,r}(\mathscr{H}om_X(P(E_l), E_{m-l-s})),$$

we define

$$g^r_{s,l} \star_\psi f^p_{q,k} = \begin{cases} (-1)^{sp} g^r_{s,l} \cdot \psi_{m-k-q} \cdot f^p_{q,k} & \text{for } l = k+q, \\ 0 & \text{otherwise,} \end{cases} \qquad (3.5)$$

where $\cdot$ denotes exterior multiplication of bundle-valued differential forms.

With this definition, the map (3.3) becomes a quasi-isomorphism of DGAs. The sign $(-1)^{sp}$ in (3.5) is determined by the sign rule (2.1) for composition of morphisms in $\widetilde{\mathcal{C}}$. This construction has the virtue that it makes both the algebra structure and the orientifold projection manifest. Note that the differential Q satisfies the graded Leibniz rule with respect to the product $\star_\psi$ because ψ is a Q-closed element of $\mathcal{C}(P(\mathfrak{E}), \mathfrak{E})$ of degree zero.

Next we check two compatibility conditions between the involution (3.4) and the DGA structure.

Lemma 3.1. *For any cochain $f \in \mathcal{C}(P(\mathfrak{E}), \mathfrak{E})$,*

$$J_* P(Q(f)) = Q(J_* P(f)). \qquad (3.6)$$

Proof. Using equation (2.18), the explicit expression for the differential Q acting on a homogeneous element $f^p_{q,k}$ as above is

$$Q(f^p_{q,k}) = \overline{\partial}_{P(E_k)E_{m-k-q}}(f^p_{q,k}) + (d_{m-k-q} \circ f^p_{q,k})_{\widetilde{\mathcal{C}}} - (-1)^{p+q}(f^p_{q,k} \circ P(d_k))_{\widetilde{\mathcal{C}}}.$$

According to equation (2.17), we have

$$\begin{aligned} P(Q(f^p_{q,k})) = {} & \overline{\partial}_{P(E_{m-k-q})P^2(E_k)}(P(f^p_{q,k})) + (P^2(d_k) \circ P(f^p_{q,k}))_{\widetilde{\mathcal{C}}} \\ & - (-1)^{|P(f)|}(P(f^p_{q,k}) \circ P(d_{m-k-q}))_{\widetilde{\mathcal{C}}}. \end{aligned} \tag{3.7}$$

The commutative diagram (2.20) shows that

$$J \circ P^2(d_k) = d_k \circ J.$$

Then, equation (3.7) yields

$$\begin{aligned} J_*P(Q(f^p_{q,k})) = {} & \overline{\partial}_{P(E_{m-k-q})E_k}(J_*P(f^p_{q,k})) + (d_k \circ J_*P(f^p_{q,k}))_{\widetilde{\mathcal{C}}} \\ & - (-1)^{|f|}(J_*P(f^p_{q,k}) \circ P(d_{m-k-q}))_{\widetilde{\mathcal{C}}}, \end{aligned}$$

which proves (3.6). □

Lemma 3.2. *For any two elements $f, g \in \mathcal{C}(P(\mathfrak{E}), \mathfrak{E})$,*

$$J_*P(g \star_\psi f) = -(-1)^{|f||g|} J_*P(f) \star_\psi J_*P(g). \tag{3.8}$$

Proof. Written in terms of homogeneous elements, (3.8) reads

$$J_*P(g^r_{s,l} \star_\psi f^p_{q,k}) = -(-1)^{(r+s)(p+q)} J_*P(f^p_{q,k}) \star_\psi J_*P(g^r_{s,l}) \tag{3.9}$$

where $l = k+q$. Using equations (2.13), (3.5) and the definition of (2.20) of J, we compute

$$\begin{aligned} J_*P(g^r_{s,l} \star_\psi f^p_{q,k}) = {} & (-1)^{(m-s-l)(m+1)+\omega}(-1)^{(s+q)(m-s-l)+1}(-1)^{sp} \\ & \sigma^*(g^r_{s,l} \cdot \psi_{m-k-q} \cdot f^p_{q,k})^\vee \\ = {} & (-1)^{(m-s-l)(m+1)+\omega}(-1)^{(s+q)(m-s-l)+1}(-1)^{sp}(-1)^{rp} \\ & \sigma^*(f^p_{q,k})^\vee \cdot \sigma^*(\psi^\vee_{m-k-q}) \cdot \sigma^*(g^r_{s,l})^\vee \\ = {} & (-1)^{(m-s-l)(m+1)+\omega}(-1)^{(s+q)(m-s-l)+1}(-1)^{sp}(-1)^{rp} \\ & (-1)^{(m-k-q)(m+1)+\omega}(-1)^{q(q+k-m)+1}(-1)^{(m-s-l)(m+1)+\omega} \\ & (-1)^{s(s+l-m)+1} \\ & J_*P(f^p_{q,k}) \cdot \sigma^*(\psi^\vee_{m-k-q}) \cdot J_*P(g^r_{s,l}), \\ & -(-1)^{(r+s)(p+q)} J_*P(f^p_{q,k}) \star_\psi J_*P(g^r_{s,l}) \\ = {} & -(-1)^{(r+s)(p+q)}(-1)^{qr} J_*P(f^p_{q,k}) \cdot \psi_l \cdot J_*P(g^r_{s,l}). \end{aligned}$$

These expressions are in agreement with equation (3.9) if and only if ψ satisfies a symmetry condition of the form

$$J^* P(\psi_{m-l}) = -\psi_l \quad \Longleftrightarrow \quad \sigma^*(\psi_{m-l})^\vee = (-1)^{(m+1)l+\omega}\psi_l. \qquad (3.10)$$

□

We saw in the last proof that compatibility of the orientifold projection with the algebraic structure imposes condition (3.10) on ψ. From now on, we assume this condition to be satisfied. Although we do not know a general existence result for a quasi-isomorphism satisfying (3.10), we will show that such a choice is possible in all the examples considered in this paper. We will also see that symmetry of ψ, which is determined by $\omega = 0, 1$ in (3.10), determines whether the orientifold projection is of type SO or Sp.

Granting such a quasi-isomorphism, it follows that the cochain space $\mathcal{C}(P(\mathfrak{E}), \mathfrak{E})$ satisfies all the conditions required for the computation of the superpotential, which is the subject of the next subsection.

3.2 The superpotential

In the absence of an orientifold projection, the computation of the superpotential can be summarized as follows [34]. Suppose we are searching for formal deformations of the differential Q of the form

$$Q_{\text{def}} = Q + f_1(\phi) + f_2(\phi) + f_3(\phi) + \dots \qquad (3.11)$$

where

$$f_1(\phi) = \phi$$

is a cochain of degree 1, which represents an infinitesimal deformation of Q. The terms $f_k(\phi)$, for $k \geq 2$, are homogeneous polynomials of degree k in ϕ corresponding to higher order deformations. We want to impose the integrability condition

$$(Q_{\text{def}})^2 = 0 \qquad (3.12)$$

order by order in ϕ. In doing so, one encounters certain obstructions, which are systematically encoded in a minimal A_∞ model of the DGA $\mathcal{C}(P(\mathfrak{E}), \mathfrak{E})$. The superpotential is essentially a primitive function for the obstructions and exists under certain cyclicity conditions.

In the orientifold model we have to solve a similar deformation problem, except that now the deformations of Q have to be invariant under the orientifold action. We will explain below that this is equivalent to the construction of a minimal L_∞ model.

Let us first consider the integrability conditions (3.12) in more detail in the absence of orientifolding. Suppose we are given an associative $\mathbb{Z}$-graded DGA $(\mathcal{C}, Q, \cdot)$, and let H denote the cohomology of Q. In order to construct an A_∞ structure on H, we need the following data.

(*i*) A $\mathbb{Z}$-graded linear subspace $\mathcal{H} \subset \mathcal{C}$ isomorphic to the cohomology of Q. In other words, $\mathcal{H}$ is spanned in each degree by representatives of the cohomology classes of Q.

(*ii*) A linear map $\eta : \mathcal{C} \to \mathcal{C}[-1]$ mapping $\mathcal{H}$ to itself such that

$$\Pi = \mathbb{I} - [Q, \eta] \tag{3.13}$$

is a projector $\Pi \colon \mathcal{C} \to \mathcal{H}$, where $[\quad , \quad]$ is the graded commutator. Moreover, we assume that the following conditions are satisfied

$$\eta|_{\mathcal{H}} = 0, \quad \eta^2 = 0. \tag{3.14}$$

Using the data (*i*), (*ii*) one can develop a recursive approach to obstructions in the deformation theory of Q [34]. The integrability condition (3.12) yields

$$\sum_{n=1}^{\infty} [Q(f_n(\phi)) + B_{n-1}(\phi)] = 0, \tag{3.15}$$

where

$$\begin{aligned} B_0 &= 0, \\ B_{n-1} &= \phi f_{n-1}(\phi) + f_{n-1}(\phi)\phi + \sum_{\substack{k+l=n \\ k,l\geq 2}} f_k(\phi) f_l(\phi), \quad n \geq 2. \end{aligned}$$

Using equation (3.13), we can rewrite equation (3.15) as

$$\sum_{n=1}^{\infty} [Q(f_n(\phi)) + ([Q, \eta] + \Pi)B_{n-1}(\phi)] = 0. \tag{3.16}$$

We claim that the integrability condition (3.15) can be solved recursively [34] provided that

$$\sum_{n=1}^{\infty} \Pi(B_{n-1}) = 0. \tag{3.17}$$

To prove this claim, note that if condition (3.17) is satisfied, equation (3.16) becomes

$$\sum_{n=1}^{\infty} (Q(f_n(\phi)) + [Q, \eta]B_{n-1}(\phi)) = 0. \tag{3.18}$$

This equation can be solved by setting recursively

$$f_n(\phi) = -\eta(B_{n-1}(\phi)). \tag{3.19}$$

One can show that this is a solution to (3.19) by proving inductively that

$$Q(B_n(\phi)) = 0.$$

In conclusion, the obstructions to the integrability condition (3.15) are encoded in the formal series

$$\sum_{n=2}^{\infty} \Pi \left(\phi f_{n-1}(\phi) + f_{n-1}(\phi)\phi + \sum_{\substack{k+l=n \\ k,l\geq 2}} f_k(\phi) f_l(\phi) \right) \tag{3.20}$$

where the $f_n(\phi)$, $n \geq 1$, are determined recursively by (3.19).

The algebraic structure emerging from this construction is a minimal A_∞ structure for the DGA $(\mathcal{C}, Q)$ [52, 53]. Merkulov [53] constructs an A_∞ structure by defining the linear maps

$$\lambda_n \colon \mathcal{C}^{\otimes n} \to \mathcal{C}[2-n], \quad n \geq 2$$

recursively

$$\begin{aligned} \lambda_n(c_1, \ldots, c_n) = & (-1)^{n-1}(\eta\lambda_{n-1}(c_1, \ldots, c_{n-1})) \cdot c_n \\ & - (-1)^{n|c_1|} c_1 \cdot \eta\lambda_{n-1}(c_2, \ldots, c_n) \\ & - \sum_{\substack{k+l=n \\ k,l\geq 2}} (-1)^r [\eta\lambda_k(c_1, \ldots, c_k)] \cdot [\eta\lambda_l(c_{k+1}, \ldots, c_n)], \end{aligned} \tag{3.21}$$

where $|c|$ denotes the degree of an element $c \in \mathcal{C}$, and

$$r = k + 1 + (l-1)(|c_1| + \cdots + |c_k|).$$

Now define the linear maps

$$\mathfrak{m}_n \colon \mathcal{H}^{\otimes n} \to \mathcal{H}[2-n], \quad n \geq 1$$

by

$$\begin{aligned} \mathfrak{m}_1 &= \eta \\ \mathfrak{m}_n &= \Pi\lambda_n. \end{aligned} \tag{3.22}$$

The products (3.22) define an A_∞ structure on $\mathcal{H} \simeq H$. If conditions (3.14) are satisfied, this A_∞ structure is a minimal model for the DGA $(\mathcal{C}, Q, \cdot)$. The products $\mathfrak{m}_n$, $n \geq 2$ agree up to sign with the obstructions $\Pi(B_n)$ found above.

The products $\mathfrak{m}_n$ determine the local equations of the D-brane moduli space, which in physics language are called F-term equations. If

$$\phi = \sum_{i=1}^{\dim(\mathcal{H})} \phi^i u_i$$

is an arbitrary cohomology element written in terms of some generators $\{u_i\}$, the F-term equations are

$$\sum_{n=2}^{\infty} (-1)^{n(n+1)/2} \mathfrak{m}_n(\phi^{\otimes n}) = 0. \tag{3.23}$$

If the products are cyclic, these equations admit a primitive

$$W = \sum_{n=2}^{\infty} \frac{(-1)^{n(n+1)/2}}{n+1} \langle \phi, \mathfrak{m}_n(\phi^{\otimes n}) \rangle, \tag{3.24}$$

where

$$\langle \quad , \quad \rangle \colon \mathcal{C} \to \mathbb{C}$$

is a bilinear form on $\mathcal{C}$ compatible with the DGA structure. The cyclicity property reads

$$\langle c_1, \mathfrak{m}_n(c_2, \ldots, c_{n+1}) \rangle = (-1)^{n|c_2|+1} \langle c_2, \mathfrak{m}_n(c_3, \ldots, c_{n+1}, c_1) \rangle.$$

Let us now examine the above deformation problem in the presence of an orientifold projection. Suppose we have an involution $\tau \colon \mathcal{C} \to \mathcal{C}$ such that the following conditions are satisfied:

$$\begin{aligned} \tau(Q(f)) &= Q(\tau(f)), \\ \tau(fg) &= -(-1)^{|f||g|} \tau(g)\tau(f). \end{aligned} \tag{3.25}$$

As explained below equation (3.12), in this case we would like to study deformations

$$Q_{\mathrm{def}} = Q + f_1(\phi) + f_2(\phi) + \cdots$$

of Q such that

$$\tau(f_n(\phi)) = f_n(\phi) \tag{3.26}$$

for all $n \geq 1$.

In order to set this problem in the proper algebraic context, note that the DG algebra $\mathcal{C}$ decomposes into a direct sum of τ-invariant and anti-invariant parts

$$\mathcal{C} \simeq \mathcal{C}^+ \oplus \mathcal{C}^-. \tag{3.27}$$

There is a similar decomposition

$$H = H^+ \oplus H^- \tag{3.28}$$

for the Q-cohomology.

Conditions (3.25) imply that Q preserves $\mathcal{C}^\pm$, but the associative algebra product is not compatible with the decomposition (3.27). There is however another algebraic structure which is preserved by τ, namely the graded commutator

$$[f, g] = fg - (-1)^{|f||g|}\, gf. \tag{3.29}$$

This follows immediately from the second equation in (3.25). The graded commutator (3.29) defines a differential graded Lie algebra structure on $\mathcal{C}$. By restriction, it also defines a DG Lie algebra structure on the invariant part $\mathcal{C}^+$. In this context, our problem reduces to the deformation theory of the restriction $Q^+ = Q|_{\mathcal{C}^+}$ as a differential operator on $\mathcal{C}^+$.

Fortunately, this problem can be treated by analogy with the previous case, except that we have to replace A_∞ structures by L_∞ structures, see, for example [33, 34, 54]. In particular, the obstructions to the deformations of Q^+ can be systematically encoded in a minimal L_∞ model, and one can similarly define a superpotential if certain cyclicity conditions are satisfied.

Note that any associative DG algebra can be naturally endowed with a DG Lie algebra structure using the graded commutator (3.29). In this case, the A_∞ and the L_∞ approach to the deformation of Q are equivalent [33] and they yield the same superpotential. However, the L_∞ approach is compatible with the involution, while the A_∞ approach is not.

To summarize this discussion, we have a DG Lie algebra on $\mathcal{C}$ which induces a DG Lie algebra of Q. The construction of a minimal L_∞ model for $\mathcal{C}$ requires the same data (i), (ii) as in the case of a minimal A_∞ model and yields the same F-term equations and the same superpotential. In order to determine the F-term equations and superpotential for the invariant part $\mathcal{C}^+$, we need again a set of data (i), (ii) as described above (3.13). This data can be naturally obtained by restriction from $\mathcal{C}$, provided that the propagator η in equation (3.13) can be chosen compatible with the involution τ, i.e.,

$$\tau(\eta(f)) = \eta(\tau(f)).$$

This condition is easily satisfied in geometric situations, hence we will assume that this is the case from now on. Then the propagator $\eta^+ : \mathcal{C}^+ \to \mathcal{C}^+[-1]$ is obtained by restricting η to the invariant part $\eta^+ = \eta|_{\mathcal{C}^+}$. Given this data, we construct a minimal L_∞ model for the DGL algebra $\mathcal{C}^+$, which yields F-term equations and, if the cyclicity condition is satisfied, a superpotential W^+.

Theorem 3.3. *The superpotential W^+ obtained by constructing the minimal L_∞ model for the DGL $\mathcal{C}^+$ is equal to the restriction of the superpotential W corresponding to $\mathcal{C}$ evaluated on τ-invariant field configurations:*

$$W^+ = W|_{H^+}. \tag{3.30}$$

In the remaining part of this section, we will give a formal argument for this claim. According to [54], the data (i), (ii) above equation (3.13) also determine an L_∞ structure on $\mathcal{H}$ as follows. First we construct a series of linear maps

$$\rho_n \colon \mathcal{C}^{\otimes n} \to \mathcal{C}[2-n], \quad n \geq 2$$

by anti-symmetrizing (in the graded sense) the maps (3.21). That is, the recursion relation becomes

$$\begin{aligned}\rho_n(c_1, \ldots, c_n) = & \sum_{\sigma \in \mathrm{Sh}(n-1,1)} (-1)^{n-1+|\sigma|} e(\sigma) \\ & \times [\eta\rho_{n-1}(c_{\sigma(1)}, \ldots, c_{\sigma(n-1)}), c_{\sigma(n)}] \\ & - \sum_{\sigma \in \mathrm{Sh}(1,n)} (-1)^{n|c_1|+|\sigma|} e(\sigma)[c_1, \eta\rho_{n-1}(c_{\sigma(2)}] \\ & - \sum_{\substack{k+l=n \\ k,l\geq 2}} \sum_{\sigma \in \mathrm{Sh}(k,n)} (-1)^{r+|\sigma|} e(\sigma)[\eta\rho_k(c_{\sigma(1)}, \ldots, c_{\sigma(k)}), \\ & \times \eta\rho_l(c_{\sigma(k+1)}, \ldots, c_{\sigma(n)})],\end{aligned} \tag{3.31}$$

where $\mathrm{Sh}(k,n)$ is the set of all permutations $\sigma \in S_n$ such that

$$\sigma(1) < \cdots < \sigma(k) \quad \text{and} \quad \sigma(k+1) < \cdots < \sigma(n)$$

and $|\sigma|$ is the signature of a permutation $\sigma \in S_n$. The symbol $e(\sigma)$ denotes the Koszul sign defined by

$$c_{\sigma(1)} \wedge \cdots \wedge c_{\sigma(n)} = (-1)^{|\sigma|} e(\sigma) c_1 \wedge \cdots \wedge c_n.$$

Then we define the L_∞ products

$$\mathfrak{l}_n \colon \mathcal{H}^{\otimes n} \to \mathcal{H}$$

by

$$\mathfrak{l}_1 = \eta, \quad \mathfrak{l}_n = \Pi\rho_n. \tag{3.32}$$

One can show that these products satisfy a series of higher Jacobi identities analogous to the defining associativity conditions of A_∞ structures. If the conditions (3.14) are also satisfied, the resulting L_∞ structure is a minimal model for the DGL algebra $\mathcal{C}$.

Finally, note that the A_∞ products (3.22) and the L_∞ products (3.32) are related by

$$\mathfrak{l}_n(c_1, \ldots, c_n) = \sum_{\sigma \in S_n} (-1)^{|\sigma|} e(\sigma) \mathfrak{m}_n(c_{\sigma(1)}, \ldots, c_{\sigma(n)}). \tag{3.33}$$

In particular, one can rewrite the F-term equations (3.23) and the superpotential (3.24) in terms of L_∞ products [33, 34].

The construction of the minimal L_∞ model of the invariant part $\mathcal{C}^+$ is analogous. Since we are working under assumption that the propagator η^+ is the restriction of η to $\mathcal{C}^+$, it is clear that the linear maps $\rho_n^+(c_1, \ldots, c_n)$ are also equal to the restriction $\rho_n|_{(\mathcal{C}^+)^n}$. The same will be true for the products $\mathfrak{l}_n^+$, i.e.,

$$\mathfrak{l}_n^+ = \mathfrak{l}_n|_{(H^+)^n}.$$

Therefore, the F-term equations and the superpotential in the orientifold model can be obtained indeed by restriction to the invariant part.

Now that we have the general machinery at hand, we can turn to concrete examples of superpotential computations.

4 Computations for obstructed curves

In this section, we perform detailed computations of the superpotential for D-branes wrapping holomorphic curves in Calabi–Yau orientifolds.

So far we have relied on the Dolbeault cochain model, which serves as a good conceptual framework for our constructions. However, a Čech cochain model is clearly preferred for computational purposes [39]. The simple prescription found above for the orientifold superpotential allows us to switch from the Dolbeault to the Čech model with little effort. Using the same definition for the action of the orientifold projection P on locally free complexes $\mathfrak{E}$, we will adopt a cochain model of the form

$$\mathcal{C}(P(\mathfrak{E}), \mathfrak{E}) = \check{C}(\mathfrak{U}, \mathscr{H}om_X(P(\mathfrak{E}), \mathfrak{E})), \tag{4.1}$$

where $\mathfrak{U}$ is a fine open cover of X. The differential Q is given by

$$Q(f) = \delta(f) + (-1)^{c(f)} \mathfrak{d}(f), \tag{4.2}$$

where δ is the Čech differential, $\mathfrak{d}$ is the differential of the local Hom complex and $c(f)$ is the Čech degree of f.

In order to obtain a well-defined involution on the complex (4.1), we have to choose the open cover $\mathfrak{U}$ so that the holomorphic involution $\sigma\colon X \to X$ maps any open set $U \in \mathfrak{U}$ isomorphically to another open set $U_{s(\alpha)} \in \mathfrak{U}$, where s is an involution on the set of indices $\{\alpha\}$. Moreover, the holomorphic involution should also be compatible with intersections. That is, if $U_\alpha, U_\beta \in \mathfrak{U}$ are mapped to $U_{s(\alpha)}, U_{s(\beta)} \in \mathfrak{U}$, then $U_{\alpha\beta}$ should be mapped isomorphically to $U_{s(\alpha)s(\beta)}$. Analogous properties should hold for arbitrary multiple intersections. Granting such a choice of a fine open cover, we have a natural involution J_*P acting on the cochain complex (4.1), defined as in (3.4).

According to the prescription derived in the previous section, the orientifold superpotential can be obtained by applying the computational scheme of [39] to invariant Q-cohomology representatives. Since the computation depends only on the infinitesimal neighborhood of the curve, it suffices to consider local Calabi–Yau models as in [39]. We will consider two representative cases, namely obstructed $(0,-2)$ curves and local conifolds, i.e., $(-1,-1)$ curves.

4.1 Obstructed $(0,-2)$ curves in $O5$ models

In this case, the local Calabi–Yau X can be covered by two coordinate patches (x, y_1, y_2), (w, z_1, z_2) with transition functions

$$\begin{aligned} w &= x^{-1}, \\ z_1 &= x^2 y_1 + x y_2^n, \\ z_2 &= y_2. \end{aligned} \tag{4.3}$$

The $(0,-2)$ curve is given by the equations

$$C\colon \quad y_1 = y_2 = 0 \quad \text{resp.} \quad z_1 = z_2 = 0 \tag{4.4}$$

in the two patches. The holomorphic involution acts as

$$\begin{aligned} (x, y_1, y_2) &\mapsto (x, -y_1, -y_2), \\ (w, z_1, z_2) &\mapsto (w, -z_1, -z_2). \end{aligned} \tag{4.5}$$

This is compatible with the transition functions if and only if n is odd. We will assume that this is the case from now on. Using (2.31), the Chan–Paton bundles

$$V_N = \mathcal{O}_C(-1)^{\oplus N} \tag{4.6}$$

define invariant D-brane configurations under the orientifold projection.

The on-shell open string states are in one-to-one correspondence with elements of the global Ext group $\mathrm{Ext}^1(i_*V_N, i_*V_N)$. Given two bundles V, W

supported on a curve $i\colon C \hookrightarrow X$, there is a spectral sequence [31]

$$E_2^{p,q} = H^p(C, V^\vee \otimes W \otimes \Lambda^q N_{C/X}) \implies \mathrm{Ext}_X^{p+q}(i_*V, i_*W) \tag{4.7}$$

which degenerates at E_2. This yields

$$\mathrm{Ext}^1(i_*\mathcal{O}_C(-1), i_*\mathcal{O}_C(-1)) \simeq H^0(C, N_{C/X}) = \mathbb{C},$$

since $N_{C/X} \simeq \mathcal{O}_C \oplus \mathcal{O}_C(-2)$. Therefore, a D5-brane with multiplicity $N = 1$ has a single normal deformation. For higher multiplicity, the normal deformations will be parameterized by an $(N \times N)$ complex matrix.

In order to apply the computational algorithm developed in Section 3, we have to find a locally free resolution $\mathfrak{E}$ of $i_*\mathcal{O}_C(-1)$ and an explicit generator of

$$\mathrm{Ext}^1(i_*\mathcal{O}_C(-1), i_*\mathcal{O}_C(-1)) \simeq \mathrm{Ext}^1(P(\mathfrak{E}), \mathfrak{E})$$

in the cochain space $\check{C}(\mathfrak{U}, \mathscr{H}om(P(\mathfrak{E}), \mathfrak{E}))$. We take $\mathfrak{E}$ to be the locally free resolution from [39] multiplied by $\mathcal{O}_C(-1)$, i.e.,

$$0 \longrightarrow \mathcal{O}(-1) \xrightarrow{\begin{pmatrix} y_2 \\ -1 \\ x \end{pmatrix}} \begin{array}{c} \mathcal{O}(-1) \\ \oplus \\ \mathcal{O} \\ \oplus \\ \mathcal{O} \end{array} \xrightarrow{\begin{pmatrix} 1 & y_2 & 0 \\ -x & 0 & y_2 \\ -y_2^{n-1} & -s & -y_1 \end{pmatrix}} \begin{array}{c} \mathcal{O} \\ \oplus \\ \mathcal{O} \\ \oplus \\ \mathcal{O}(-1) \end{array} \xrightarrow{(\,s\ y_1\ y_2\,)} \mathcal{O}(-1). \tag{4.8}$$

The quasi-isomorphism $\psi\colon \mathfrak{E} \to P(\mathfrak{E})$ is given by

$$\begin{array}{ccccc}
\begin{array}{c} \mathcal{O}(-1) \\ \oplus \\ \mathcal{O}^{\oplus 2} \end{array} & \xrightarrow{\begin{pmatrix} 1 & y_2 & 0 \\ -x & 0 & y_2 \\ -y_2^{n-1} & -s & -y_1 \end{pmatrix}} & \begin{array}{c} \mathcal{O}^{\oplus 2} \\ \oplus \\ \mathcal{O}(-1) \end{array} & \xrightarrow{(\,s\ y_1\ y_2\,)} & \mathcal{O}(-1) \\
\Big\downarrow{\scriptstyle (\,0\ x\ 1\,)} & & \Big\downarrow{\scriptstyle \begin{pmatrix} 0 & y_2^{n-1} & -x \\ -y_2^{n-1} & 0 & -1 \\ x & 1 & 0 \end{pmatrix}} & & \Big\downarrow{\scriptstyle \begin{pmatrix} 0 \\ x \\ 1 \end{pmatrix}} \\
\mathcal{O}(1) & \xrightarrow{\begin{pmatrix} s \\ y_1 \\ y_2 \end{pmatrix}} & \begin{array}{c} \mathcal{O}^{\oplus 2} \\ \oplus \\ \mathcal{O}(1) \end{array} & \xrightarrow{\begin{pmatrix} 1 & -x & -y_2^{n-1} \\ -y_2 & 0 & s \\ 0 & -y_2 & y_1 \end{pmatrix}} & \begin{array}{c} \mathcal{O}(1) \\ \oplus \\ \mathcal{O}^{\oplus 2} \end{array}
\end{array} \tag{4.9}$$

Note that ψ satisfies the symmetry condition (3.10) with $\omega = 0$, which in this case reduces to

$$\sigma^*(\psi_{2-l})^\vee = (-1)^l \psi_l. \tag{4.10}$$

We are searching for a generator $\mathsf{c} \in \check{C}(\mathfrak{U}, \mathscr{H}om(P(\mathfrak{E}), \mathfrak{E}))$ of the form $\mathsf{c} = \mathsf{c}^{1,0} + \mathsf{c}^{0,1}$ for two homogenous elements

$$\mathsf{c}^{p,1-p} \in \check{C}^p(\mathfrak{U}, \mathscr{H}om^{1-p}(P(\mathfrak{E}), \mathfrak{E})), \quad p = 0, 1.$$

The cocycle condition $Q(\mathsf{c}) = 0$ is equivalent to

$$\begin{aligned} \mathfrak{d}\mathsf{c}^{0,1} &= \delta\mathsf{c}^{1,0} = 0, \\ Q(\mathsf{c}^{0,1} + \mathsf{c}^{1,0}) &= \delta\mathsf{c}^{0,1} - \mathfrak{d}\mathsf{c}^{1,0} = 0. \end{aligned} \tag{4.11}$$

A solution to these equations is given by

$$\mathsf{c}^{1,0} := \begin{array}{ccccc} \mathcal{O}(1) & \longrightarrow & \begin{matrix}\mathcal{O}^{\oplus 2}\\ \oplus \\ \mathcal{O}(1)\end{matrix} & \longrightarrow & \begin{matrix}\mathcal{O}(1)\\ \oplus \\ \mathcal{O}^{\oplus 2}\end{matrix} \\ \Big\downarrow {\scriptstyle \begin{pmatrix} x^{-1} \\ 0 \\ 0 \end{pmatrix}_{01}} & & \Big\downarrow {\scriptstyle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -x^{-1}y_2^{n-2} \end{pmatrix}_{01}} & & \Big\downarrow {\scriptstyle (x^{-1}\ 0\ 0)_{01}} \\ \begin{matrix}\mathcal{O}(-1)\\ \oplus \\ \mathcal{O}^{\oplus 2}\end{matrix} & \longrightarrow & \begin{matrix}\mathcal{O}^{\oplus 2}\\ \oplus \\ \mathcal{O}(-1)\end{matrix} & \longrightarrow & \mathcal{O}(-1) \end{array}$$

$$\mathsf{c}^{0,1} := \begin{array}{ccc} \mathcal{O}(1) & \longrightarrow & \begin{matrix}\mathcal{O}^{\oplus 2}\\ \oplus \\ \mathcal{O}(1)\end{matrix} \\ \Big\downarrow {\scriptstyle \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}_0 + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_1} & & \Big\downarrow {\scriptstyle (0\ 1\ 0)_0 + (-1\ 0\ 0)_1} \\ \begin{matrix}\mathcal{O}^{\oplus 2}\\ \oplus \\ \mathcal{O}(-1)\end{matrix} & \longrightarrow & \mathcal{O}(-1) \end{array} \tag{4.12}$$

These satisfy the symmetry conditions

$$J_* P(\mathsf{c}^{p,1-p}) = -(-1)^\omega \mathsf{c}^{p,1-p}, \quad p = 0, 1, \tag{4.13}$$

For multiplicity $N > 1$, we have the locally free resolution $\mathfrak{E}_N = \mathfrak{E} \otimes \mathbb{C}^N$. The quasi-isomorphism $\psi_N : \mathfrak{E}_N \to P(\mathfrak{E}_N)$ is of the form $\psi_N = \psi \otimes M$, where $M \in \mathcal{M}_N(\mathbb{C})$ is an $N \times N$ complex matrix. Note that ψ_N induces the isomorphism (2.32) in cohomology. Moreover, we have

$$\sigma^*(\psi_{N,m-l})^\vee = (-1)^{l+\omega}\psi_{N,l}.$$

Referring back to (4.10), we see that this last equation constrains the matrix M:

$$\omega = \begin{cases} 0, & \text{if } M = M^{\mathrm{tr}}, \\ 1, & \text{if } M = -M^{\mathrm{tr}}. \end{cases} \tag{4.14}$$

The first case corresponds to an $SO(N)$ gauge group, while the second case corresponds to $Sp(N/2)$ (N even). This confirms the correlation between the symmetry of ψ_N and the SO/Sp projection, as we alluded to after (3.10).

The infinitesimal deformations of the D-brane are now parameterized by a matrix-valued field

$$\phi = \mathsf{C}(\mathsf{c}^{1,0} + \mathsf{c}^{0,1}),$$

where $\mathsf{C} \in \mathcal{M}_N(\mathbb{C})$ is the $N \times N$ Chan–Paton matrix. Taking (4.13) into account, invariance under the orientifold projection yields the following condition on C

$$\mathsf{C} = -(-1)^\omega \mathsf{C}^{\mathrm{tr}}. \tag{4.15}$$

For $\omega = 1$, this condition does not look like the usual one defining the Lie algebra of $Sp(N/2)$ because we are working in a non-usual basis of fields, namely $\mathcal{C}(P(\mathfrak{E}_N), \mathfrak{E}_N)$. By composing with the quasi-isomorphism ψ_N, we find the Chan–Paton matrix in $\mathcal{C}(P(\mathfrak{E}_N), P(\mathfrak{E}_N))$ to be $M\mathsf{C}$. By performing a change of basis in the space of Chan–Paton indices, we can choose M to be

$$M = \begin{cases} \mathrm{I}_N, & \text{if } \omega = 0, \\ i\begin{pmatrix} & \mathrm{I}_{N/2} \\ -\mathrm{I}_{N/2} & \end{pmatrix}, & \text{if } \omega = 1, \end{cases}$$

and so the Chan–Paton matrices satisfy the well-known conditions [48]

$$(M\mathsf{C})^{\mathrm{tr}} = -(M\mathsf{C}), \quad \text{for } \omega = 0,$$
$$(M\mathsf{C})^{\mathrm{tr}} = -M(M\mathsf{C})M, \quad \text{for } \omega = 1.$$

The superpotential is determined by the A_∞ products (3.22) evaluated on ϕ. According to Theorem 3.3, the final result is obtained by the superpotential of the underlying unprojected theory evaluated on invariant field configurations. Therefore, the computations are identical in both cases ($\omega = 0, 1$) and the superpotential is essentially determined by the A_∞ products of a single D-brane with multiplicity $N = 1$.

Proceeding by analogy with [39], let us define the cocycles

$$\mathsf{a}_p \in \check{C}^1(\mathfrak{U}, \mathscr{H}om^0(P(\mathfrak{E}), \mathfrak{E})), \quad \mathsf{b}_p \in \check{C}^1(\mathfrak{U}, \mathscr{H}om^1(P(\mathfrak{E}), \mathfrak{E}))$$

as follows

$$\mathsf{a}_p := \begin{array}{ccccc} \mathcal{O}(1) & \longrightarrow & \begin{matrix}\mathcal{O}^{\oplus 2}\\ \oplus \\ \mathcal{O}(1)\end{matrix} & \longrightarrow & \begin{matrix}\mathcal{O}(1)\\ \oplus \\ \mathcal{O}^{\oplus 2}\end{matrix} \\ \Big\downarrow {\scriptstyle (0)_{01}} & & \Big\downarrow {\scriptstyle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -x^{-1}y_2^p \end{pmatrix}_{01}} & & \Big\downarrow {\scriptstyle (0)_{01}} \\ \begin{matrix}\mathcal{O}(-1)\\ \oplus \\ \mathcal{O}^{\oplus 2}\end{matrix} & \longrightarrow & \begin{matrix}\mathcal{O}^{\oplus 2}\\ \oplus \\ \mathcal{O}(-1)\end{matrix} & \longrightarrow & \mathcal{O}(-1) \end{array} \tag{4.16}$$

$$\mathsf{b}_p := \begin{array}{ccc} \mathcal{O}(1) & \longrightarrow & \begin{matrix}\mathcal{O}^{\oplus 2}\\ \oplus \\ \mathcal{O}(1)\end{matrix} \\ \Big\downarrow {\scriptstyle \begin{pmatrix} 0 \\ 0 \\ x^{-1}y_2^p \end{pmatrix}_{01}} & & \Big\downarrow {\scriptstyle \begin{pmatrix} 0 & 0 & -x^{-1}y_2^p \end{pmatrix}_{01}} \\ \begin{matrix}\mathcal{O}^{\oplus 2}\\ \oplus \\ \mathcal{O}(-1)\end{matrix} & \longrightarrow & \mathcal{O}(-1) \end{array}$$

One shows by direct computation that they satisfy the relations

$$\begin{aligned} \mathsf{b}_p &= Q(\mathsf{a}_{p-1}), \\ \mathsf{b}_p &= \mathsf{c} \star_\psi \mathsf{a}_p + \mathsf{a}_p \star_\psi \mathsf{c}. \end{aligned} \tag{4.17}$$

Moreover, we have

$$\begin{aligned} \mathsf{c} \star_\psi \mathsf{c} &= \mathsf{b}_{n-2}, \\ \mathsf{b}_p \star_\psi \mathsf{b}_p &= 0, \end{aligned} \tag{4.18}$$

for any p. Therefore, the computation of the A_∞ products is identical to [39]. We find only one non-trivial product

$$\mathfrak{m}_n(\mathsf{c}, \ldots, \mathsf{c}) = -(-1)^{n(n-1)/2}\mathsf{b}_0. \tag{4.19}$$

If we further compose with c, we obtain

$$\mathsf{b}_0 \star_\psi \mathsf{c} := \quad \begin{array}{c} \mathcal{O}(1) \\ \Big\downarrow {\scriptstyle (-x^{-1})_{01}} \\ \mathcal{O}(-1) \end{array}$$

which is a generator of $\mathrm{Ext}^3(i_*\mathcal{O}_C(-1), i_*\mathcal{O}_C(-1))$. Therefore, we obtain a superpotential of the form

$$W = \frac{(-1)^n}{n+1}\,\mathsf{C}^{n+1}$$

where C satisfies the invariance condition (4.15).

4.2 Local conifold $O3/O7$ models

In this case, the local Calabi–Yau threefold X is isomorphic to the crepant resolution of a conifold singularity, i.e., the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$. X can be covered with two coordinate patches (x, y_1, y_2), (w, z_1, z_2) with transition functions

$$\begin{aligned} w &= x^{-1}, \\ z_1 &= xy_1, \\ z_2 &= xy_2. \end{aligned} \tag{4.20}$$

The $(-1,-1)$ curve C is given by

$$x = y_1 = y_2 = 0, \quad w = z_1 = z_2 = 0 \tag{4.21}$$

and the holomorphic involution takes

$$\begin{aligned} (x, y_1, y_2) &\mapsto (-x, -y_1, -y_2), \\ (w, z_1, z_2) &\mapsto (-w, z_1, z_2). \end{aligned} \tag{4.22}$$

In this case, we have an $O3$ plane at

$$x = y_1 = y_2 = 0$$

and a noncompact $O7$ plane at $w = 0$. The invariant D5-brane configurations are of the form $\mathcal{E}_n^{\oplus N}$, where

$$\mathcal{E}_n = i_*\mathcal{O}_C(-1+n) \oplus i_*(\sigma^*\mathcal{O}_C(-1-n))[1], \quad n \geq 1. \tag{4.23}$$

We have a global Koszul resolution of the structure sheaf $\mathcal{O}_C$

$$0 \longrightarrow \mathcal{O}(2) \xrightarrow{\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}} \mathcal{O}(1)^{\oplus 2} \xrightarrow{(y_1\ y_2)} \mathcal{O} \longrightarrow 0. \tag{4.24}$$

Therefore, the locally free resolution of $\mathcal{E}_n$ is a complex $\mathfrak{E}_n$ of the form

$$\sigma^*\mathcal{O}(1-n) \xrightarrow{\begin{pmatrix} 0 \\ y_1 \\ y_2 \end{pmatrix}} \begin{matrix} \mathcal{O}(1+n) \\ \oplus \\ \sigma^*\mathcal{O}(-n)^{\oplus 2} \end{matrix} \xrightarrow{\begin{pmatrix} -y_2 & 0 & 0 \\ y_1 & 0 & 0 \\ 0 & y_2 & -y_1 \end{pmatrix}} \begin{matrix} \mathcal{O}(n)^{\oplus 2} \\ \oplus \\ \sigma^*\mathcal{O}(-1-n) \end{matrix} \xrightarrow{(\,y_1\ y_2\ 0\,)} \mathcal{O}(-1+n) \tag{4.25}$$

in which the last term to the right has degree 0, and the last term to the left has degree -3. The quasi-isomorphism $\psi\colon \mathfrak{E}_n \to P(\mathfrak{E}_n)$ is given by

$$\begin{array}{ccccccc}
\sigma^*\mathcal{O}(1-n) & \xrightarrow{\begin{pmatrix} 0 \\ y_1 \\ y_2 \end{pmatrix}} & \begin{matrix} \mathcal{O}(1+n) \\ \oplus \\ \sigma^*\mathcal{O}(-n)^{\oplus 2} \end{matrix} & \xrightarrow{\begin{pmatrix} -y_2 & 0 & 0 \\ y_1 & 0 & 0 \\ 0 & y_2 & -y_1 \end{pmatrix}} & \begin{matrix} \mathcal{O}(n)^{\oplus 2} \\ \oplus \\ \sigma^*\mathcal{O}(-1-n) \end{matrix} & \xrightarrow{(\,y_1\ y_2\ 0\,)} & \mathcal{O}(-1+n) \\
\Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}} & & \Big\downarrow{\scriptstyle \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}} & & \Big\downarrow{\scriptstyle 1} \\
\sigma^*\mathcal{O}(1-n) & \xrightarrow{\begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix}} & \begin{matrix} \sigma^*\mathcal{O}(-n)^{\oplus 2} \\ \oplus \\ \mathcal{O}(1+n) \end{matrix} & \xrightarrow{\begin{pmatrix} y_2 & -y_1 & 0 \\ 0 & 0 & -y_2 \\ 0 & 0 & y_1 \end{pmatrix}} & \begin{matrix} \sigma^*\mathcal{O}(-1-n) \\ \oplus \\ \mathcal{O}(n)^{\oplus 2} \end{matrix} & \xrightarrow{(\,0\ y_1\ y_2\,)} & \mathcal{O}(-1+n)
\end{array} \tag{4.26}$$

and satisfies $\sigma^*(\psi_{3-l})^\vee = \psi_l$, that is, the symmetry condition (3.10) with $\omega = 0$. The on-shell open string states $\mathrm{Ext}^1_X(\mathfrak{E}_n, \mathfrak{E}_n)$ are computed by the spectral sequence (4.7):

$$\begin{aligned}
\mathrm{Ext}^1_X(\mathcal{O}_C(-1+n), \mathcal{O}_C(-1+n)) &= 0, \\
\mathrm{Ext}^1_X(\sigma^*\mathcal{O}_C(-1-n)[1], \sigma^*\mathcal{O}_C(-1-n)[1]) &= 0, \\
\mathrm{Ext}^1_X(\mathcal{O}_C(-1+n), \sigma^*\mathcal{O}_C(-1-n)[1]) &= \mathbb{C}^{4n}, \\
\mathrm{Ext}^1_X(\sigma^*\mathcal{O}_C(-1-n)[1], \mathcal{O}_C(-1+n)) &= \mathbb{C}^{2n+1},
\end{aligned} \tag{4.27}$$

where in the last two lines we have used the condition $n \geq 1$.

To compute the superpotential, we work with the cochain model $\check{C}(\mathfrak{U}, \mathscr{H}om(P(\mathfrak{E}_n), \mathfrak{E}_n))$. The direct sum of the above Ext groups represents the degree 1 cohomology of this complex with respect to the differential (4.2). The first step is to find explicit representatives for all degree 1 cohomology classes with well-defined transformation properties under the orientifold projection. We list all generators below on a case by case basis.

a) $\mathrm{Ext}^1(\sigma^*\mathcal{O}_C(-1-n)[1], \mathcal{O}_C(-1+n))$

We have $2n+1$ generators $\mathsf{a}_i \in \check{C}^0(\mathfrak{U}, \mathscr{H}om^1(\mathcal{P}(\mathfrak{E}_n), \mathfrak{E}_n), \quad i = 0, \dots, 2n$, given by

$$\mathsf{a}_i := x^i \mathsf{a}, \tag{4.28}$$

where

$$\mathsf{a} := \quad \begin{array}{ccccc} \sigma^*\mathcal{O}(1-n) & \longrightarrow & \begin{array}{c}\sigma^*\mathcal{O}(-n)^{\oplus 2} \\ \oplus \\ \mathcal{O}(1+n)\end{array} & \longrightarrow & \begin{array}{c}\sigma^*\mathcal{O}(-1-n) \\ \oplus \\ \mathcal{O}(n)^{\oplus 2}\end{array} \\ \Big\downarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & & \Big\downarrow \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & & \Big\downarrow \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \\ \begin{array}{c}\mathcal{O}(1+n) \\ \oplus \\ \sigma^*\mathcal{O}(-n)^{\oplus 2}\end{array} & \longrightarrow & \begin{array}{c}\mathcal{O}(n)^{\oplus 2} \\ \oplus \\ \sigma^*\mathcal{O}(-1-n)\end{array} & \longrightarrow & \mathcal{O}(-1+n) \end{array} \tag{4.29}$$

Note that we have written down the expressions of the generators only in the U_0 patch.[5] The transformation properties under the orientifold projection are

$$J_* P(\mathsf{a}_i) = -(-1)^{i+\omega} \mathsf{a}_i, \quad 0 \le i \le 2n. \tag{4.30}$$

b) $\mathrm{Ext}^1(\mathcal{O}_C(-1+n), \sigma^*\mathcal{O}_C(-1-n)[1])$

We have $4n$ generators $\mathsf{b}_i, \mathsf{c}_i \in \check{C}^1(\mathfrak{U}, \mathscr{H}om^0(P(\mathfrak{F}_n), \mathfrak{F}_n), \quad i = 1, \dots, 2n$, given by

$$\mathsf{b}_i := x^{-i}\mathsf{b}, \quad \mathsf{c}_i := x^{-i}\mathsf{c}, \tag{4.31}$$

where

$$\mathsf{b} := \quad \begin{array}{ccc} \begin{array}{c}\sigma^*\mathcal{O}(-n)^{\oplus 2} \\ \oplus \\ \mathcal{O}(1+n)\end{array} & \longrightarrow & \begin{array}{c}\sigma^*\mathcal{O}(-1-n) \\ \oplus \\ \mathcal{O}(n)^{\oplus 2}\end{array} \\ \Big\downarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}_{01} & & \Big\downarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}_{01} \\ \begin{array}{c}\mathcal{O}(1+n) \\ \oplus \\ \sigma^*\mathcal{O}(-n)^{\oplus 2}\end{array} & \longrightarrow & \begin{array}{c}\mathcal{O}(n)^{\oplus 2} \\ \oplus \\ \sigma^*\mathcal{O}(-1-n)\end{array} \end{array} \tag{4.32}$$

[5]The expressions in the U_1 patch can be obtained using the transition functions (4.20) since the a_i are Čech closed. They will not be needed in the computation.

$$
\mathsf{c} := \quad
\begin{array}{ccc}
\begin{matrix}\sigma^*\mathcal{O}(-n)^{\oplus 2}\\ \oplus \\ \mathcal{O}(1+n)\end{matrix} & \longrightarrow & \begin{matrix}\sigma^*\mathcal{O}(-1-n)\\ \oplus \\ \mathcal{O}(n)^{\oplus 2}\end{matrix} \\
\Big\downarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}_{01} & & \Big\downarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{01} \\
\begin{matrix}\mathcal{O}(1+n)\\ \oplus \\ \sigma^*\mathcal{O}(-n)^{\oplus 2}\end{matrix} & \longrightarrow & \begin{matrix}\mathcal{O}(n)^{\oplus 2}\\ \oplus \\ \sigma^*\mathcal{O}(-1-n)\end{matrix}
\end{array}
\tag{4.33}
$$

The action of the orientifold projection is

$$
J_*P(\mathsf{b}_i) = (-1)^{i+\omega}\mathsf{b}_i, \quad J_*P(\mathsf{c}_i) = (-1)^{i+\omega}\mathsf{c}_i. \tag{4.34}
$$

For multiplicity $N \geq 1$, we work as in the last subsection, taking the locally free resolution $\mathfrak{E}_{n,N} = \mathfrak{E}_n \otimes \mathbb{C}^N$, together with the quasi-isomorphism $\psi_N :$ $\mathfrak{E}_{n,N} \to P(\mathfrak{E}_{n,N})$; $\psi_N = \psi \otimes M$. Again, M is a symmetric matrix for $\omega = 0$ and antisymmetric for $\omega = 1$. A general invariant degree 1 cocycle ϕ will be a linear combination

$$
\phi = \sum_{i=0}^{2n} \mathsf{A}^i \mathsf{a}_i + \sum_{i=1}^{2n} (\mathsf{B}^i \mathsf{b}_i + \mathsf{C}^i \mathsf{c}_i), \tag{4.35}
$$

where A^i, B^i, C^i are $N \times N$ matrices satisfying

$$
(\mathsf{A}^i)^{\mathrm{tr}} = -(-1)^{i+\omega}\mathsf{A}^i, \quad (\mathsf{B}^i)^{\mathrm{tr}} = (-1)^{i+\omega}\mathsf{B}^i, \quad (\mathsf{C}^i)^{\mathrm{tr}} = (-1)^{i+\omega}\mathsf{C}^i. \tag{4.36}
$$

In the following, we will let the indices $i, j, k, \ldots$ run from 0 to $2n$ with the convention $\mathsf{B}^0 = \mathsf{C}^0 = 0$.

The multiplication table of the above generators with respect to the product (3.5) is

$$
\begin{aligned}
&\mathsf{a}_i \star_\psi \mathsf{a}_j = \mathsf{b}_i \star_\psi \mathsf{b}_j = \mathsf{c}_i \star_\psi \mathsf{c}_j = 0, \\
&\mathsf{b}_i \star_\psi \mathsf{c}_j = \mathsf{c}_i \star_\psi \mathsf{b}_j = 0.
\end{aligned} \tag{4.37}
$$

The remaining products are all Q-exact:

$$
\begin{aligned}
\mathsf{a}_i \star_\psi \mathsf{b}_j &= Q(f_2(\mathsf{a}_i, \mathsf{b}_j)), \\
\mathsf{b}_i \star_\psi \mathsf{a}_j &= Q(f_2(\mathsf{b}_i, \mathsf{a}_j)),
\end{aligned}
$$

as required in (3.15). Let us show a sample computation.

$$\mathsf{b}_i \star \mathsf{a}_j = \begin{array}{ccc} \sigma^*\mathcal{O}(1-n) & \longrightarrow & \begin{array}{c}\sigma^*\mathcal{O}(-n)^{\oplus 2}\\ \oplus \\ \mathcal{O}(1+n)\end{array} \\ \Big\downarrow {\scriptstyle x^{-i+j}\begin{pmatrix}0\\-1\\0\end{pmatrix}_{01}} & & \Big\downarrow {\scriptstyle x^{-i+j}\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}_{01}} \\ \begin{array}{c}\mathcal{O}(1+n)\\ \oplus \\ \sigma^*\mathcal{O}(-n)^{\oplus 2}\end{array} & \longrightarrow & \begin{array}{c}\mathcal{O}(n)^{\oplus 2}\\ \oplus \\ \sigma^*\mathcal{O}(-1-n)\end{array} \end{array} \tag{4.38}$$

For $j \geq i$,

$$f_2(\mathsf{b}_i, \mathsf{a}_j) = \begin{array}{ccc} \sigma^*\mathcal{O}(1-n) & \longrightarrow & \begin{array}{c}\sigma^*\mathcal{O}(-n)^{\oplus 2}\\ \oplus \\ \mathcal{O}(1+n)\end{array} \\ \Big\downarrow {\scriptstyle x^{-i+j}\begin{pmatrix}0\\-1\\0\end{pmatrix}_{0}} & & \Big\downarrow {\scriptstyle x^{-i+j}\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}_{0}} \\ \begin{array}{c}\mathcal{O}(1+n)\\ \oplus \\ \sigma^*\mathcal{O}(-n)^{\oplus 2}\end{array} & \longrightarrow & \begin{array}{c}\mathcal{O}(n)^{\oplus 2}\\ \oplus \\ \sigma^*\mathcal{O}(-1-n)\end{array} \end{array} \tag{4.39}$$

For $j < i$,

$$f_2(\mathsf{b}_i, \mathsf{a}_j) = \begin{array}{ccc} \sigma^*\mathcal{O}(1-n) & \longrightarrow & \begin{array}{c}\sigma^*\mathcal{O}(-n)^{\oplus 2}\\ \oplus \\ \mathcal{O}(1+n)\end{array} \\ \Big\downarrow {\scriptstyle (-1)x^{-i+j+1}\begin{pmatrix}0\\1\\0\end{pmatrix}_{1}} & & \Big\downarrow {\scriptstyle (-1)x^{-i+j+1}\begin{pmatrix}0&0&0\\0&0&0\\0&-1&0\end{pmatrix}_{1}} \\ \begin{array}{c}\mathcal{O}(1+n)\\ \oplus \\ \sigma^*\mathcal{O}(-n)^{\oplus 2}\end{array} & \longrightarrow & \begin{array}{c}\mathcal{O}(n)^{\oplus 2}\\ \oplus \\ \sigma^*\mathcal{O}(-1-n)\end{array} \end{array} \tag{4.40}$$

$$\mathsf{a}_i \star \mathsf{b}_j = \begin{array}{ccc} \begin{matrix}\sigma^*\mathcal{O}(-n)^{\oplus 2}\\ \oplus \\ \mathcal{O}(1+n)\end{matrix} & \longrightarrow & \begin{matrix}\sigma^*\mathcal{O}(-1-n)\\ \oplus \\ \mathcal{O}(n)^{\oplus 2}\end{matrix} \\ \Big\downarrow x^{i-j}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 0 & 0\end{pmatrix}_{01} & & \Big\downarrow x^{i-j}\begin{pmatrix} 0 & -1 & 0\end{pmatrix}_{01} \\ \begin{matrix}\mathcal{O}(n)^{\oplus 2}\\ \oplus \\ \sigma^*\mathcal{O}(-1-n)\end{matrix} & \longrightarrow & \mathcal{O}(-1+n) \end{array} \tag{4.41}$$

For $i \geq j$,

$$f_2(\mathsf{a}_i, \mathsf{b}_j) = \begin{array}{ccc} \begin{matrix}\sigma^*\mathcal{O}(-n)^{\oplus 2}\\ \oplus \\ \mathcal{O}(1+n)\end{matrix} & \longrightarrow & \begin{matrix}\sigma^*\mathcal{O}(-1-n)\\ \oplus \\ \mathcal{O}(n)^{\oplus 2}\end{matrix} \\ \Big\downarrow x^{i-j}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 0 & 0\end{pmatrix}_{0} & & \Big\downarrow x^{i-j}\begin{pmatrix} 0 & -1 & 0\end{pmatrix}_{0} \\ \begin{matrix}\mathcal{O}(n)^{\oplus 2}\\ \oplus \\ \sigma^*\mathcal{O}(-1-n)\end{matrix} & \longrightarrow & \mathcal{O}(-1+n) \end{array} \tag{4.42}$$

For $i < j$,

$$f_2(\mathsf{a}_i, \mathsf{b}_j) = \begin{array}{ccc} \begin{matrix}\sigma^*\mathcal{O}(-n)^{\oplus 2}\\ \oplus \\ \mathcal{O}(1+n)\end{matrix} & \longrightarrow & \begin{matrix}\sigma^*\mathcal{O}(-1-n)\\ \oplus \\ \mathcal{O}(n)^{\oplus 2}\end{matrix} \\ \Big\downarrow x^{i-j+1}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}_{1} & & \Big\downarrow x^{i-j+1}\begin{pmatrix} 0 & 1 & 0\end{pmatrix}_{1} \\ \begin{matrix}\mathcal{O}(n)^{\oplus 2}\\ \oplus \\ \sigma^*\mathcal{O}(-1-n)\end{matrix} & \longrightarrow & \mathcal{O}(-1+n) \end{array} \tag{4.43}$$

Since all pairwise products of generators are Q-exact, it follows that the obstruction $\Pi(B_1(\phi)) = \Pi(\phi \star \phi)$ vanishes. Moreover, the second order deformation $f_2(\phi)$ is given by

$$f_2(\phi) = \sum_{i,j} \left(\mathsf{A}^i\mathsf{B}^j f_2(\mathsf{a}_i, \mathsf{b}_j) + \mathsf{B}^i\mathsf{A}^j f_2(\mathsf{b}_i, \mathsf{a}_j) + \mathsf{A}^i\mathsf{C}^j f_2(\mathsf{a}_i, \mathsf{c}_j) + \mathsf{C}^i\mathsf{A}^j f_2(\mathsf{c}_i, \mathsf{a}_j)\right). \tag{4.44}$$

Following the recursive algorithm discussed in Section 3, we compute the next obstruction $\Pi(\phi \star f_2(\phi) + f_2(\phi) \star \phi)$. For this, we have to compute products of the form

$$\alpha_i \star f_2(\alpha_j, \alpha_k), \quad f_2(\alpha_j, \alpha_k) \star \alpha_i.$$

Again we present a sample computation in detail. For $i \geq j$,

$$-\mathsf{a}_k \star f_2(\mathsf{b}_j, \mathsf{a}_i) = \begin{array}{ccc} \sigma^*\mathcal{O}(1-n) & \longrightarrow & \begin{matrix}\sigma^*\mathcal{O}(-n)^{\oplus 2} \\ \oplus \\ \mathcal{O}(1+n)\end{matrix} \\ \Big\downarrow {\scriptstyle x^{i-j+k}\begin{pmatrix}0\\-1\\0\end{pmatrix}_0} & & \Big\downarrow {\scriptstyle x^{i-j+k}\begin{pmatrix}0 & -1 & 0\end{pmatrix}_0} \\ \begin{matrix}\mathcal{O}(n)^{\oplus 2} \\ \oplus \\ \sigma^*\mathcal{O}(-1-n)\end{matrix} & \longrightarrow & \mathcal{O}(-1+n) \end{array} \tag{4.45}$$

and, for $i < j$,

$$-\mathsf{a}_k \star f_2(\mathsf{b}_j, \mathsf{a}_i) = \begin{array}{ccc} \sigma^*\mathcal{O}(1-n) & \longrightarrow & \begin{matrix}\sigma^*\mathcal{O}(-n)^{\oplus 2} \\ \oplus \\ \mathcal{O}(1+n)\end{matrix} \\ \Big\downarrow {\scriptstyle (-1)^{n+1}x^{i-j+k-2n+1}\begin{pmatrix}0\\1\\0\end{pmatrix}_1} & & \Big\downarrow {\scriptstyle (-1)^{n+2}x^{i-j+k-2n+1}\begin{pmatrix}0 & 1 & 0\end{pmatrix}_1} \\ \begin{matrix}\mathcal{O}(n)^{\oplus 2} \\ \oplus \\ \sigma^*\mathcal{O}(-1-n)\end{matrix} & \longrightarrow & \mathcal{O}(-1+n) \end{array} \tag{4.46}$$

For $k \geq j$,

$$-f_2(\mathsf{a}_k, \mathsf{b}_j) \star \mathsf{a}_i = \begin{array}{ccc} \sigma^*\mathcal{O}(1-n) & \longrightarrow & \begin{matrix}\sigma^*\mathcal{O}(-n)^{\oplus 2} \\ \oplus \\ \mathcal{O}(1+n)\end{matrix} \\ \Big\downarrow {\scriptstyle x^{i-j+k}\begin{pmatrix}0\\1\\0\end{pmatrix}_0} & & \Big\downarrow {\scriptstyle x^{i-j+k}\begin{pmatrix}0 & 1 & 0\end{pmatrix}_0} \\ \begin{matrix}\mathcal{O}(n)^{\oplus 2} \\ \oplus \\ \sigma^*\mathcal{O}(-1-n)\end{matrix} & \longrightarrow & \mathcal{O}(-1+n) \end{array} \tag{4.47}$$

and, for $k < j$,

$$-f_2(\mathsf{a}_k, \mathsf{b}_j) \star \mathsf{a}_i = \begin{array}{ccc} \sigma^*\mathcal{O}(1-n) & \longrightarrow & \begin{array}{c} \sigma^*\mathcal{O}(-n)^{\oplus 2} \\ \oplus \\ \mathcal{O}(1+n) \end{array} \\ \Big\downarrow {\scriptstyle (-1)^{n-1} x^{i-j+k+1-2n} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}_1} & & \Big\downarrow {\scriptstyle (-1)^{n} x^{i-j+k+1-2n} \begin{pmatrix} 0 & -1 & 0 \end{pmatrix}_1} \\ \begin{array}{c} \mathcal{O}(n)^{\oplus 2} \\ \oplus \\ \sigma^*\mathcal{O}(-1-n) \end{array} & \longrightarrow & \mathcal{O}(-1+n) \end{array} \tag{4.48}$$

Then the third order products are the following. For $k < j \le i$,

$$\mathfrak{m}_3(\mathsf{a}_k, \mathsf{b}_j, \mathsf{a}_i) = \begin{array}{ccc} \sigma^*\mathcal{O}(1-n) & \longrightarrow & \begin{array}{c} \sigma^*\mathcal{O}(-n)^{\oplus 2} \\ \oplus \\ \mathcal{O}(1+n) \end{array} \\ \Big\downarrow {\scriptstyle x^{i-j+k} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}} & & \Big\downarrow {\scriptstyle x^{i-j+k} \begin{pmatrix} 0 & -1 & 0 \end{pmatrix}} \\ \begin{array}{c} \mathcal{O}(n)^{\oplus 2} \\ \oplus \\ \sigma^*\mathcal{O}(-1-n) \end{array} & \longrightarrow & \mathcal{O}(-1+n) \end{array} \tag{4.49}$$

and, for $i < j \le k$,

$$\mathfrak{m}_3(\mathsf{a}_k, \mathsf{b}_j, \mathsf{a}_i) = \begin{array}{ccc} \sigma^*\mathcal{O}(1-n) & \longrightarrow & \begin{array}{c} \sigma^*\mathcal{O}(-n)^{\oplus 2} \\ \oplus \\ \mathcal{O}(1+n) \end{array} \\ \Big\downarrow {\scriptstyle x^{i-j+k} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} & & \Big\downarrow {\scriptstyle x^{i-j+k} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}} \\ \begin{array}{c} \mathcal{O}(n)^{\oplus 2} \\ \oplus \\ \sigma^*\mathcal{O}(-1-n) \end{array} & \longrightarrow & \mathcal{O}(-1+n) \end{array} \tag{4.50}$$

According to [39], the corresponding terms in the superpotential can be obtained by taking products of the form $\mathfrak{m}_2(\mathfrak{m}_3(\alpha_i, \alpha_j, \alpha_k), \alpha_l)$, which take values in $\mathrm{Ext}^3(\mathcal{E}_n, \mathcal{E}_n)$. For $k < j \leq i$ and $i - j + k - l = -1$, we have

$$\mathfrak{m}_2(\mathfrak{m}_3(\mathsf{a}_k, \mathsf{b}_j, \mathsf{a}_i), \mathsf{c}_l) = \begin{array}{c} \sigma^*\mathcal{O}(-n)^{\oplus 2} \\ \oplus \\ \mathcal{O}(1+n) \\ \Big\downarrow {\scriptstyle x^{i-j+k-l}(0\ 0\ 1)_{01}} \\ \mathcal{O}(-1+n) \end{array} \tag{4.51}$$

The expression obtained in the right hand side of equation (4.51) is a generator for

$$\mathrm{Ext}^3(\sigma^*\mathcal{O}_C(-1-n)[1], \sigma^*\mathcal{O}_C(-1-n)[1]) = \mathbb{C}. \tag{4.52}$$

For $i < j \leq k$ and $i - j + k - l = -1$,

$$\mathfrak{m}_2(\mathfrak{m}_3(\mathsf{a}_k, \mathsf{b}_j, \mathsf{a}_i), \mathsf{c}_l) = \begin{array}{c} \sigma^*\mathcal{O}(-n)^{\oplus 2} \\ \oplus \\ \mathcal{O}(1+n) \\ \Big\downarrow {\scriptstyle x^{i-j+k-l}(0\ 0\ -1)_{01}} \\ \mathcal{O}(-1+n) \end{array} \tag{4.53}$$

Note that the expression in the right hand side of (4.53) is the same generator of (4.52) multiplied by (-1). The first product (4.51) gives rise to superpotential terms of the form

$$\mathrm{Tr}(C^l A^k B^j A^i)$$

with

$$(i+k) - (j+l) = -1, \quad k < j \leq i.$$

The second product (4.53) gives rise to terms in the superpotential of the form

$$-\mathrm{Tr}(C^l A^k B^j A^i)$$

with

$$(i+k) - (j+l) = -1, \quad i < j \leq k.$$

If we consider the case $n = 1$ for simplicity, the superpotential interactions resulting from these two products are

$$\begin{aligned} W = \mathrm{Tr}(&C^1A^0B^1A^1 - C^1A^1B^1A^0 + C^2A^0B^1A^2 - C^2A^2B^1A^0 \\ &+ C^1A^0B^2A^2 - C^1A^2B^2A^0 + C^2A^1B^2A^2 - C^2A^2B^2A^1). \end{aligned} \tag{4.54}$$

A An alternative derivation

In this appendix, we give an alternative derivation of Lemma 2.2. This approach relies on one of the most powerful results in algebraic geometry, namely the Grothendieck duality. Let us start by recalling the latter. Consider $f\colon X \to Y$ to be a proper morphism of smooth varieties.[6] Choose $\mathcal{F} \in D^b(X)$ and $\mathcal{G} \in D^b(Y)$ to be objects in the corresponding bounded derived categories. Then one has the following isomorphism (see, e.g., III.11.1 of [55]):

$$\mathbf{R}f_*\mathbf{R}\,\mathscr{H}om_X(\mathcal{F}, f^!\mathcal{G}) \cong \mathbf{R}\,\mathscr{H}om_Y(\mathbf{R}f_*\mathcal{F}, \mathcal{G}). \tag{A.1}$$

Now it is true that $f^!$ in general is a complicated functor, in particular it is *not* the total derived functor of a classical functor, i.e., a functor between the category of coherent sheaves, but in our context it will have a very simple form.

The original problem that lead to Lemma 2.2 was to determine the derived dual, a.k.a, the Verdier dual, of a torsion sheaf. Let $i\colon E \to X$ be the embedding of a codimension d subvariety E into a smooth variety X, and let V be a vector bundle on E. We want to determine $\mathbf{R}\,\mathscr{H}om_X(i_*V, \mathcal{O}_X)$. Using (A.1), we have

$$\mathbf{R}\,\mathscr{H}om_X(i_*V, \mathcal{O}_X) \cong i_*\mathbf{R}\,\mathscr{H}om_E(V, i^!\mathcal{O}_X), \tag{A.2}$$

where we used the fact that the higher direct images of i vanish. Furthermore, since V is locally free, we have that

$$\mathbf{R}\,\mathscr{H}om_E(V, i^!\mathcal{O}_X) = \mathbf{R}\,\mathscr{H}om_E(\mathcal{O}_E, V^\vee \otimes i^!\mathcal{O}_X) = V^\vee \otimes i^!\mathcal{O}_X, \tag{A.3}$$

where $V^\vee$ is the dual of V on E, rather than on X. On the other hand, for an embedding

$$i^!\mathcal{O}_X = K_{E/X}[-d]\,, \tag{A.4}$$

where $K_{E/X}$ is the relative canonical bundle. Now if we assume that the ambient space X is a Calabi–Yau variety, then $K_{E/X} = K_E$. We can summarize this

Proposition A.1. *For the embedding $i\colon E \to X$ of a codimension d subvariety E in a smooth Calabi–Yau variety X, and a vector bundle V on E, we have that*

$$\mathbf{R}\,\mathscr{H}om_X(i_*V, \mathcal{O}_X) \cong i_*\left(V^\vee \otimes K_E\right)[-d]. \tag{A.5}$$

[6] The Grothendieck duality applies to more general schemes than varieties, but we limit ourselves to the cases considered in this paper.

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