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Vector bundle extensions, sheaf cohomology, and the heterotic standard model

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Abstract

Stable, holomorphic vector bundles are constructed on a torus fibered, non-simply connected Calabi–Yau three-fold using the method of bundle extensions. Since the manifold is multiply connected, we work with equivariant bundles on the elliptically fibered covering space. The cohomology groups of the vector bundle, which yield the low energy spectrum, are computed using the Leray spectral sequence and fit the requirements of particle phenomenology. The physical properties of these vacua were discussed previously. In this paper, we systematically compute all relevant cohomology groups and explicitly prove the existence of the necessary vector bundle extensions. All mathematical details are explained in a pedagogical way, providing the technical framework for constructing heterotic standard model vacua.

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Contents

1 Introduction

The ultimate goal of string theory is to completely describe the known forces and particles. While string theory itself is basically unique, the possible choice of vacua is not. Since low energy physics is determined by the compactification, the question of whether string theory has phenomenologically viable vacua is one of the key issues today. We are not yet able to answer this in full generality. We are, however, able to claim an encouraging success. A long-standing problem in string theory is whether or not one can find compactifications that produce the correct low energy spectrum, without any exotic matter. By "exotic" we mean not only matter fields transforming in representations that are not in the standard model, but also additional replicas of quarks and leptons beyond three families. Note that we are including the right-handed neutrino as a member of each standard model family, see [1–3].

To date, most attempts at model building used Type II orientifolds, see [4–17]. The advantage of this approach is the availability of a conformal field theory description, in particular Gepner models [18–21]. This is, however, also the biggest drawback, as one is always forced to work at special points in the moduli space with enhanced symmetries and extra massless fields. Going to more generic points in this context is very difficult [22, 23]. The same problem is implicit in heterotic orbifold constructions [24–29].

Instead, our construction employs the $E_8 \times E_8$ heterotic string, both in the strong coupling [30–36] regime of heterotic M-theory and in the weakly coupled [37, 38] regime. Moreover, we allow for arbitrary vector bundles in the $(0, 2)$ model instead of restricting ourselves to the so-called[1] "standard embedding" [39–43]. To find a realistic, $\mathcal{N} = 1$ supersymmetric vacuum of this theory, one needs to specify a six-dimensional Ricci-flat manifold and an $E_8 \times E_8$ gauge connection satisfying the hermitian Yang–Mills equations. Fortunately, we do not actually have to solve the equations of motion. The results of [44, 45] guarantee that any solution is equivalent to constructing a Calabi–Yau three-fold together with a stable, holomorphic vector bundle. Until now, the standard way to construct such bundles was to use spectral covers on elliptically fibered three-folds, see [46–48]. However, it turned out to be difficult to construct realistic matter spectra in this context, see [48–53]. Mixing spectral covers with vector bundle extensions was attempted in [54] for $SU(5)$ bundles, but failed to yield a phenomenologically viable model.

In this work, we will give a detailed mathematical analysis of the heterotic standard model that we presented previously in [55, 56]. For the above reasons, we will not employ spectral covers to construct vector bundles. Rather, we use the method of "bundle extensions" alone. This method is discussed in detail, and we give a careful computation of the low energy spectrum [57–60] via the Leray spectral sequence. We have already constructed a suitable Calabi–Yau three-fold in [61] and will take this manifold as the base space for the necessary vector bundles. This three-fold is torus fibered [62–68], which gives us good control over the bundles. By choosing

[1]This is really a misnomer, there is nothing intrinsically "standard" here.

a suitable bundle and Wilson lines [69–74], we are able to find a compactification which is devoid of any exotic matter fields, except for an additional Higgs–Higgs conjugate pair. A second Higgs pair is not ruled out experimentally and may be viewed as a prediction of this class of models.

2 Overview

The goal of model building is to construct realistic compactifications of the string theory. In this paper, we focus on finding the standard model with two extra symmetries. Specifically, in addition to the usual $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetry, we impose

- $\mathcal{N} = 1$ supersymmetry in four dimensions,
- an additional $U(1)_{B-L}$ gauge symmetry. This extra symmetry naturally suppresses proton decay.

We work in the context of the $E_8 \times E_8$ heterotic string and choose the first E_8 factor to be in the observable sector. This factor is then broken down to the desired low energy gauge group. In principle, there are a number of breaking patterns one could try, but the minimal pattern in our context is obtained by choosing an

$$SU(4) \times \mathbb{Z}_3 \times \mathbb{Z}_3 \;\subset\; E_8 \tag{2.1}$$

instanton on the internal Calabi–Yau manifold. In other words, we compactify via a "nonstandard embedding" rank 4 gauge bundle together with $\mathbb{Z}_3 \times \mathbb{Z}_3$ Wilson lines. In [61], we constructed a Calabi–Yau three-fold X, which allows for $\mathbb{Z}_3 \times \mathbb{Z}_3$ Wilson lines, and we will review its most important properties in Section 3.

However, the Calabi–Yau manifold alone does not determine the heterotic string compactification. One must, in addition, construct a gauge bundle with a hermitian Yang–Mills connection. This connection satisfies a complicated non-linear system of differential equations, but, fortunately, these can be replaced by an algebraic geometric criterion, see [44, 45]. That is, it suffices to construct a (rank 4 in our case) stable, holomorphic vector bundle on X. Technically, it is easier to work on the simply connected covering space $\widetilde{X}$, and we will do so throughout this work. The price one has to pay, however, is that one must construct equivariant vector bundles on $\widetilde{X}$. The precise definition and relationship of these to vector bundles on X will be developed in Section 4, together with some notation.

One way to obtain such bundles is by the so-called "spectral cover construction". While based on a clever trick and exploited thoroughly in the recent years, this method has always failed to yield vector bundles for realistic compactifications. Instead, we take the following starting point. There are two kinds of vector bundles that one has good control over:

- Line bundles on the Calabi–Yau three-fold $\widetilde{X}$.
- Rank 2 vector bundles on the base surface of the elliptic fibration $\widetilde{X}$.

The trivial equation $4 = 2 \cdot 1 + 2 \cdot 1$ then suggests that a certain combination of tensor products would have the desired rank 4, and this is precisely the basis for our construction. Of course, direct sums of vector bundles are never stable. So, we have to take nontrivial extensions.

Once one has constructed a stable, holomorphic bundle, one can then proceed to determine the low energy particle spectrum in the heterotic compactification. By a standard identification, this is determined by the sheaf cohomology groups of some associated holomorphic vector bundles. Computing these requisite cohomology groups is going to be the main part of this work.

Since there are considerable technical difficulties, we start by constructing a rank 2 instanton in the hidden E_8 sector. As it turns out, this is needed for heterotic anomaly cancellation in the presence of five-branes later on. It also serves as a simple introduction to our technology. This will be the subject of Section 5. Since the Calabi–Yau manifold together with the vector bundle completely determines the compactification, we are then in a position to read off the low energy spectrum. By a standard identification between zero modes of the Dirac operator and sheaf cohomology groups, see [57, 58], this again reduces to a question in algebraic geometry. We first apply this to the hidden gauge bundle in Section 5.3 and conclude that there are no hidden matter fields.

At this point we are ready to describe the center piece of our work, the construction of the visible E_8 bundle. After a long search, we found precisely one rank 4 vector bundle which yields a phenomenologically viable low energy spectrum. This is described in Subsection 6.1. Our bundle cancels the heterotic anomaly in a nice way, as we show in Subsection 6.2. In the following two subsections, we compute the requisite cohomology groups, check for the existence of extensions in Subsection 6.5, and make sure that the possible torsion part of the first Chern class vanishes in Subsection 6.6. Using these mathematical results, one can then determine the low energy spectrum and we proceed to do so in Section 7. One can think of the gauge symmetry breaking as first taking only the $SU(4) \subset E_8$ instanton into account, and then subsequently add the effect of the Wilson lines. We do that in Subsections 7.1, 7.2 and present the resulting spectrum in 7.3.

This concludes the main part of our work, but we would still like to discuss a modification of our vacuum. So far, we have utilized five-branes in the bulk to cancel the heterotic anomaly. This is only possible within the context of the strongly coupled heterotic string. One might ask whether one could perform a small instanton transition [75–77] and absorb the five-branes into the hidden E_8 bundle and, indeed, this is possible. Therefore, by a modification of the hidden sector, which we present in Section 8, one can work in the weak coupling regime at the expense of introducing two hidden matter multiplets. In that case, an $SU(2) \times SU(2)$ bundle is used to break E_8 to Spin(12), and we find two $\underline{\mathbf{12}}$ matter multiplets in the hidden Spin(12).

3 The Calabi-Yau manifold

3.1 Fiber products and group actions

To compactify the heterotic string so as to preserve $N = 1$ supersymmetry, we have to specify two geometric data. First, we must pick a spacetime background geometry $\mathbb{R}^{3,1} \times X$, where X is a Calabi–Yau manifold. This is what we describe in this section. Second, we must construct an $E_8 \times E_8$ gauge bundle with a suitable connection. That will be done in the following sections.

We take the Calabi–Yau manifold X to be the space constructed in [61]. Let us review these results, in as far as we are going to need them for the remainder of this paper. First of all, X is not simply connected. Rather,

$$\pi_1(X) = \mathbb{Z}_3 \times \mathbb{Z}_3. \tag{3.1}$$

This means that there is another Calabi–Yau manifold $\widetilde{X}$ whose $G \stackrel{\text{def}}{=} \mathbb{Z}_3 \times \mathbb{Z}_3$ quotient is X. That is,

$$X \stackrel{\text{def}}{=} \frac{\widetilde{X}}{G} = \frac{\widetilde{X}}{\mathbb{Z}_3 \times \mathbb{Z}_3}. \tag{3.2}$$

In [61], we constructed $\widetilde{X}$ as a fiber product of two dP_9 surfaces and then showed that for special values of the moduli there is a discrete $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry.

The fiber product $B_1 \times_{\mathbb{P}^1} B_2$ of two elliptic del Pezzo surfaces B_1 and B_2 is defined as follows. We already have fibrations $\beta_i : B_i \to \mathbb{P}^1$ such that the fiber over a generic point $x \in \mathbb{P}^1$ is a smooth elliptic curve,

$$\beta_1^{-1}(x) \simeq T^2 \simeq \beta_2^{-1}(x). \tag{3.3}$$

The fiber product is the fibration over $\mathbb{P}^1$ with fiber $\beta_1^{-1}(x) \times \beta_2^{-1}(x)$, $x \in \mathbb{P}^1$. Over a generic point, the fiber is a smooth Abelian surface (a T^4). Note that the base is (complex) one-dimensional and the fiber is two-dimensional, so we constructed a three-fold as desired. If the singular fibers of β_1 and β_2 do not collide, then the fiber product $B_1 \times_{\mathbb{P}^1} B_2$ is again a smooth variety.

To summarize, the Calabi–Yau manifold $\widetilde{X}$ comes with the following chains of fibrations

$$\begin{tikzcd}[row sep=small]
\dim_{\mathbb{C}} = 3: & & \widetilde{X} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\dim_{\mathbb{C}} = 2: & B_1 \arrow[dr, "\beta_1"'] & & B_2 \arrow[dl, "\beta_2"] \\
\dim_{\mathbb{C}} = 1: & & \mathbb{P}^1 \arrow[d, "\pi"'] & \\
\dim_{\mathbb{C}} = 0: & & \{\text{pt.}\}. &
\end{tikzcd} \tag{3.4}$$

The maps π_1, π_2, β_1, and β_2 are elliptic fibrations, and π is trivially a $\mathbb{P}^1$ fibration.

The Hodge diamond of the Calabi–Yau manifold X is given by

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 3 & & 0 & \\
1 & & 3 & & 3 & & 1\,. \\
 & 0 & & 3 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array} \tag{3.5}$$

3.2 Homology ring

Any dP_9 surface B has $H_2(B, \mathbb{Z}) = \mathbb{Z}^{10}$. In [61], we restricted ourselves to dP_9 surfaces with $3I_1$ and $3I_3$ singular fibers. In that case, we defined three special rational curves $\mathbb{P}^1 \subset B$:

- The 0-section σ of the elliptic fibration $\beta : B \to \mathbb{P}^1$.
- The section η, which generates the torsion part of the Mordell–Weil group.

- A section ξ, which, together with its $\mathbb{Z}_3 \times \mathbb{Z}_3$ images, generates the remainder of the Mordell–Weil group.

It turned out that the $\mathbb{Z}_3 \times \mathbb{Z}_3$-invariant part of the homology group has rank 2.

$$H_2(B, \mathbb{Z})^{\mathbb{Z}_3 \times \mathbb{Z}_3} = \mathbb{Z}f \oplus \mathbb{Z}t, \tag{3.6}$$

where f is the class of a fiber of the elliptic fibration and t is the homology sum of three sections[2]

$$t = \xi + \alpha_B \xi + (\eta \boxplus \xi). \tag{3.7}$$

By definition, a section intersects a fiber at a point. Hence, t and f intersect at three points,

$$ft = 3\{\text{pt.}\} = 3t^2. \tag{3.8}$$

The $\mathbb{Z}_3 \times \mathbb{Z}_3$-invariant part of the homology ring is therefore

$$H_*(B, \mathbb{Q})^{\mathbb{Z}_3 \times \mathbb{Z}_3} = \frac{\mathbb{Q}[f, t]}{\langle f^2, ft = 3t^2 \rangle}. \tag{3.9}$$

Now, let us return to the Calabi–Yau manifold $\widetilde{X}$. Its $\mathbb{Z}_3 \times \mathbb{Z}_3$-invariant divisors are the pullbacks of the invariant divisors[3] on B_1 and B_2, which we label as

$$\begin{aligned} \tau_1 &\stackrel{\text{def}}{=} \pi_1^{-1}(t) \\ \tau_2 &\stackrel{\text{def}}{=} \pi_2^{-1}(t) \\ \phi &\stackrel{\text{def}}{=} \pi_1^{-1}(f) = \pi_2^{-1}(f). \end{aligned} \tag{3.10}$$

The intersection numbers on $\widetilde{X}$ then follow from equation (3.9). We find that the invariant part of the homology groups in even degrees is

$$H_{\text{ev}}(\widetilde{X}, \mathbb{Q})^{\mathbb{Z}_3 \times \mathbb{Z}_3} = \frac{\mathbb{Q}[\phi, \tau_1, \tau_2]}{\langle \phi^2,\ \phi\tau_1 = 3\tau_1^2, \phi\tau_2 = 3\tau_2^2 \rangle}. \tag{3.11}$$

For practical purposes, it is useful to switch to a Gröbner basis, for example, with the lexicographic term ordering $\phi \succ \tau_1 \succ \tau_2$,

$$\langle \phi^2,\ \phi\tau_1 = 3\tau_1^2,\ \phi\tau_2 = 3\tau_2^2 \rangle = \langle \phi^2,\ \phi\tau_1 = 3\tau_1^2,\ \phi\tau_2 = 3\tau_2^2,\ \tau_1^3,\ \tau_1^2\tau_2 = \tau_1\tau_2^2,\ \tau_2^3 \rangle. \tag{3.12}$$

[2]Here, α_B is a $\mathbb{Z}_3$ action related to the overall $\mathbb{Z}_3 \times \mathbb{Z}_3$ action. And $\boxplus$ denotes the addition in the Mordell–Weil group, that is, addition of sections by point-wise addition in each fiber.

[3]We denote the invariant divisors on B_1 and B_2 by f and t and elect to not index them separately. It will always be clear from the context which surface we are referring to.

Then, one can easily bring any polynomial in ϕ, τ_1, τ_2 into the standard form

$$\begin{aligned}
H_0(\widetilde{X},\mathbb{Q})^{\mathbb{Z}_3\times\mathbb{Z}_3} &\simeq H^6(\widetilde{X},\mathbb{Q})^{\mathbb{Z}_3\times\mathbb{Z}_3} = \mathbb{Q}\tau_1\tau_2^2,\\
H_2(\widetilde{X},\mathbb{Q})^{\mathbb{Z}_3\times\mathbb{Z}_3} &\simeq H^4(\widetilde{X},\mathbb{Q})^{\mathbb{Z}_3\times\mathbb{Z}_3} = \mathbb{Q}\tau_1^2 \oplus \mathbb{Q}\tau_1\tau_2 \oplus \mathbb{Q}\tau_2^2,\\
H_4(\widetilde{X},\mathbb{Q})^{\mathbb{Z}_3\times\mathbb{Z}_3} &\simeq H^2(\widetilde{X},\mathbb{Q})^{\mathbb{Z}_3\times\mathbb{Z}_3} = \mathbb{Q}\phi \oplus \mathbb{Q}\tau_1 \oplus \mathbb{Q}\tau_2,\\
H_6(\widetilde{X},\mathbb{Q})^{\mathbb{Z}_3\times\mathbb{Z}_3} &\simeq H^0(\widetilde{X},\mathbb{Q})^{\mathbb{Z}_3\times\mathbb{Z}_3} = \mathbb{Q}.
\end{aligned} \tag{3.13}$$

Note that the natural generator of $H^6\big(\widetilde{X},\mathbb{Q}\big)^{\mathbb{Z}_3\times\mathbb{Z}_3}$ is not primitive, that is, it is a multiple of the generator in integral cohomology. In fact, it is three times the generator of $H^6\big(\widetilde{X},\mathbb{Z}\big)^{\mathbb{Z}_3\times\mathbb{Z}_3}$,

$$\tau_1^2\tau_2 = \tau_1\tau_2^2 = 3\{\mathrm{pt.}\}. \tag{3.14}$$

Getting the normalization correct is, of course, important for index computations in the following.

4 Notation and conventions for bundles

4.1 Line bundles, ideal sheaves, and extensions

Having studied the Calabi–Yau three-fold $\widetilde{X}$, we now want to construct vector bundles. We start with some basics, which will help to set the notation in the remainder of this work.

All vector bundles that we consider are holomorphic, that is, the defining transition functions are holomorphic. The simplest vector bundle is just the trivial line bundle $\widetilde{X} \times \mathbb{C} \to \widetilde{X}$. The sections of the trivial line bundle are simply holomorphic functions $f : \widetilde{X} \to \mathbb{C}$, and any such function must be constant since $\widetilde{X}$ is compact. But on each coordinate chart $U \subset \widetilde{X}$, there are many holomorphic functions. So, while considering holomorphic functions that are defined everywhere on $\widetilde{X}$ is unenlightening, the holomorphic functions on open subsets can be interesting. This is why one works with the sheaf of holomorphic functions, $\mathcal{O}_{\widetilde{X}}$, which assigns to each open set $U \subset \widetilde{X}$, the holomorphic functions on U.

Now, technically, $\widetilde{X} \times \mathbb{C}$ is the trivial line bundle and $\mathcal{O}_{\widetilde{X}}$ is the sheaf of local sections of the trivial line bundle. We will not make that distinction in the following and use either to denote the line bundle.

Just like the sheaf of local holomorphic functions, one can define the sheaf of local holomorphic functions which vanish at some points. This is called the ideal sheaf of the set of points. In particular, we will use an ideal sheaf on the surface B_2 in the following. We write I_k for the functions on B_2 vanishing at a giving set of k points. Note that an ideal sheaf on a surface is not quite a vector bundle, but, rather, it contains "point defects".

Coming back to bundles, there is a simple way to describe all line bundles on any variety Y using the correspondence

$$\Big\{\text{Divisors } D\Big\}\Big/\sim \quad \stackrel{1:1}{\longleftrightarrow} \quad \Big\{\text{Line bundles } \mathcal{O}_Y(D)\Big\}$$

between divisors and line bundles. On $\widetilde{X}$, B_1, B_2, $\mathbb{P}^1$, the "linear equivalence" relation $\sim$ just amounts to taking the homology class of the divisor. Hence, every line bundle is of the form

- $\mathcal{O}_{\widetilde{X}}(x_1\tau_1 + x_2\tau_2 + x_3\phi)$, $x_1, x_2, x_3 \in \mathbb{Z}$.
- $\mathcal{O}_{B_i}(y_1 t + y_2 f)$, $y_1, y_2 \in \mathbb{Z}$.
- $\mathcal{O}_{\mathbb{P}^1}(n) \stackrel{\text{def}}{=} \mathcal{O}_{\mathbb{P}^1}(n\{\text{pt.}\})$, $n \in \mathbb{Z}$.

4.2 Equivariant structures

Our ultimate goal is, of course, to construct holomorphic vector bundles on the quotient Calabi–Yau three-fold $X = \widetilde{X}/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. But, unfortunately, vector bundles on X and vector bundles on $\widetilde{X}$ are only distantly related. Really, we want to exploit the one-to-one correspondence

$$\left\{\begin{matrix}\text{Vector bundles}\\ \text{on } X = \widetilde{X}\Big/(\mathbb{Z}_3\times\mathbb{Z}_3)\end{matrix}\right\} \quad \stackrel{1:1}{\longleftrightarrow} \quad \left\{\begin{matrix}\mathbb{Z}_3\times\mathbb{Z}_3 \text{ Equivariant vector}\\ \text{bundles on } \widetilde{X}\end{matrix}\right\} \tag{4.1}$$

Let us pause to define equivariant vector bundles. First of all, just as vector bundles are defined over a fixed base space, equivariant vector bundles are defined over a fixed G-space, that is, a topological space with an action of the group G. An equivariant vector bundle is then a pair $(\mathcal{E}, \phi)$ consisting of an ordinary vector bundle $\mathcal{E}$ together with an action of the group $\phi_g : \mathcal{E} \to \mathcal{E}$, $g \in G$. It is crucial that the $|G|$ maps ϕ_g represent the group, that is $\phi_g\phi_{g'} = \phi_{gg'}$. Finally, the group action on the vector bundle must cover the action on the base space, that is, map the fiber $\mathcal{E}_p$ over the point p to the fiber over the image point $g(p)$, in other words $\phi_g(\mathcal{E}_p) = \mathcal{E}_{g(p)}$.

An important point is that this group action is not unique. Consider (the sheaf of sections of) the trivial line bundle $\mathcal{O}_{\widetilde{X}}$ on the Calabi–Yau three-fold $\widetilde{X}$. It is clearly invariant under the $\mathbb{Z}_3 \times \mathbb{Z}_3$ action, but there are different choices for how the group acts on $\mathcal{O}_{\widetilde{X}}$. To illustrate this, let us look at a single $\mathbb{Z}_3$ action $g : \widetilde{X} \to \widetilde{X}$. Now a $\mathbb{Z}_3$ equivariant structure on $\mathcal{O}_{\widetilde{X}}$ is a map

$\gamma : \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_{\widetilde{X}}$ covering g. That is, γ maps elements in the vector space over a point to the vector space over the g-image of that point. In other words, the diagram

$$\begin{array}{ccc} \mathcal{O}_{\widetilde{X}} & \xrightarrow[\gamma]{} & \mathcal{O}_{\widetilde{X}} \\ \downarrow & & \downarrow \\ \widetilde{X} & \xrightarrow[g]{} & \widetilde{X} \end{array} \tag{4.2}$$

commutes. There is an obvious such map γ: identify $\mathcal{O}_{\widetilde{X}} \simeq \widetilde{X} \times \mathbb{C}$ and let γ not act on the vector space at all. That is,

$$\gamma : \widetilde{X} \times \mathbb{C} \longrightarrow \widetilde{X} \times \mathbb{C}, (p, v) \longmapsto (g(p), v). \tag{4.3}$$

But this is not the only choice, and we could combine it with any third root of unity multiplying the vector component. In other words, for any character[4] χ of $\mathbb{Z}_3$, there is another equivariant structure

$$\chi\gamma : \widetilde{X} \times \mathbb{C} \longrightarrow \widetilde{X} \times \mathbb{C}, (p, v) \longmapsto \Big(g(p), \chi(g)v\Big). \tag{4.4}$$

This is why, in the following, we need a notation to express the equivariant structure on a line bundle. We fix generators g_1 and g_2 of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ group,

$$G = G_1 \times G_2 = \{e, g_1, g_1^2\} \times \{e, g_2, g_2^2\} \simeq \mathbb{Z}_3 \times \mathbb{Z}_3, \tag{4.5}$$

and choose the following generators of the character ring,

$$\begin{aligned} \chi_1(g_1) &= \omega \qquad \chi_1(g_2) = 1 \\ \chi_2(g_1) &= 1 \qquad \chi_2(g_2) = \omega, \end{aligned} \tag{4.6}$$

where $\omega = \mathrm{e}^{2\pi i/3}$. We then always take the G action on the trivial line bundle $\mathcal{O}_Y$ to be the pure translation and write $\chi\mathcal{O}_Y$ for the translation composed with multiplication by a character,

$$g : \chi\mathcal{O}_Y \longrightarrow \chi\mathcal{O}_Y, \qquad (p, v) \longmapsto \Big(g_i(p), \chi(g)v\Big). \tag{4.7}$$

This uniquely determines the action on trivial bundles. In addition, we need to consider vertical bundles, that is, line bundles whose associated divisor is a multiple of the elliptic fiber. For that, we restrict to one of the two G

[4]A character of a group G is a homomorphism $\chi : G \to \mathbb{C}^\times$. Since the group is finite in our case, it is actually a map $G \to U(1)$. Note that this is not quite the same as the character of a representation, the latter being the traces of representation matrices.

fixed points on the base $\mathbb{P}^1$. So pick once and for all

$$0 \in \mathbb{P}^1, \qquad G \cdot 0 = 0. \tag{4.8}$$

Then, any vertical bundle $\mathcal{O}_{B_i}(nf)$ restricts to a trivial bundle on $f = \beta_i^{-1}(0)$ and, hence, is of the form

$$\mathcal{O}_{B_i}(nf)\big|_{f=\beta_i^{-1}(0)} = \chi \mathcal{O}_f. \tag{4.9}$$

We then label the equivariant structure of $\mathcal{O}_{B_i}(nf)$ by the character χ, that is,

$$\mathcal{L} = \chi \mathcal{O}_{B_i}(nf) \overset{\text{def}}{\Longleftrightarrow} \mathcal{L} \simeq \mathcal{O}_{B_i}(nf) \qquad \text{and} \qquad \mathcal{L}\big|_{f=\beta_i^{-1}(0)} = \chi \mathcal{O}_f. \tag{4.10}$$

In the same way, we label the equivariant structure on line bundles on $\mathbb{P}^1$ and of vertical line bundles on $\widetilde{X}$ as

$$\mathcal{L} = \chi \mathcal{O}_{\mathbb{P}^1}(n) \overset{\text{def}}{\Longleftrightarrow} \mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^1}(n) \qquad \text{and} \qquad \mathcal{L}\big|_0 = \chi, \tag{4.11}$$

$$\mathcal{L} = \chi \mathcal{O}_{\widetilde{X}}(n\phi) \overset{\text{def}}{\Longleftrightarrow} \mathcal{L} \simeq \mathcal{O}_{\widetilde{X}}(n\phi) \qquad \text{and} \qquad \mathcal{L}\big|_{\phi=(\beta_i\circ\pi_i)^{-1}(0)} = \chi \mathcal{O}_\phi. \tag{4.12}$$

Here, we also identified the one-dimensional representation of G determined by χ with the character $\chi : G \to U(1)$. This abuse of notation will be continued throughout this paper.

4.3 Equivariant versus invariant bundles

Our insistence on explicitly denoting the group action on every bundle is important. Most vector bundles on X do not admit a $\mathbb{Z}_3 \times \mathbb{Z}_3$ action, even if their Chern classes are invariant and do not correspond to vector bundles on the quotient X/G. This is why we must be very careful to construct equivariant vector bundles.

The underlying problem can be made very explicit. Consider the line bundle $\mathcal{O}_{\widetilde{X}}(\tau_1)$, one of the simplest invariant line bundles one can possibly write down. Yet $\mathcal{O}_{\widetilde{X}}(\tau_1)$ does not have any $\mathbb{Z}_3 \times \mathbb{Z}_3$ action and cannot be made into an equivariant line bundle. This can be seen as follows. The elliptic fibration $\pi_2 : \widetilde{X} \to B_1$ has one elliptic fiber which is mapped to itself under the $\mathbb{Z}_3 \times \mathbb{Z}_3$ action and, moreover, on which both generators act as translation by an order three point. Call this elliptic fiber $E \simeq \mathbb{C}/\Lambda$. The divisor

τ_1 intersects this elliptic curve in three points

$$\tau_1 \cap E = \{P_1, P_2, P_3\}, \tag{4.13}$$

and, hence, the restriction of the line bundle $\mathcal{O}_{\widetilde{X}}(\tau_1)$ to E is

$$\mathcal{O}_{\widetilde{X}}(\tau_1)\big|_E = \mathcal{O}_E(P_1 + P_2 + P_3). \tag{4.14}$$

Now the moduli space of line bundles on an elliptic curve (the Picard variety) looks like

$$\operatorname{Pic}(E) \simeq \mathbb{Z} \times T^2. \tag{4.15}$$

In other words, the line bundles are determined by one integer (the first Chern class) and one complex number. The latter is a continuous modulus, which is just the sum of the coordinates (modulo Λ) of the points of the corresponding divisor. That is, in our case

$$P_1 \boxplus P_2 \boxplus P_3 \in E. \tag{4.16}$$

Obviously, if we pick an order three point and translate P_1, P_2, and P_3 by it then the sum does not change (again, modulo Λ). Hence $g^*\mathcal{O}_E(P_1 + P_2 + P_3) \simeq \mathcal{O}_E(P_1 + P_2 + P_3)$ for each $g \in \mathbb{Z}_3 \times \mathbb{Z}_3$, and we can pick one such map

$$\phi_{g_i} \colon \mathcal{O}_E(P_1 + P_2 + P_3) \xrightarrow{\sim} \mathcal{O}_E(P_1 + P_2 + P_3) \tag{4.17}$$

for each of the two generators $g_1, g_2 \in \mathbb{Z}_3 \times \mathbb{Z}_3$. Now one might think that these maps would turn $\mathcal{O}_E(P_1 + P_2 + P_3)$ into an equivariant line bundle, and it is indeed $\mathbb{Z}_3$ equivariant for the $\mathbb{Z}_3$ subgroups generated by g_1 or g_2. But one cannot turn it into a $\mathbb{Z}_3 \times \mathbb{Z}_3$ equivariant line bundle, simply because its first Chern class

$$c_1\big(\mathcal{O}_E(P_1 + P_2 + P_3)\big) = 3 \tag{4.18}$$

is not divisible by $|\mathbb{Z}_3 \times \mathbb{Z}_3| = 9$. The maps ϕ_{g_1}, ϕ_{g_2} fail to define an equivariant line bundle because they do not commute, whereas g_1, g_2 of course do commute. At most, one can choose them to commute up to a third root of unity $\omega = e^{2\pi i/3}$,

$$\phi_{g_1} \circ \phi_{g_2} = \omega\, \phi_{g_2} \circ \phi_{g_1}. \tag{4.19}$$

Put differently, the line bundle $\mathcal{O}_E(P_1 + P_2 + P_3)$ can only be equivariant under the Heisenberg group G_{H}, that is, the central extension

$$0 \longrightarrow \mathbb{Z}_3 \longrightarrow G_{\mathrm{H}} \longrightarrow \mathbb{Z}_3 \times \mathbb{Z}_3 \longrightarrow 0. \tag{4.20}$$

Since we will make use of it in the following, let us mention an elementary fact from the representation theory of the Heisenberg group. There is only one irreducible representation such that the central $\mathbb{Z}_3$ acts by multiplication with ω. This representation is three dimensional. In terms of matrices, it is generated by

$$\rho(g_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \rho(g_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \tag{4.21}$$

4.4 Canonical bundles

One basic and technique for computing cohomology groups is to use Serre duality. For any variety Y, it relates the cohomology groups

$$H^{\dim_{\mathbb{C}}(Y)-i}\Big(Y, \mathcal{F}\Big)^{\vee} = H^i\Big(Y, \mathcal{F}^{\vee} \otimes K_Y\Big) \tag{4.22}$$

if $\mathcal{F}$ is vector bundle and

$$H^{\dim_{\mathbb{C}}(Y)-i}\Big(Y, \mathcal{F}\Big)^{\vee} = \operatorname{Ext}^i\Big(\mathcal{F}, K_Y\Big) \tag{4.23}$$

for arbitrary sheaves $\mathcal{F}$. Here, K_Y is the canonical bundle of Y, that is, the sheaf of $\dim_{\mathbb{C}}(Y)$-forms. The duality follows from a perfect pairing defined by integrating over the manifold Y. Hence, the appearance of the canonical bundle, that is, the line bundle of top-dimensional holomorphic differentials.

More important for our purposes, there is a relative (fiberwise) version which, for any elliptic fibration $\pi : Y \to Z$, yields

$$\pi_*(\mathcal{F}^{\vee}) = \Big(R^1\pi_*(\mathcal{F} \otimes K_{Y|Z})\Big)^{\vee}, \tag{4.24}$$

where $K_{Y|Z}$ denotes the relative canonical bundle

$$K_{Y|Z} \stackrel{\text{def}}{=} K_Y \otimes \pi^* K_Z^{\vee}. \tag{4.25}$$

To make use of these dualities, we have to know how the canonical bundles transform under the $G = G_1 \times G_2 \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$ action. It is well known that, up to the character coming from the group action,

$$K_{\widetilde{X}} \simeq \mathcal{O}_{\widetilde{X}}, \qquad K_{B_i} \simeq \mathcal{O}_{B_i}(-f), \qquad K_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-2). \tag{4.26}$$

To determine the extra phase, all one has to do is to look at the behavior in one of the two G-stable fibers. Without loss of generality, we restrict our attention to $E_i = \beta_i^{-1}(0)$. From the analysis of the G action in [61], we know that G_1 acts as an order 3 translation on one of the fibers and complex multiplication on the other fiber. Again without loss of generality, we assume that G_1 acts via complex multiplication on E_1, thereby fixing the complex structure modulus of that elliptic curve. The complex structure on E_2, on the other hand, remains unconstrained and is one of the moduli of the

Calabi–Yau three-fold. To summarize, the fibers over $0 \in \mathbb{P}^1$ are as depicted in figure 1.

The G action can easily be written down in terms of local coordinates (z, u, v). First, G_1 acts as rotation on the base $\mathbb{P}^1$. Therefore, in terms of the local coordinate around the fixed point, it acts by

$$g_1 : z \longmapsto \omega z, \qquad g_2 : z \longmapsto z, \tag{4.27}$$

where ω is a third root of unity, which, without loss of generality, can be taken to be $\omega \stackrel{\text{def}}{=} e^{2\pi i/3}$. Second, consider the G action on the elliptic curve E_2. Since its complex structure λ_2 is arbitrary, each generator of G must act by translation. Hence, the action must be

$$g_1 : v \longmapsto v + \frac{1}{3}, \qquad g_2 : v \longmapsto v + \frac{\lambda_2}{3}. \tag{4.28}$$

Finally, let us examine the G action on E_1. By definition, g_2 acts on every fiber as a translation. In contrast, g_1 acts by complex multiplication on the elliptic curve. Moreover, this phase is coupled to the phase in the transformation of the coordinate z on the base $\mathbb{P}^1$, equation (4.27), since the holomorphic volume form $\Omega \sim du\, dv\, dz$ must be invariant under the G action.

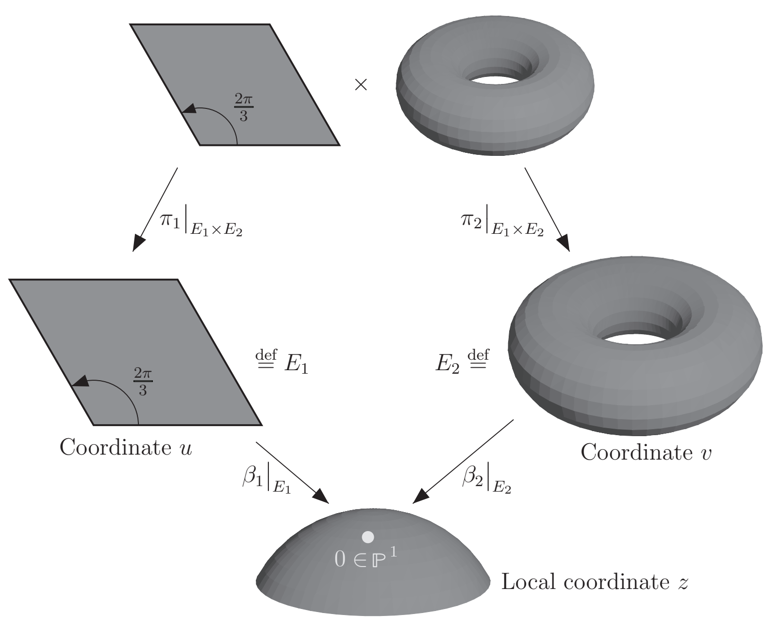

Figure 1: Fibers over $0 \in \mathbb{P}^1$.

Therefore, the G action on the u coordinate is as follows,

$$g_1 : u \longmapsto \omega^2 u, \qquad g_2 \colon u \longmapsto u + \frac{1}{3}. \tag{4.29}$$

Now we know the action on the local coordinates u, v, and z. Hence, we also know the G action on the top-dimensional holomorphic differentials, that is, the local sections of the canonical bundle. This fixes the missing characters in equation (4.26) to be

$$
\begin{tikzcd}[column sep=small]
\dim_{\mathbb{C}} = 3 : & & \left(\widetilde{X},\, K_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}\right) \arrow[dl, "\pi_1", "K_{\widetilde{X}|B_1} = \mathcal{O}_{\widetilde{X}}(\phi)"'] \arrow[dr, "\pi_2"', "K_{\widetilde{X}|B_2} = \chi_1^2 \mathcal{O}_{\widetilde{X}}(\phi)"] & \\
\dim_{\mathbb{C}} = 2 : & \left(B_1,\, K_{B_1} = \mathcal{O}_{B_1}(-f)\right) \arrow[dr, "\beta_1", "K_{B_1|\,\mathbb{P}^1} = \chi_1^2 \mathcal{O}_{B_1}(f)"'] & & \left(B_2,\, K_{B_2} = \chi_1 \mathcal{O}_{B_2}(-f)\right) \arrow[dl, "\beta_2"', "K_{B_2|\,\mathbb{P}^1} = \mathcal{O}_{B_2}(f)"] \\
\dim_{\mathbb{C}} = 1 : & & \left(\mathbb{P}^1,\, K_{\mathbb{P}^1} = \chi_1 \mathcal{O}_{\mathbb{P}^1}(-2)\right). &
\end{tikzcd}
\tag{4.30}
$$

5 The hidden E_8 bundle

5.1 Constructing vector bundles by extension

For simplicity, we start by constructing an $SU(2)$ instanton on the hidden brane. This allows us to introduce the techniques we are going to use later on in a simpler setting.

We define $\mathcal{H}$ to be an extension of the line bundle $\mathcal{O}_{\widetilde{X}}(-2\tau_1 - \tau_2 + \phi)$ by $\mathcal{O}_{\widetilde{X}}(2\tau_1 + \tau_2 - \phi)$. That is, by definition, $\mathcal{H}$ is the middle term in a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}}(2\tau_1 + \tau_2 - \phi) \longrightarrow \mathcal{H} \longrightarrow \mathcal{O}_{\widetilde{X}}(-2\tau_1 - \tau_2 + \phi) \longrightarrow 0. \tag{5.1}$$

The first Chern classes of the line bundles obviously add up to zero. Hence, we really obtain an $SU(2)$ rather than just a $U(2)$ bundle.

We furthermore demand that the extension be generic, ruling out the (slope-unstable) direct sum $\mathcal{O}_{\widetilde{X}}(2\tau_1 + \tau_2 - \phi) \oplus \mathcal{O}_{\widetilde{X}}(-2\tau_1 - \tau_2 + \phi)$. Apart from disallowing special cases, we are not going to impose any further restrictions on the extensions. That is, we are not constraining the vector bundle moduli to specific values.

Of course, this is only possible if non-trivial extensions exist. So, we must compute the possible extensions and show that there exist more than the trivial extension in order to justify our assumptions. The space of extensions is

$$\operatorname{Ext}^1\Big(\mathcal{O}_{\widetilde{X}}(-2\tau_1 - \tau_2 + \phi), \mathcal{O}_{\widetilde{X}}(2\tau_1 + \tau_2 - \phi)\Big). \tag{5.2}$$

However, not every such extension gives rise to an equivariant vector bundle. The line bundles are equivariant, but if the class of the extension changes under the group action then we do not obtain a group action on the extension. Only the G-invariant part of the Ext^1 yields an equivariant vector bundle.

We are going to compute the space of extensions in Subsection 5.4 and find for its G-invariant part that

$$\operatorname{Ext}^1\Big(\mathcal{O}_{\widetilde{X}}(-2\tau_1 - \tau_2 + \phi), \mathcal{O}_{\widetilde{X}}(2\tau_1 + \tau_2 - \phi)\Big)^G = 6. \tag{5.3}$$

Hence, the assumption that a generic (nontrivial) extension of the form, equation (5.1) exists is justified.

5.2 Sheaf cohomology on elliptic fibrations

5.2.1 Cohomology of line bundles

In order to determine the low energy spectrum, we must compute the cohomology groups of the vector bundle $\mathcal{H}$. The general way to do this is as follows. First, we can exploit the fibration structure, equation (3.4), and compute the cohomology of line bundles by successively pushing down all the way to a point. In the second step, we then determine the cohomology of an extension of line bundles by the associated long exact sequence in cohomology.

For example, let us start with the line bundle $\mathcal{O}_{\widetilde{X}}(2\tau_1 + \tau_2 - \phi)$. First, we use the elliptic fibration $\pi_1 : \widetilde{X} \to B_1$ to relate the cohomology groups $H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(2\tau_1 + \tau_2 - \phi))$ on the three-fold $\widetilde{X}$ to cohomology groups on the complex surface B_1. In general, for any fibration, the cohomology groups on the whole space are determined by a combination of the cohomology of the fibers and the cohomology of the base. This is made precise by the Leray spectral sequence:

$$\begin{aligned} E_2^{p,q}(\widetilde{X}|B_1) = H^p\Big(B_1,\, R^q\pi_{1*}\mathcal{O}_{\widetilde{X}}(2\tau_1+\tau_2-\phi)\Big) \\ \Longrightarrow H^{p+q}\Big(\widetilde{X},\, \mathcal{O}_{\widetilde{X}}(2\tau_1+\tau_2-\phi)\Big). \end{aligned} \tag{5.4}$$

The derived pushdown $R^q\pi_{1*}$ is nothing but the cohomology along the fiber, that is, some sheaves on B_1 that have to be computed first. Then, we must determine their cohomology groups (on the surface B_1). Starting with cohomology on B_1, the Leray spectral sequence then converges (denoted by "$\Rightarrow$") to the cohomology on the three-fold $\widetilde{X}$. By dimension, the terms of the first quadrant spectral sequence $E_2^{p,q}(\widetilde{X}|B_1)$ vanish for $p > \dim_{\mathbb{C}}(B_1) = 2$ and for $q > \dim_{\mathbb{C}}(\widetilde{X}) - \dim_{\mathbb{C}}(B_1) = 1$.

To completely determine the ingredients in the spectral sequence, we will use the following two facts.

- The projection formula, which in general reads

$$R^q\pi_*\Big(\mathcal{E}\otimes\pi^*\mathcal{F}\Big) = R^q\pi_*(\mathcal{E})\otimes\mathcal{F} \tag{5.5}$$

  for any fibration π, arbitrary sheaf $\mathcal{E}$, and vector bundle $\mathcal{F}$. In the case at hand, we conclude that

$$\begin{aligned} R^q\pi_{1*}\mathcal{O}_{\widetilde{X}}(2\tau_1+\tau_2-\phi) &= R^q\pi_{1*}\Big(\pi_1^*(\mathcal{O}_{B_1}(2t-f)) \\ &\qquad + \pi_2^*(\mathcal{O}_{B_2}(t))\Big) \\ &= \mathcal{O}_{B_1}(2t-f)\otimes R^q\pi_{1*}\pi_2^*(\mathcal{O}_{B_2}(t)). \end{aligned} \tag{5.6}$$

- The commutativity of the projections in equation (3.4), which implies

$$\begin{aligned} (R^q\pi_{1*})\circ(\pi_2^*) &= (\beta_1^*)\circ(R^q\beta_{2*}), \\ (R^q\pi_{2*})\circ(\pi_1^*) &= (\beta_2^*)\circ(R^q\beta_{1*}). \end{aligned} \tag{5.7}$$

Using these, the starting point of the spectral sequence, equation (5.4), is completely determined in terms of line bundles on B_1 alone,

$$\begin{aligned} H^p\Big(B_1, R^q\pi_{1*}\mathcal{O}_{\widetilde{X}}(2\tau_1+\tau_2-\phi)\Big) &= H^p\Big(B_1,\, \mathcal{O}_{B_1}(2t-f)\otimes R^q\pi_{1*}\pi_2^*(\mathcal{O}_{B_2}(t))\Big) \\ &= H^p\Big(B_1, \mathcal{O}_{B_1}(2t-f)\otimes\beta_1^*R^q\beta_{2*}(\mathcal{O}_{B_2}(t))\Big). \end{aligned} \tag{5.8}$$

It remains to compute the cohomology groups on B_1. We again proceed by pushing down one step to the base $\mathbb{P}^1$. But now we end up with two spectral sequences corresponding to the case $q=0$ and $q=1$ in the previous equation. In the following, we will describe the $q = 0$ case, the computation for $q = 1$ being completely analogous.

The Leray spectral sequence now reads

$$\begin{aligned} E_2^{p,q}(B_1|\,\mathbb{P}^1) &= H^p\Big(\,\mathbb{P}^1, R^q\beta_{1*}\Big[\mathcal{O}_{B_1}(2t-f)\otimes\beta_1^*\beta_{2*}\mathcal{O}_{B_2}(t)\Big]\Big) \\ &= H^p\Big(\,\mathbb{P}^1,\, R^q\beta_{1*}\mathcal{O}_{B_1}(2t-f)\otimes\beta_{2*}\mathcal{O}_{B_2}(t)\Big) \\ &= H^p\Big(\,\mathbb{P}^1, R^q\beta_{1*}\mathcal{O}_{B_1}(2t)\otimes\beta_{2*}\mathcal{O}_{B_2}(t)\otimes\mathcal{O}_{\mathbb{P}^1}(-1)\Big) \\ &\Longrightarrow H^{p+q}\Big(B_1, \pi_{1*}\mathcal{O}_{\widetilde{X}}(2\tau_1+\tau_2-\phi)\Big). \end{aligned} \tag{5.9}$$

The pushdown of the line bundles on B_1 can be found by a straightforward application of the long exact sequence for pushdowns. We defer the details to Appendix A, here only listing the result for convenience.

$$\mathcal{O}_{B_i}(nf) = \beta_i^*\mathcal{O}_{\mathbb{P}^1}(n), \quad n\in\mathbb{Z} \tag{5.10}$$

$$\beta_{i*}\mathcal{O}_{B_i}(2t) = 6\mathcal{O}_{\mathbb{P}^1} \qquad \beta_{i*}\mathcal{O}_{B_i}(-2t) = 0 \qquad R^1\beta_{i*}\mathcal{O}_{B_i}(2t) = 0 \tag{5.11}$$

$$\beta_{i*}\mathcal{O}_{B_i}(t) = 3\mathcal{O}_{\mathbb{P}^1} \qquad \beta_{i*}\mathcal{O}_{B_i}(-t) = 0 \qquad R^1\beta_{i*}\mathcal{O}_{B_i}(t) = 0 \tag{5.12}$$

$$R^1\beta_{1*}\mathcal{O}_{B_1}(-t) = 3\chi_1\mathcal{O}_{\mathbb{P}^1}(-1) \qquad R^1\beta_{1*}\mathcal{O}_{B_1}(-2t) = 6\chi_1\mathcal{O}_{\mathbb{P}^1}(-1) \tag{5.13}$$

$$R^1\beta_{2*}\mathcal{O}_{B_2}(-t) = 3\mathcal{O}_{\mathbb{P}^1}(-1) \qquad R^1\beta_{2*}\mathcal{O}_{B_2}(-2t) = 6\mathcal{O}_{\mathbb{P}^1}(-1). \tag{5.14}$$

Hence, the E_2 term of the spectral sequence is

$$E_2^{p,q}(B_1|\,\mathbb{P}^1) = \begin{cases} 0 & q=1 \\ H^p\big(\,\mathbb{P}^1,\, 18\mathcal{O}_{\mathbb{P}^1}(-1)\big) & q=0. \end{cases} \tag{5.15}$$

Sheaves on $\mathbb{P}^1$ are particularly simple. All vector bundles split into the sum of line bundles. Furthermore, the global sections of a line bundle are polynomials of the given degree. The number of monomials and their transformation under the $\mathbb{Z}_3\times\mathbb{Z}_3$ action can be found easily, determining the zeroth cohomology group. The first cohomology group is then Serre dual to the zeroth cohomology group. It follows that

$$\begin{aligned} H^0\big(\,\mathbb{P}^1,\, \chi\mathcal{O}_{\mathbb{P}^1}(n)\big) &= \begin{cases} 0, & n<0 \\ \chi\sum_{i=0}^n\chi_1^i, & n\geq 0 \end{cases} \\ H^1\big(\,\mathbb{P}^1, \chi\mathcal{O}_{\mathbb{P}^1}(n)\big) &= H^0\Big(\,\mathbb{P}^1, \big(\chi\mathcal{O}_{\mathbb{P}^1}(n)\big)^\vee\otimes K_{\mathbb{P}^1}\Big)^\vee. \end{aligned} \tag{5.16}$$

Putting everything together, we see that all cohomology groups vanish and, hence, $E_2^{p,q}(\widetilde{X}|B_1) = 0$. There is no possibility for nonvanishing differentials and, therefore, all cohomology groups of the line bundle $\mathcal{O}_{\widetilde{X}}(2\tau_1+\tau_2-\phi)$

actually vanish,

$$H^*\Big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(2\tau_1 + \tau_2 - \phi)\Big) = 0. \tag{5.17}$$

Similarly, one can compute the cohomology of the dual line bundle or simply invoke the Serre duality. In either case, one finds

$$H^*\Big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-2\tau_1 - \tau_2 + \phi)\Big) = 0. \tag{5.18}$$

5.2.2 Cohomology of an extension

We can now compute the cohomology of the $SU(2)$ bundle $\mathcal{H}$. By definition, it is the middle term of a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\widetilde{X}}(2\tau_1 + \tau_2 - \phi) \longrightarrow \mathcal{H} \longrightarrow \mathcal{O}_{\widetilde{X}}(-2\tau_1 - \tau_2 + \phi) \longrightarrow 0. \tag{5.19}$$

The associated long exact sequence of cohomology groups reads

$$\cdots \underbrace{H^i\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(2\tau_1 + \tau_2 - \phi)\big)}_{=0} \longrightarrow H^i\big(\widetilde{X}, \mathcal{H}\big) \longrightarrow \underbrace{H^i\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-2\tau_1 - \tau_2 + \phi)\big)}_{=0} \ldots, \tag{5.20}$$

so we immediately obtain

$$H^*\big(\widetilde{X}, \mathcal{H}\big) = 0. \tag{5.21}$$

5.3 Absence of hidden matter

Let us return to the physical application of the $SU(2)$ bundle $\mathcal{H}$ which we just constructed. We are going to take the usual regular embedding of $SU(2)$ in E_8 with commutant E_7. The fiber product of the $SU(2)$ principal bundle together with the trivial E_7 principal bundle then determines an E_8 principal bundle, which we take to be our hidden E_8 gauge bundle.

Now, the gauge fermions in the heterotic string transform in the adjoint representation of E_8, which branches as

$$R[E_8] \ni \underline{\mathbf{248}} = (\underline{\mathbf{3}}, \underline{\mathbf{1}}) \oplus (\underline{\mathbf{1}}, \underline{\mathbf{133}}) \oplus (\underline{\mathbf{2}}, \underline{\mathbf{56}}) \in R\big[SU(2) \times E_7\big]. \tag{5.22}$$

Correspondingly, the fermions, that is, the rank 248 vector bundle $\mathcal{E}_8^H$ associated to the hidden E_8 principal bundle, decompose as

$$\mathcal{E}_8^H = \Big(\operatorname{Sym}^2(\mathcal{H}) \otimes \theta(1)\Big) \oplus \Big(\theta(1) \otimes \theta(133)\Big) \oplus \Big(\mathcal{H} \otimes \theta(56)\Big), \tag{5.23}$$

where $\theta(n) \stackrel{\text{def}}{=} \widetilde{X} \times \mathbb{C}^n$ denotes the rank n trivial vector bundle.

5.4 The space of extensions

It remains to determine the space of extensions for the short exact sequence, eq. (5.1). Using elementary properties of the global Ext, this is given by

$$\begin{aligned}&\operatorname{Ext}^1\Big(\mathcal{O}_{\widetilde{X}}(-2\tau_1-\tau_2+\phi),\mathcal{O}_{\widetilde{X}}(2\tau_1+\tau_2-\phi)\Big)\\&\quad=\operatorname{Ext}^1\Big(\mathcal{O}_{\widetilde{X}},\mathcal{O}_{\widetilde{X}}(4\tau_1+2\tau_2-2\phi)\Big)=H^1\Big(\widetilde{X},\mathcal{O}_{\widetilde{X}}(4\tau_1+2\tau_2-2\phi)\Big).\end{aligned}\tag{5.24}$$

From a Leray spectral sequence, we can easily compute the cohomology groups of this line bundle and obtain

$$\dim_{\mathbb{C}} H^p\Big(\widetilde{X},\mathcal{O}_{\widetilde{X}}(4\tau_1+2\tau_2-2\phi)\Big)=\begin{cases}54 & p=1\\ 0 & p\neq 1.\end{cases}\tag{5.25}$$

We could now trace the $\mathbb{Z}_3\times\mathbb{Z}_3$ action through the Leray spectral sequence and determine directly which 54-dimensional representation occurs. However, there is a simple shortcut which we will employ instead.

For this, note that the index

$$\begin{aligned}\operatorname{Index}\Big(\mathcal{O}_{\widetilde{X}}(4\tau_1+2\tau_2-2\phi)\Big)&=\sum_{p=0}^{3}(-1)^p\dim_{\mathbb{C}} H^p\Big(\widetilde{X},\mathcal{O}_{\widetilde{X}}(4\tau_1+2\tau_2-2\phi)\Big)\\&=-54\end{aligned}\tag{5.26}$$

is simply divided by the order of the group $G=\mathbb{Z}_3\times\mathbb{Z}_3$ when descending to the quotient. That is,

$$\begin{aligned}\operatorname{Index}\left(\frac{\mathcal{O}_{\widetilde{X}}(4\tau_1+2\tau_2-2\phi)}{G}\right)&=\sum_{p=0}^{3}(-1)^p\dim_{\mathbb{C}} H^p\Big(\widetilde{X},\mathcal{O}_{\widetilde{X}}(4\tau_1+2\tau_2-2\phi)\Big)^G\\&=-9\end{aligned}\tag{5.27}$$

Since from equation (5.25) all the other cohomology groups vanish, only the $p=1$ term can have an invariant part which, moreover, must be nine-dimensional.

The same reasoning can be applied to the line bundles $\chi\mathcal{O}_{\widetilde{X}}(4\tau_1+2\tau_2-2\phi)$ for any character χ of G. The G-invariant subspace must always be nine-dimensional. Of course, the dimension of this invariant subspace is nothing else but the multiplicity of the representation χ^{-1} in $H^1(\widetilde{X},\mathcal{O}_{\widetilde{X}}(4\tau_1+2\tau_2-2\phi))$. It follows that this cohomology group decomposes as the sum of all nine irreducible $\mathbb{Z}_3 \times \mathbb{Z}_3$ representations, each with multiplicity 6:

$$H^p\Big(\widetilde{X},\, \mathcal{O}_{\widetilde{X}}(4\tau_1 + 2\tau_2 - 2\phi)\Big) = \begin{cases} 0, & p = 3 \\ 0, & p = 2 \\ 6\ \mathrm{Reg}(\mathbb{Z}_3 \times \mathbb{Z}_3), & p = 1 \\ 0, & p = 0. \end{cases} \tag{5.28}$$

Here, we used the fact that the *regular* representation of $G = \mathbb{Z}_3 \times \mathbb{Z}_3$, that is, the representation of G on its group ring $\mathbb{C}[G]$, decomposes as

$$\mathrm{Reg}(\mathbb{Z}_3 \times \mathbb{Z}_3) = \bigoplus_{i,j=0}^{2} \chi_1^i \chi_2^j, \tag{5.29}$$

the sum of every irreducible G representation with multiplicity one.

To summarize, the Ext group in question is

$$\mathrm{Ext}^1\left(\mathcal{O}_{\widetilde{X}}(-2\tau_1 - \tau_2 + \phi), \mathcal{O}_{\widetilde{X}}(2\tau_1 + \tau_2 - \phi)\right) = 6\ \mathrm{Reg}(\mathbb{Z}_3 \times \mathbb{Z}_3). \tag{5.30}$$

5.5 Checks on stability

As we stated in the introduction, in this paper we are not going to prove stability of the vector bundles in a mathematically rigorous sense. We will, however, subject them to the following two important and nontrivial tests.

The first is that a stable vector bundle is necessarily simple [78], that is, has no endomorphisms except multiplication by a constant. This can be expressed as

$$\mathrm{End}(\mathcal{H}) \simeq \mathbb{C} \Longleftrightarrow H^0\Big(\widetilde{X}, \mathcal{H} \otimes \mathcal{H}^\vee\Big) = 1. \tag{5.31}$$

The other test for stability is

$$\mathcal{H} \text{ stable} \Longrightarrow H^0\Big(\widetilde{X}, \mathcal{H}^\vee\Big) = 0, \tag{5.32}$$

which follows from the following contradiction. Assume that $H^0(\widetilde{X}, \mathcal{H}^\vee) \neq 0$. This means that there is a global section of $\mathcal{H}^\vee$. But a global section of $\mathcal{H}^\vee$ is a map $s : \mathcal{H} \to \mathcal{O}$, which is necessarily surjective and can be completed to a short exact sequence

$$0 \longrightarrow \ker(s) \longrightarrow \mathcal{H} \xrightarrow{s} \mathcal{O} \longrightarrow 0. \tag{5.33}$$

Since $\mathcal{H}$ has vanishing first Chern class, the slopes μ of the bundles all vanish,

$$\mu\big(\ker(s)\big) = \frac{c_1\big(\ker(s)\big)}{\operatorname{rank}\big(\ker(s)\big)} = 0 = \mu(\mathcal{H}). \tag{5.34}$$

Therefore $\ker(s)$ would be a destabilizing subsheaf of $\mathcal{H}$, and $\mathcal{H}$ could at most be semistable.

Finally, note that the dual bundle of a stable bundle is a again stable. Therefore, (slope-)stability of $\mathcal{H}$ implies the following three constraints on cohomology groups:

$$H^0\Big(\widetilde{X}, \mathcal{H} \otimes \mathcal{H}^\vee\Big) = 1, \qquad H^0\Big(\widetilde{X}, \mathcal{H}\Big) = 0, \qquad H^0\Big(\widetilde{X}, \mathcal{H}^\vee\Big) = 0. \tag{5.35}$$

We already computed the cohomology of $\mathcal{H}$ (and $\mathcal{H}^\vee$ by the Serre duality) and found that it vanishes, see equation (5.21). It remains to compute the cohomology of $\mathcal{H} \otimes \mathcal{H}^\vee$, which we will do in the following subsection. We find that, indeed, the vector bundle $\mathcal{H}$ is simple, that is, $H^0\big(\widetilde{X}, \mathcal{H} \otimes \mathcal{H}^\vee\big) = 1$. Hence, $\mathcal{H}$ passes all the checks on stability.

5.6 Simplicity

Let us perform the promised computation of the endomorphisms of the extension equation (5.1). We will first discuss the general case of an arbitrary extension of vector bundles

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0. \tag{5.36}$$

Then, the dual vector bundle $\mathcal{B}^\vee$ fits into the short exact sequence

$$0 \longrightarrow \mathcal{C}^\vee \longrightarrow \mathcal{B}^\vee \longrightarrow \mathcal{A}^\vee \longrightarrow 0 \tag{5.37}$$

and the tensor product $\mathcal{B} \otimes \mathcal{B}^\vee$ fits into the commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{A} \otimes \mathcal{C}^\vee \arrow[r] \arrow[d] & \mathcal{A} \otimes \mathcal{B}^\vee \arrow[r] \arrow[d] & \mathcal{A} \otimes \mathcal{A}^\vee \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{B} \otimes \mathcal{C}^\vee \arrow[r] \arrow[d] & \mathcal{B} \otimes \mathcal{B}^\vee \arrow[r] \arrow[d] & \mathcal{B} \otimes \mathcal{A}^\vee \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{C} \otimes \mathcal{C}^\vee \arrow[r] \arrow[d] & \mathcal{C} \otimes \mathcal{B}^\vee \arrow[r] \arrow[d] & \mathcal{C} \otimes \mathcal{A}^\vee \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{5.38}
$$

with all rows and columns exact. These nested short exact sequences yield interrelated long exact sequences of cohomology groups which we are going to use.

Let us apply these general considerations to the extension equation (5.1). The tensor products of the line bundles are either the trivial line bundle or the line bundle $\mathcal{O}_{\widetilde{X}}(4\tau_1 + 2\tau_2 - 2\phi)$, whose cohomology is noted in equation (5.28). Using these values, the commutative diagram of long exact sequences simplifies to

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \bigcirc \arrow[r] \arrow[d] & 1 \arrow[r, "\delta"] \arrow[d] & 6\ \mathrm{Reg}(\mathbb{Z}_3 \times \mathbb{Z}_3) \arrow[r] & \cdots \\
0 \arrow[r] & \bigcirc \arrow[r] \arrow[d] & H^0\Big(\widetilde{X},\, \mathcal{H} \otimes \mathcal{H}^\vee\Big) \arrow[r] \arrow[d] & \bigcirc \arrow[r] \arrow[d] & \cdots & \\
0 \arrow[r] & 1 \arrow[r] \arrow[d, "\delta"] & \bigcirc \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots & \\
 & 6\ \mathrm{Reg}(\mathbb{Z}_3 \times \mathbb{Z}_3) \arrow[d] & \vdots & \vdots & & \\
 & \vdots & & & &
\end{tikzcd}
\tag{5.39}
$$

where the empty circles are cohomology groups (of tensor products of $\mathcal{H}$ and line bundles) that are yet to be determined. To do this, we must analyze the coboundary map

$$\delta\colon H^0\Big(\widetilde{X},\, \mathcal{O}_{\widetilde{X}}\Big) \longrightarrow H^1\Big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(4\tau_1 + 2\tau_2 - 2\phi)\Big). \tag{5.40}$$

This is simply multiplication by the extension class[5]

$$\begin{aligned} \epsilon \in \operatorname{Ext}^1\Big(&\mathcal{O}_{\widetilde{X}}(-2\tau_1 - \tau_2 + \phi), \mathcal{O}_{\widetilde{X}}(2\tau_1 + \tau_2 - \phi)\Big) \\ &= H^1\Big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(4\tau_1 + 2\tau_2 - 2\phi)\Big), \end{aligned} \tag{5.41}$$

that is, the cohomology class encoding the choice of extension in equation (5.1). By our assumption that the extension class is generic (that is, not zero), the coboundary map δ is injective. This determines the missing entries in the long exact sequences, equation (5.39) to be

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 1 \arrow[r, "\delta"] \arrow[d] & \cdots \\ 0 \arrow[r] & 0 \arrow[r] \arrow[d] & H^0\Big(\widetilde{X},\, \mathcal{H}\otimes\mathcal{H}^\vee\Big) \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & 1 \arrow[r] \arrow[d, "\delta"'] & 1 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ & \vdots & \vdots & \vdots & \end{tikzcd} \tag{5.42}$$

Exactness then implies that the desired H^0 is either zero or one dimensional. But there is always a global section of $\mathcal{H}\otimes\mathcal{H}^\vee$ corresponding to multiplication by an overall constant. Hence,

$$H^0\Big(\widetilde{X},\, \mathcal{H}\otimes\mathcal{H}^\vee\Big) = 1. \tag{5.43}$$

[5]The discussion of the extension class is unavoidable at this point. If the extension were trivial, that is, $\epsilon = 0$, then $\mathcal{H}$ would be a sum of two line bundles. But, such a sum is never simple as the two line bundles can be scaled independently.

6 The visible E_8 bundle

6.1 The $SU(4)$ bundle

Let us now construct the $SU(4)$ instanton inside the visible E_8 gauge group. We first define an auxiliary bundle $\mathcal{W}$ on the dP_9 surface B_2. For that, we take an extension of the form

$$0 \longrightarrow \mathcal{O}_{B_2}(-2f) \longrightarrow \mathcal{W} \longrightarrow \chi_2 \mathcal{O}_{B_2}(2f) \otimes I_9 \longrightarrow 0, \tag{6.1}$$

where I_9 denotes the ideal sheaf of nine points, which we take to be one generic G orbit. Then the nine points end up in three distinct fibers of the elliptic fibration β_2: $B_2 \to \mathbb{P}^1$, each containing three points. Furthermore, we remark that the first Chern class of $\mathcal{W}$ vanishes, and hence $\mathcal{W} \simeq \mathcal{W}^\vee$.

Using the pullback of $\mathcal{W}$, we now define two $U(2)$ bundles on $\widetilde{X}$ as

$$\begin{aligned} \mathcal{V}_1 &\stackrel{\text{def}}{=} \chi_2 \mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2) \oplus \chi_2 \mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2) = 2\chi_2 \mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2), \\ \mathcal{V}_2 &\stackrel{\text{def}}{=} \mathcal{O}_{\widetilde{X}}(\tau_1-\tau_2) \otimes \pi_2^*(\mathcal{W}). \end{aligned} \tag{6.2}$$

Finally, we define the rank 4 bundle $\mathcal{V}$ as a generic extension of $\mathcal{V}_2$ with $\mathcal{V}_1$, that is,

$$0 \longrightarrow \mathcal{V}_1 \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}_2 \longrightarrow 0. \tag{6.3}$$

Because the first Chern classes of the $U(2)$ bundles add up to zero, $c_1(\mathcal{V}_1) + c_1(\mathcal{V}_2) = 0 = c_1(\mathcal{V})$, the bundle $\mathcal{V}$ defines an $SU(4)$ gauge bundle.

Simply computing the Chern character, see equation (6.13), we immediately can conclude that the net number of generations (the index) on the covering space is

$$\begin{aligned} N_{\text{gen}}(\widetilde{X}) = \operatorname{Index}(\mathcal{V}) \int_{\widetilde{X}} \operatorname{ch}(\mathcal{V})\, Td(T\widetilde{X}) &= -\int_{\widetilde{X}} 9\,\mathrm{PD}(\tau_1\tau_2^2) \\ = -\int_{\widetilde{X}} 9\,\mathrm{PD}(3\{\text{pt.}\}) = -27, & \end{aligned} \tag{6.4}$$

where PD denotes the Poincaré dual. Therefore, the net number of generations on the quotient is

$$N_{\text{gen}}(X) = N_{\text{gen}}(\widetilde{X}/G) = \frac{1}{|G|} N_{\text{gen}}(\widetilde{X}) = -3, \tag{6.5}$$

and we do get three net generations (the sign of the index is irrelevant). Of course, obtaining three net generations is necessary but not sufficient to get a standard model spectrum. In general, one can expect a whole zoo of exotic matter accompanying these three generations. To discuss these, we must compute the actual cohomology groups which correspond to the massless modes of the Dirac operator and not just their alternating sum.

We will compute the cohomology groups of $\mathcal{V}$ and $\wedge^2\mathcal{V}$ in the remainder of this section and then extract the complete low energy spectrum in Section 7.

6.2 Anomaly cancellation with five-branes

First of all, let us check the heterotic anomaly cancellation. It requires that

$$[\operatorname{tr} R^2] - \frac{1}{30}[\operatorname{tr} F^2] = 0 \in H^4(\widetilde{X}, \mathbb{Z}), \tag{6.6}$$

where F is in the adjoint representation of the $E_8 \times E_8$ gauge group. For any regular $SU(n)$ subgroup of E_8, the second Chern class of an $SU(n)$ vector bundle $\mathcal{F}$ and the second Chern class of the associated adjoint E_8 bundle $\mathcal{E}_8$ are related by

$$c_2(\mathcal{F}) = \frac{1}{60} c_2(\mathcal{E}_8). \tag{6.7}$$

Now, we pick an $SU(n_V)$ bundle $\mathcal{V}$ and an $SU(n_H)$ bundle $\mathcal{H}$ and construct an $E_8^V \times E_8^H$ bundle from the regular embeddings of $SU(n_i) \subset E_8^i$. Then, we can rewrite the anomaly cancellation in terms of characteristic classes as

$$c_2(T\widetilde{X}) - \frac{1}{60}c_2(\mathcal{E}_8^V) - \frac{1}{60}c_2(\mathcal{E}_8^H) = c_2(T\widetilde{X}) - c_2(\mathcal{V}) - c_2(\mathcal{H}) = 0 \in H^4(\widetilde{X}, \mathbb{Z}). \tag{6.8}$$

This condition can be slightly relaxed if one allows for five-branes, which also contribute to the anomaly. To preserve supersymmetry, the five-branes must be wrapped on an effective curve, that is an actual holomorphic curve rather than a sum of curves and orientation-reversed curves. The Poincaré dual of the curve C then contributes to the anomaly as

$$c_2(T\widetilde{X}) - c_2(\mathcal{V}) - c_2(\mathcal{H}) = \mathrm{PD}(C) \in H^4(\widetilde{X}, \mathbb{Z}). \tag{6.9}$$

If this equation holds, then wrapping a five-brane on the curve C cancels the heterotic anomaly. Now, the Chern classes of (the tangent bundle of) a fiber product of dP_9 surfaces was already computed in [79]. One finds that

$$c_1(T\widetilde{X}) = 0, \qquad c_2(T\widetilde{X}) = 12(\tau_1^2 + \tau_2^2), \qquad c_3(T\widetilde{X}) = 0. \tag{6.10}$$

The Chern classes for the visible and the hidden gauge bundle are also easy to compute. For simplicity, we work with the Chern character. The Chern character of the hidden bundle is

$$\begin{aligned} \mathrm{ch}(\mathcal{H}) &= \mathrm{ch}\Big(\mathcal{O}(2\tau_1 + \tau_2 - \phi)\Big) + \mathrm{ch}\Big(\mathcal{O}(-2\tau_1 - \tau_2 + \phi)\Big) \\ &= \mathrm{e}^{2\tau_1+\tau_2-\phi} + e^{-2\tau_1-\tau_2+\phi} = 2 - 8\tau_1^2 - 5\tau_2^2 + 4\tau_1\tau_2, \end{aligned} \tag{6.11}$$

where we have used the relations in the intersection ring, equation (3.11). The second Chern class is then

$$c_2(\mathcal{H}) = \frac{1}{2}c_1(\mathcal{H})^2 - \mathrm{ch}_2(\mathcal{H}) = 8\tau_1^2 + 5\tau_2^2 - 4\tau_1\tau_2. \tag{6.12}$$

Similarly, the Chern character of the visible bundle is given by

$$\begin{aligned} \mathrm{ch}(\mathcal{V}) = \mathrm{ch}(\mathcal{V}_1) + \mathrm{ch}(\mathcal{V}_2) &= 2\mathrm{ch}\Big(\mathcal{O}(-\tau_1 + \tau_2)\Big) + \mathrm{ch}\Big(\mathcal{O}(\tau_1 - \tau_2)\Big)\mathrm{ch}(\pi_2^* W) \\ &= 2\mathrm{e}^{-\tau_1+\tau_2} + \mathrm{e}^{+\tau_1-\tau_2}\Big(\mathrm{e}^{-2\phi} + \mathrm{e}^{2\phi}(1 - 9\tau_2^2)\Big) \\ &= 4 + 2\tau_1^2 - 7\tau_2^2 - 4\tau_1\tau_2 - 9\tau_1\tau_2^2 \end{aligned} \tag{6.13}$$

and its second Chern class is

$$c_2(\mathcal{V}) = -2\tau_1^2 + 7\tau_2^2 + 4\tau_1\tau_2. \tag{6.14}$$

Combining everything, the combined gravity and gauge contribution to the anomaly is

$$c_2(T\widetilde{X}) - c_2(\mathcal{V}) - c_2(\mathcal{H}) = 6\tau_1^2. \tag{6.15}$$

The homology class

$$\mathrm{PD}(\tau_1^2) = \mathrm{PD}(\pi_1^{-1}(t^2)) = \mathrm{PD}\Big(\pi_1^{-1}(\{\mathrm{pt.}\})\Big) \tag{6.16}$$

is effective, indeed it is a multiple of an elliptic fiber of π_1. Wrapping five-branes on this homology class cancels the anomaly, equation (6.15), and yields a completely well defined, albeit strongly coupled, compactification.

So far, we really worked on the universal covering space $\widetilde{X}$, whereas we ultimately want to compactify on the $G \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$ quotient $X = \widetilde{X}/G$. This is justified as follows. By definition, we have a quotient map

$$q : \widetilde{X} \longrightarrow X, \quad p \longmapsto Gp. \tag{6.17}$$

Moreover, the vector bundle $\mathcal{V}$ on $\widetilde{X}$ is really the pull back of a vector bundle $\mathcal{V}/G$ on X. The Chern classes are natural, that is

$$\begin{tikzcd} \mathcal{V} \arrow[d] & \mathcal{V}/G \arrow[l, "q^*"'] \arrow[d] \\ \widetilde{X} \arrow[r, "q"] & X \end{tikzcd} \implies c_i(\mathcal{V}) = c_i\Big(q^*\left(\frac{\mathcal{V}}{G}\right)\Big) = q^* c_i\left(\frac{\mathcal{V}}{G}\right). \tag{6.18}$$

At least rationally[6] , the pullback of cohomology classes

$$q^*\colon H^{2i}\Big(X,\,\mathbb{Q}\Big) \longrightarrow H^{2i}\Big(\widetilde{X},\mathbb{Q}\Big) \tag{6.19}$$

is just the inclusion of the G-invariant cohomology,

$$H^{2i}\Big(X,\,\mathbb{Q}\Big) = H^{2i}\Big(\widetilde{X},\,\mathbb{Q}\Big)^G \subset H^{2i}\Big(\widetilde{X},\mathbb{Q}\Big). \tag{6.20}$$

In particular, q^* is injective. Hence, the vanishing of a sum of Chern classes on $\widetilde{X}$ is sufficient to conclude that the same sum on the quotient also vanishes.

6.3 Cohomology of $\mathcal{V}$

6.3.1 Cohomology of $\mathcal{V}_1$

First, let us consider the cohomology of the line bundle $\chi_2\mathcal{O}(-\tau_1+\tau_2)$, one-half of the vector bundle $\mathcal{V}_1$. We compute the cohomology by pushing down to B_1 and obtain for the E_2 tableau of the Leray spectral sequence

$$\begin{aligned} E_2^{p,q}(\widetilde{X}|B_1) &= H^p\Big(B_1, R^q\pi_{1*}(\chi_2\mathcal{O}(-\tau_1+\tau_2))\Big) \\ &= H^p\Big(B_1, \mathcal{O}_{B_1}(-t)\otimes R^q\pi_{1*}\pi_2^*(\chi_2\mathcal{O}_{B_2}(t))\Big) \\ &= H^p\Big(B_1, \mathcal{O}_{B_1}(-t)\otimes \beta_1^* R^q\beta_{2*}(\chi_2\mathcal{O}_{B_2}(t))\Big) \\ &\Longrightarrow \quad H^{p+q}\Big(\widetilde{X},\,\chi_2\mathcal{O}(-\tau_1+\tau_2)\Big). \end{aligned} \tag{6.21}$$

Because the fiber degree of $\mathcal{O}_{B_2}(t)$ is positive, the $q=1$ row vanishes, while for $q=0$, we obtain

$$\begin{aligned} H^p\Big(B_1, \mathcal{O}_{B_1}(-t)\otimes \beta_1^*\beta_{2*}(\chi_2\mathcal{O}_{B_2}(t))\Big) &= H^p\Big(B_1, \mathcal{O}_{B_1}(-t)\otimes\beta_1^*(3\chi_2\mathcal{O}_{\mathbb{P}^1})\Big) \\ &= H^p\Big(B_1, 3\chi_2\mathcal{O}_{B_1}(-t)\Big). \end{aligned} \tag{6.22}$$

We compute this cohomology group by another Leray spectral sequence

$$\begin{aligned} E_2^{p,q}(B_1|\,\mathbb{P}^1\,) &= H^p\Big(\,\mathbb{P}^1, 3R^q\beta_{1*}(\chi_2\mathcal{O}_{B_1}(-t))\Big) \\ &= \begin{cases} H^p\Big(\,\mathbb{P}^1, 9\chi_1\chi_2\mathcal{O}_{\mathbb{P}^1}(-1)\Big) & q=1 \\ 0, & q=0. \end{cases} \end{aligned} \tag{6.23}$$

---

[6]We are ignoring possible torsion issues for the purposes of this paper.

Every entry in the second tableau vanishes and, therefore, all cohomology groups vanish,

$$H^*\Big(\widetilde{X}, \mathcal{V}_1\Big) = 0. \tag{6.24}$$

6.3.2 Cohomology of $\mathcal{V}_2$

We continue with the cohomology of the rank 2 bundle $\mathcal{V}_2$. Its dual[7] is

$$\mathcal{V}_2^\vee = \mathcal{O}(-\tau_1 + \tau_2) \otimes \pi_2^*(\mathcal{W}^\vee), \tag{6.26}$$

and, because the degree on the fiber is negative, the pushdowns

$$\pi_{1*}(\mathcal{V}_2) = 0, \qquad \pi_{2*}(\mathcal{V}_2^\vee) = 0 \tag{6.27}$$

automatically vanish. But a global sections on $\widetilde{X}$ is a choice of section on every fiber, that is,

$$H^0\Big(\widetilde{X}, \mathcal{V}_i\Big) = H^0\Big(B_j, \pi_{j*}(\mathcal{V}_i)\Big), \qquad i = 1, 2, \;\; j = 1, 2, \tag{6.28}$$

and we immediately conclude that

$$H^0\Big(\widetilde{X}, \mathcal{V}_2\Big) = 0, \qquad H^3\Big(\widetilde{X}, \mathcal{V}_2\Big) \simeq H^0\Big(\widetilde{X}, \mathcal{V}_2^\vee\Big)^\vee = 0. \tag{6.29}$$

Next, let us compute $H^2(\widetilde{X}, \mathcal{V}_2)$ or, rather, its Serre dual $H^1(\widetilde{X}, \mathcal{V}_2^\vee)$. Using the Leray spectral sequence, one immediately shows that

$$H^1\Big(\widetilde{X}, \mathcal{V}_2^\vee\Big) = H^0\Big(B_2, R^1\pi_{2*}\mathcal{V}_2^\vee\Big) = H^0\Big(\mathbb{P}^1, \beta_{2*}(R^1\pi_{2*}\mathcal{V}_2^\vee)\Big). \tag{6.30}$$

We have to determine the higher pushdown of $\mathcal{V}_2^\vee$. For that, we start with the projection formula and arrive at

$$\begin{aligned}
\beta_{2*}(R^1\pi_{2*}\mathcal{V}_2^\vee) &= \beta_{2*}\Big(R^1\pi_{2*}(\mathcal{O}_{\widetilde{X}}(-\tau_1)) \otimes \mathcal{O}_{B_2}(t) \otimes \mathcal{W}^\vee\Big) \\
&= \beta_{2*}\Big(R^1\pi_{2*} \circ \pi_1^*(\mathcal{O}_{B_1}(-t)) \otimes \mathcal{O}_{B_2}(t) \otimes \mathcal{W}^\vee\Big) \\
&= \beta_{2*}\Big(\beta_2^* \circ R^1\beta_{1*}(\mathcal{O}_{B_1}(-t)) \otimes \mathcal{O}_{B_2}(t) \otimes \mathcal{W}^\vee\Big) \\
&= R^1\beta_{1*}\Big(\mathcal{O}_{B_1}(-t)\Big) \otimes \beta_{2*}\Big(\mathcal{O}_{B_2}(t) \otimes \mathcal{W}^\vee\Big).
\end{aligned} \tag{6.31}$$

Since the pushdown does not distribute over tensor products, we have to compute the second factor separately. For that, we use the long exact sequence associated to the pushdown of the short exact sequence, equation (6.1). But that sequence contains the ideal sheaf, whose pushdown we

[7] We remark that $\underline{\mathbf{2}} = \overline{\underline{\mathbf{2}}} \in R[SU(2)]$ and, therefore, $\mathcal{W} \simeq \mathcal{W}^\vee$ as vector bundles. However, the dual equivariant structure differs and $\mathcal{W}^\vee$ is an extension

$$0 \longrightarrow \chi_2^2 \mathcal{O}_{B_2}(-2f) \longrightarrow \mathcal{W}^\vee \longrightarrow \mathcal{O}_{B_2}(2f) \otimes I_9 \longrightarrow 0. \tag{6.25}$$

have to determine first. By definition, the ideal sheaf is the sheaf of functions vanishing at the given nine points, that is, the kernel of the restriction to the skyscraper sheaf on these nine points,

$$0 \longrightarrow I_9 \longrightarrow \mathcal{O}_{B_2} \longrightarrow \bigoplus_{i=1}^{9} \mathcal{O}_{p_i} \longrightarrow 0. \tag{6.32}$$

Since we ultimately want to compute the pushdown of $\mathcal{O}_{B_2}(t) \otimes \mathcal{W}$ instead of $\mathcal{W}$, we tensor everything with $\mathcal{O}_{B_2}(t)$ and obtain

$$0 \longrightarrow I_9 \otimes \mathcal{O}_{B_2}(t) \longrightarrow \mathcal{O}_{B_2}(t) \longrightarrow \bigoplus_{i=1}^{9} \mathcal{O}_{p_i} \longrightarrow 0. \tag{6.33}$$

Now a generic G orbit consists of nine points, living on three distinct fibers of the elliptic fibration B_2. The generator g_1, acting nontrivially on the base, permutes the three fibers. And the generator g_2, acting by translation along each fiber separately, permutes the triple of points on each fiber. We label the points such that

$$g_1: \begin{tikzcd} p_1 \arrow[rr] & & p_2 \arrow[dl] \\ & p_3 \arrow[ul] & \end{tikzcd}, \quad \begin{tikzcd} p_4 \arrow[rr] & & p_5 \arrow[dl] \\ & p_6 \arrow[ul] & \end{tikzcd}, \quad \begin{tikzcd} p_7 \arrow[rr] & & p_8 \arrow[dl] \\ & p_9 \arrow[ul] & \end{tikzcd}$$

$$g_2: \begin{tikzcd} p_1 \arrow[rr] & & p_4 \arrow[dl] \\ & p_7 \arrow[ul] & \end{tikzcd}, \quad \begin{tikzcd} p_2 \arrow[rr] & & p_5 \arrow[dl] \\ & p_8 \arrow[ul] & \end{tikzcd}, \quad \begin{tikzcd} p_3 \arrow[rr] & & p_6 \arrow[dl] \\ & p_9 \arrow[ul] & \end{tikzcd}. \tag{6.34}$$

The image under the projection map β_2 is then

$$\beta_2(p_k) = \beta_2(p_{3+k}) = \beta_2(p_{6+k}), \quad k = 1, 2, 3, \tag{6.35}$$

and accordingly

$$\beta_{2*}\Big(\mathcal{O}_{p_k}\Big) = \beta_{2*}\Big(\mathcal{O}_{p_{3+k}}\Big) = \beta_{2*}\Big(\mathcal{O}_{p_{6+k}}\Big), \quad k = 1, 2, 3. \tag{6.36}$$

Furthermore, the fiber degree of $I_9 \otimes \mathcal{O}_{B_2}(t)$ is always positive or zero. This means that, for generic positions of the points p_k, the first cohomology group of the restriction $\big(I_9 \otimes \mathcal{O}_{B_2}(t)\big)|_f$ vanishes. Hence,

$$R^1\beta_{2*}\Big(I_9 \otimes \mathcal{O}_{B_2}(t)\Big) = 0. \tag{6.37}$$

Using all of this, the long exact sequence for the pushdown of equation (6.33) simplifies to

$$0 \longrightarrow \beta_{2*}\big(I_9 \otimes \mathcal{O}_{B_2}(t)\big) \longrightarrow 3\mathcal{O}_{\mathbb{P}^1} \longrightarrow \bigoplus_{i=1}^{3} \mathcal{O}_{3\beta_2(p_i)} \longrightarrow 0. \tag{6.38}$$

Part of the Heisenberg group action on $\beta_{2*}\mathcal{O}_{B_2}(t) = 3\mathcal{O}_{\mathbb{P}^1}$ permutes the line bundles, this uniquely fixes the multi degrees of the pushdown of $I_9 \otimes$

$\mathcal{O}_{B_2}(t)$ to be

$$\beta_{2*}\Big(I_9 \otimes \mathcal{O}_{B_2}(t)\Big) = \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^1}(-3). \tag{6.39}$$

Now we can compute the pushdown of $\mathcal{W}^\vee \otimes \mathcal{O}_{B_2}(t)$, which fits into a short exact sequence[8]

$$0 \longrightarrow \mathcal{O}_{B_2}(-2f+t) \longrightarrow \mathcal{W}^\vee \otimes \mathcal{O}_{B_2}(t) \longrightarrow \mathcal{O}_{B_2}(2f) \otimes I_9 \otimes \mathcal{O}_{B_2}(t) \longrightarrow 0. \tag{6.40}$$

Since $R^1\beta_{2*}\mathcal{O}_{B_2}(-2f+t) = 0$ for degree reasons, the associated long exact sequence for the pushdown simplifies to

$$0 \longrightarrow 3\mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow \beta_{2*}\Big(\mathcal{W}^\vee \otimes \mathcal{O}_{B_2}(t)\Big) \longrightarrow 3\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow 0. \tag{6.41}$$

There is no nontrivial extension and, hence,

$$\beta_{2*}\Big(\mathcal{W}^\vee \otimes \mathcal{O}_{B_2}(t)\Big) = 3\mathcal{O}_{\mathbb{P}^1}(-1) \oplus 3\mathcal{O}_{\mathbb{P}^1}(-2). \tag{6.42}$$

Finally, we found the pushdown in equation (6.31) to be

$$\begin{aligned}\beta_{2*}(R^1\pi_{2*}\mathcal{V}_2^\vee) &= R^1\beta_{1*}(\mathcal{O}_{B_1}(-t)) \otimes \beta_{2*}\Big(\mathcal{O}_{B_2}(t) \otimes \mathcal{W}^\vee\Big)\\ &= 3\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \Big(3\mathcal{O}_{\mathbb{P}^1}(-1) \oplus 3\mathcal{O}_{\mathbb{P}^1}(-2)\Big)\\ &= 9\mathcal{O}_{\mathbb{P}^1}(-2) \oplus 9\mathcal{O}_{\mathbb{P}^1}(-3).\end{aligned} \tag{6.43}$$

But negative degree line bundles do not have global sections, that is,

$$H^1\Big(\widetilde{X}, \mathcal{V}_2^\vee\Big) = H^0\Big(\mathbb{P}^1, \beta_{2*}(R^1\pi_{2*}\mathcal{V}_2^\vee)\Big) = 0. \tag{6.44}$$

6.3.3 Cohomology of the extension

Thus far we determined that all cohomology groups of $\mathcal{V}_1$ and $\mathcal{V}_2$, except $H^1(\widetilde{X}, \mathcal{V}_2)$, vanish. We could compute this last cohomology group again

[8]Here we are suppressing the $\mathbb{Z}_3 \times \mathbb{Z}_3$ characters, since we have not properly defined what we mean by $\chi_2\mathcal{O}_{B_2}(t)$. Indeed, there is no $\mathbb{Z}_3 \times \mathbb{Z}_3$ action on $\mathcal{O}_{B_2}(t)$, only on the bundles $\mathcal{V}_1$ and $\mathcal{V}_2$.

via the Leray spectral sequence, using the formula in equation (6.43). It is simpler to just use the index

$$\operatorname{Index}(\mathcal{V}) = \operatorname{Index}(\mathcal{V}_1) + \operatorname{Index}(\mathcal{V}_2) = -27 \Longrightarrow \operatorname{Index}(\mathcal{V}_2) = -27. \tag{6.45}$$

Together with the argument in Section 5.4, this determines the dimension and the $\mathbb{Z}_3 \times \mathbb{Z}_3$ representation to be

$$\dim_{\mathbb{C}} H^1(\widetilde{X}, \mathcal{V}_2) = 27, \qquad H^1(\widetilde{X}, \mathcal{V}_2) = 3 \operatorname{Reg}(\mathbb{Z}_3 \times \mathbb{Z}_3). \tag{6.46}$$

Now it is a simple matter to apply the long exact sequence associated to the extension equation (6.3) and find the cohomology groups of $\mathcal{V}$. Because so many entries vanish, there are no ambiguities and we immediately obtain

$$H^p\Big(\widetilde{X}, \mathcal{V}\Big) = \begin{cases} 0, & p = 3 \\ 0, & p = 2 \\ 3 \operatorname{Reg}(\mathbb{Z}_3 \times \mathbb{Z}_3), & p = 1 \\ 0, & p = 0. \end{cases} \tag{6.47}$$

6.4 Cohomology of $\wedge^2 \mathcal{V}$

6.4.1 Exact sequences

To compute the cohomology of $\wedge^2 \mathcal{V}$, we have to relate it to cohomology groups involving $\mathcal{V}_1$ and $\mathcal{V}_2$. For this, note that we are, by definition, given an injection $\mathcal{V}_1 \to \mathcal{V}$ and a surjection $\mathcal{V} \to \mathcal{V}_2$. From these, we can construct maps of various tensor operations. In particular, we get short exact sequences

$$0 \longrightarrow \wedge^2 \mathcal{V}_1 \longrightarrow \wedge^2 \mathcal{V} \longrightarrow \mathcal{Q}_1 \longrightarrow 0, \tag{6.48}$$

$$0 \longrightarrow \mathcal{Q}_2 \longrightarrow \wedge^2 \mathcal{V} \longrightarrow \wedge^2 \mathcal{V}_2 \longrightarrow 0 \tag{6.49}$$

with some cokernel $\mathcal{Q}_1$ and kernel $\mathcal{Q}_2$. Now $\wedge^2 \mathcal{V}$ contains contributions of $\wedge^2 \mathcal{V}_1$, $\mathcal{V}_1 \otimes \mathcal{V}_2$, and $\wedge^2 \mathcal{V}_2$ (but, of course, is not a direct sum of these). Keeping this in mind, we can relate the two short exact sequences in a commutative diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \wedge^2\mathcal{V}_1 \arrow[r] \arrow[d, equal] & \mathcal{Q}_1 \arrow[r] \arrow[d] & \mathcal{V}_1 \otimes \mathcal{V}_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \wedge^2\mathcal{V}_1 \arrow[r] & \wedge^2\mathcal{V} \arrow[r] \arrow[d] & \mathcal{Q}_2 \arrow[r] \arrow[d] & 0 \\
 & & \wedge^2\mathcal{V}_2 \arrow[r, equal] \arrow[d] & \wedge^2\mathcal{V}_2 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
\tag{6.50}
$$

with exact rows and columns.

Furthermore, we can easily determine the line bundles

$$\wedge^2\mathcal{V}_1 = \chi_2^2\mathcal{O}_{\widetilde{X}}(-2\tau_1 + 2\tau_2), \tag{6.51}$$

$$\wedge^2\mathcal{V}_2 = \mathcal{O}_{\widetilde{X}}(2\tau_1 - 2\tau_2) \tag{6.52}$$

simply by computing their first Chern class. As one can easily calculate using the Leray spectral sequence, it turns out that these line bundles have no cohomology,

$$H^*\Big(\widetilde{X},\, \wedge^2\mathcal{V}_1\Big) = 0 = H^*\Big(\widetilde{X},\, \wedge^2\mathcal{V}_2\Big). \tag{6.53}$$

Because of this lucky coincidence, the long exact sequences in cohomology for the diagram, equation (6.50), identify

$$H^p\Big(\widetilde{X}, \wedge^2\mathcal{V}\Big) \simeq H^p\Big(\widetilde{X}, \mathcal{Q}_1\Big) \simeq H^p\Big(\widetilde{X}, \mathcal{Q}_2\Big) \simeq H^p\Big(\widetilde{X}, \mathcal{V}_1 \otimes \mathcal{V}_2\Big). \tag{6.54}$$

Hence, we have simplified the computation of the cohomology groups of $\wedge^2\mathcal{V}$ to the cohomology of the vector bundle

$$\mathcal{V}_1 \otimes \mathcal{V}_2 = 2\chi_2\pi_2^*(\mathcal{W}) = 2\pi_2^*(\chi_2\mathcal{W}). \tag{6.55}$$

6.4.2 A pushdown formula

To compute the cohomology of $\mathcal{V}_1 \otimes \mathcal{V}_2$, we first have to determine the pushdown of $\mathcal{W}$ to the base $\mathbb{P}^1$.

We start by pushing down the ideal sheaf. The long exact sequence for the pushdown of the sequence equation (6.32) is

$$\begin{tikzcd} 0 \arrow[r] & \beta_{2*}(I_9) \arrow[r] & \mathcal{O}_{\mathbb{P}^1} \arrow[r, "r"] & \bigoplus_{k=1}^{3} \mathcal{O}_{3\beta_2(p_k)} \arrow[dlll, rounded corners] \\ R^1\beta_{2*}(I_9) \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(-1) \arrow[r] & 0 \arrow[r] & 0. \end{tikzcd} \tag{6.56}$$

The restriction map r works as follows. It takes a local function f in a neighborhood of $\beta_2(p_k) \in \mathbb{P}^1$ and pulls it back to a local function on B_2. This function is then restricted to the three points p_k, p_{3+k}, and p_{6+k}. Since all functions on an elliptic fiber are constant, the restriction to these three points yields the same value. Hence, the image of r is one dimensional inside the three-dimensional vector space over $\beta_2(p_k)$. More precisely, let

$$G_1 \stackrel{\text{def}}{=} \{e, g_1, g_1^2\}, \qquad G_2 \stackrel{\text{def}}{=} \{e, g_2, g_2^2\} \tag{6.57}$$

be the $\mathbb{Z}_3$ groups generated by g_1 and g_2. Then the three-dimensional stalk is the regular representation of G_2,

$$\operatorname{Reg}(G_2) = 1 \oplus \chi_2 \oplus \chi_2^2, \tag{6.58}$$

and the image of r is the trivial representation. Knowing the restriction map r determines the pushdown of I_9 to be

$$\begin{aligned} \beta_{2*}(I_9) &= \mathcal{O}_{\mathbb{P}^1}(-3), \\ R^1\beta_{2*}(I_9) &= \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \left[\bigoplus_{k=1}^{3} (\chi_2 \oplus \chi_2^2) \mathcal{O}_{\beta_2(p_k)} \right]. \end{aligned} \tag{6.59}$$

Now, we can calculate the pushdown of $\chi_2\mathcal{W}$ to the base $\mathbb{P}^1$. The vector bundle is defined via an extension

$$0 \longrightarrow \chi_2\mathcal{O}_{B_2}(-2f) \longrightarrow \chi_2\mathcal{W} \longrightarrow \chi_2^2\mathcal{O}_{B_2}(2f) \otimes I_9 \longrightarrow 0, \tag{6.60}$$

and the associated long exact sequence of pushdowns is

$$\begin{tikzcd} 0 \arrow[r] & \chi_2\mathcal{O}_{\mathbb{P}^1}(-2) \arrow[r] & \beta_{2*}(\chi_2\mathcal{W}) \arrow[r] & \chi_2^2\mathcal{O}_{\mathbb{P}^1}(2) \otimes \beta_{2*}(I_9) \arrow[dlll, rounded corners, "\delta"] \\ \chi_2\mathcal{O}_{\mathbb{P}^1}(-3) \arrow[r] & R^1\beta_{2*}(\chi_2\mathcal{W}) \arrow[r] & \chi_2^2\mathcal{O}_{\mathbb{P}^1}(2) \otimes R^1\beta_{2*}(I_9) \arrow[r] & 0. \end{tikzcd} \tag{6.61}$$

Since there are only maps of line bundles $\mathcal{O}_{\mathbb{P}^1}(n) \to \mathcal{O}_{\mathbb{P}^1}(m)$ for $n \leq m$, the coboundary map

$$\chi_2^2 \mathcal{O}_{\mathbb{P}^1}(2) \otimes \beta_{2*}(I_9) = \chi_2^2 \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\delta} \chi_2 \mathcal{O}_{\mathbb{P}^1}(-3) \tag{6.62}$$

has to be zero and the long exact sequence splits. There is no extension ambiguity for the direct image, and we obtain

$$\beta_{2*}(\chi_2 \mathcal{W}) = \chi_2 \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \chi_2^2 \mathcal{O}_{\mathbb{P}^1}(-1), \tag{6.63}$$

whereas the derived direct image

$$\begin{aligned} 0 \longrightarrow \chi_2 \mathcal{O}_{\mathbb{P}^1}(-3) \longrightarrow R^1\beta_{2*}(\chi_2 \mathcal{W}) \longrightarrow \chi_2^2 \mathcal{O}_{\mathbb{P}^1}(1) \\ \oplus \left[\bigoplus_{k=1}^{3} (1 \oplus \chi_2) \mathcal{O}_{\beta_2(p_k)} \right] \longrightarrow 0 \end{aligned} \tag{6.64}$$

is not uniquely determined. To disentangle the short exact sequence, we compare with relative duality. For that, we need the pushdown of $(\chi_2 \mathcal{W})^\vee$, which we compute from the short exact equation (6.25). Alternatively, we can observe that $\chi_2 \mathcal{W}$ is self-dual while $\mathcal{W}$ alone is not, see footnote 7. Either way, one finds

$$\beta_{2*}\Big[(\chi_2 \mathcal{W})^\vee \Big] = \beta_{2*}(\chi_2 \mathcal{W}). \tag{6.65}$$

Using relative duality, this implies

$$\begin{aligned} \beta_{2*}\Big[(\chi_2 \mathcal{W})^\vee \Big] = \Big(R^1\beta_{2*}(\chi_2 \mathcal{W} \otimes K_{B_2|\mathbb{P}^1}) \Big)^\vee = \Big(R^1\beta_{2*}(\chi_2 \mathcal{W}) \Big)^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \\ \iff \Big(R^1\beta_{2*}(\chi_2 \mathcal{W}) \Big)^\vee = \chi_2 \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \chi_2^2 \mathcal{O}_{\mathbb{P}^1} \\ \iff \Big(R^1\beta_{2*}(\chi_2 \mathcal{W}) \Big)^{\vee\vee} = \chi_2^2 \mathcal{O}_{\mathbb{P}^1}(1) \oplus \chi_2 \mathcal{O}_{\mathbb{P}^1}. \end{aligned} \tag{6.66}$$

Now the dual of the dual is not quite the original sheaf, since the dual of a skyscraper sheaf is zero. So, we can only conclude that

$$R^1\beta_{2*}(\chi_2 \mathcal{W}) = \chi_2^2 \mathcal{O}_{\mathbb{P}^1}(1) \oplus \chi_2 \mathcal{O}_{\mathbb{P}^1} \oplus \text{torsion} \tag{6.67}$$

However, together with the short exact sequence, equation (6.64), this singles out the unique extension

$$R^1\beta_{2*}(\chi_2 \mathcal{W}) = \chi_2^2 \mathcal{O}_{\mathbb{P}^1}(1) \oplus \chi_2 \mathcal{O}_{\mathbb{P}^1} \oplus \left[\bigoplus_{k=1}^{3} \mathcal{O}_{\beta_2(p_k)} \right]. \tag{6.68}$$

6.4.3 The cohomology

Now we have everything in place to compute the cohomology of $\mathcal{V}_1 \otimes \mathcal{V}_2$. We start with the Leray spectral sequence pushing down to B_2,

$$E_2^{p,q}(\widetilde{X}, \mathcal{V}_1 \otimes \mathcal{V}_2) = H^p\Big(B_2,\ R^q\pi_{2*}(\mathcal{V}_1 \otimes \mathcal{V}_2)\Big) = H^p\Big(B_2,\ 2\chi_2\mathcal{W} \otimes R^q\pi_{2*}\mathcal{O}_{\widetilde{X}}\Big)$$
$$= \begin{cases} H^p\Big(B_2, 2\chi_2\mathcal{W} \otimes \pi_{2*}K^\vee_{\widetilde{X}|B_2}\Big), & q = 1 \\ H^p\Big(B_2, 2\chi_2\mathcal{W}\Big), & q = 0. \end{cases} \tag{6.69}$$

We proceed by computing the cohomology of $\chi_2\mathcal{W}$ using another Leray spectral sequence,

$$E_2^{p,q}\Big(B_2, \chi_2\mathcal{W}\Big) = H^p\Big(\mathbb{P}^1, R^q\beta_{2*}(\chi_2\mathcal{W})\Big). \tag{6.70}$$

We computed the pushdowns in equations (6.63) and (6.68). The $q = 0$ cohomology groups are

$$H^0\Big(\mathbb{P}^1, \beta_{2*}(\chi_2\mathcal{W})\Big) = H^0\Big(\mathbb{P}^1, \chi_2\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \chi_2^2\mathcal{O}_{\mathbb{P}^1}(-1)\Big) = 0, \tag{6.71}$$

$$\begin{aligned} H^1\Big(\mathbb{P}^1, \beta_{2*}(\chi_2\mathcal{W})\Big) &= H^1\Big(\mathbb{P}^1, \chi_2\mathcal{O}_{\mathbb{P}^1}(-2)\Big) \oplus H^1\Big(\mathbb{P}^1, \chi_2^2\mathcal{O}_{\mathbb{P}^1}(-1)\Big) \\ &= H^0\Big(\mathbb{P}^1, (\chi_2\mathcal{O}_{\mathbb{P}^1}(-2))^\vee \otimes K_{\mathbb{P}^1}\Big)^\vee \oplus 0 \\ &= H^0\Big(\mathbb{P}^1, (\chi_2^2\mathcal{O}_{\mathbb{P}^1}(2)) \otimes (\chi_1 \otimes \mathcal{O}_{\mathbb{P}^1}(-2))\Big)^\vee \\ &= \Big(\chi_1\chi_2^2\Big)^\vee = \chi_1^2\chi_2. \end{aligned} \tag{6.72}$$

For the $q = 1$ terms, note that G_1 cyclically permutes the skyscraper sheaves, so it acts in the regular representation on the global sections of $\oplus_{k=1}^3 \mathcal{O}_{\beta_2(p_k)}$. One obtains

$$\begin{aligned} H^0\Big(\mathbb{P}^1, R^1\beta_{2*}(\chi_2\mathcal{W})\Big) &= H^0\Big(\mathbb{P}^1, \chi_2^2\mathcal{O}_{\mathbb{P}^1}(1)\Big) \oplus H^0\Big(\mathbb{P}^1, \chi_2\mathcal{O}_{\mathbb{P}^1}\Big) \\ &\quad \oplus H^0\left(\mathbb{P}^1, \bigoplus_{k=1}^{3} \mathcal{O}_{\beta_2(p_k)}\right) \\ &= \Big(\chi_2^2(1 \oplus \chi_1)\Big) \oplus \Big(\chi_2\Big) \oplus \Big(\operatorname{Reg}(G_1)\Big) \\ &= \chi_2^2 \oplus \chi_1\chi_2^2 \oplus \chi_2 \oplus 1 \oplus \chi_1 \oplus \chi_1^2 \end{aligned} \tag{6.73}$$

and

$$H^1\Big(\mathbb{P}^1, R^1\beta_{2*}(\chi_2\mathcal{W})\Big) = H^1\left(\mathbb{P}^1, \chi_2^2\mathcal{O}_{\mathbb{P}^1}(1) \oplus \chi_2\mathcal{O}_{\mathbb{P}^1} \oplus \left[\bigoplus_{k=1}^{3} \mathcal{O}_{\beta_2(p_k)}\right]\right) = 0. \tag{6.74}$$

Hence, the Leray spectral sequence for the $B_2 \to \mathbb{P}^1$ pushdown is

$$E_2^{p,q}\Big(B_2, \chi_2 \mathcal{W}\Big) = \begin{array}{c|cc} q{=}1 & \chi_2^2 \oplus \chi_1\chi_2^2 \oplus \chi_2 \oplus 1 \oplus \chi_1 \oplus \chi_1^2 & 0 \\ q{=}0 & 0 & \chi_1^2\chi_2 \\ \hline & p{=}0 & p{=}1 \end{array} \tag{6.75}$$

and we obtain

$$H^p\Big(B_2, \chi_2 \mathcal{W}\Big) = \begin{cases} 0, & p=2 \\ 1 \oplus \chi_1 \oplus \chi_2 \oplus \chi_1^2 \oplus \chi_2^2 \oplus \chi_1\chi_2^2 \oplus \chi_1^2\chi_2, & p=1 \\ 0, & p=0. \end{cases} \tag{6.76}$$

Similarly to $\chi_2\mathcal{W}$, one can also compute the cohomology of

$$\chi_2 \mathcal{W} \otimes \pi_{2*} K^\vee_{\widetilde{X}|B_2} = \chi_2 \mathcal{W} \otimes \Big(\chi_1^2 \mathcal{O}_{\widetilde{X}}(\phi)\Big)^\vee = \chi_1\chi_2 \mathcal{W} \otimes \mathcal{O}_{B_2}(-f) \tag{6.77}$$

by yet another Leray spectral sequence. However, there is a faster way to do so. We already know that the Leray spectral sequence on $\widetilde{X}$, equation (6.69), degenerates because $E_2^{2,0}(\widetilde{X}, \mathcal{V}_1 \otimes \mathcal{V}_2) = 0$. But this spectral sequence also has to yield the fact[9] that the (character-valued) Index $(\mathcal{V}_1 \otimes \mathcal{V}_2) = 0$, and the only way to accomplish that now is if the $q = 0$ and $q = 1$ terms coincide. Hence, we conclude that

$$H^p\Big(B_2, 2\chi_2 \mathcal{W} \otimes \pi_{2*} K^\vee_{\widetilde{X}|B_2}\Big) = H^p\Big(B_2, 2\chi_2 \mathcal{W}\Big). \tag{6.78}$$

Putting everything together, the $E_2^{p,q}(\widetilde{X}, \mathcal{V}_1 \otimes \mathcal{V}_2) = E_\infty^{p,q}(\widetilde{X}, \mathcal{V}_1 \otimes \mathcal{V}_2)$ tableau is

$$\begin{array}{c|ccc} q{=}1 & 0 & 2 \oplus 2\chi_1 \oplus 2\chi_2 \oplus 2\chi_1^2 \oplus 2\chi_2^2 \oplus 2\chi_1\chi_2^2 \oplus 2\chi_1^2\chi_2 & 0 \\ q{=}0 & 0 & 2 \oplus 2\chi_1 \oplus 2\chi_2 \oplus 2\chi_1^2 \oplus 2\chi_2^2 \oplus 2\chi_1\chi_2^2 \oplus 2\chi_1^2\chi_2 & 0 \\ \hline & p{=}0 & p{=}1 & p{=}2 \end{array} \tag{6.79}$$

and, therefore, the desired cohomology group is

$$\begin{aligned} H^p\Big(\widetilde{X}, \wedge^2 \mathcal{V}\Big) &= H^p\Big(\widetilde{X}, \mathcal{V}_1 \otimes \mathcal{V}_2\Big) \\ &= \begin{cases} 0, & p=3 \\ 2 \oplus 2\chi_1 \oplus 2\chi_2 \oplus 2\chi_1^2 \oplus 2\chi_2^2 \oplus 2\chi_1\chi_2^2 \oplus 2\chi_1^2\chi_2, & p=2 \\ 2 \oplus 2\chi_1 \oplus 2\chi_2 \oplus 2\chi_1^2 \oplus 2\chi_2^2 \oplus 2\chi_1\chi_2^2 \oplus 2\chi_1^2\chi_2, & p=1 \\ 0, & p=0. \end{cases} \end{aligned} \tag{6.80}$$

[9] Group theory tells us that $\wedge^2 \underline{\mathbf{4}} = \underline{\mathbf{6}} = \wedge^2 \overline{\underline{\mathbf{4}}} \in R[SU(4)]$. Therefore, $\wedge^2 \mathcal{V} = \wedge^2 \mathcal{V}^\vee$ and Serre duality forces the vanishing of the index.

6.5 Existence of extensions

In the definition of $\mathcal{V}$, we assumed the existence of a generic extension in two places. The first was the definition of the vector bundle $\mathcal{W}$ on B_2 in equation (6.1). There, the Cayley–Bacharach theorem assured us that $\mathcal{W}$ really is a vector bundle, and not just a sheaf, if only we pick a generic nonzero extension class. But, of course, we can only do so if there are any nontrivial extensions. We can easily compute the space of extensions using the Serre duality and the Leray spectral sequence,

$$\begin{aligned}
&\operatorname{Ext}^1\Big(\chi_2\mathcal{O}_{B_2}(2f)\otimes I_9, \mathcal{O}_{B_2}(-2f)\Big)\\
&\quad= \operatorname{Ext}^1\Big(\chi_1\chi_2\mathcal{O}_{B_2}(3f)\otimes I_9, K_{B_2}\Big) = H^1\Big(B_2, \chi_1\chi_2\mathcal{O}_{B_2}(3f)\otimes I_9\Big)^\vee\\
&\quad= H^0\Big(\mathbb{P}^1, \chi_1\chi_2\mathcal{O}_{\mathbb{P}^1}(3)\otimes R^1\beta_{2*}(I_9)\Big)^\vee\\
&\quad= H^0\Big(\mathbb{P}^1, \chi_1\chi_2\mathcal{O}_{\mathbb{P}^1}(2)\Big)^\vee \oplus H^0\left(\mathbb{P}^1, \bigoplus_{k=1}^{3}(\chi_1\chi_2^2\oplus\chi_1)\mathcal{O}_{\beta_2(p_k)}\right)^\vee \qquad (6.81)\\
&\quad= \Big(\chi_1\chi_2\oplus\chi_1^2\chi_2\oplus\chi_2\Big)^\vee \oplus \Big((\chi_1\chi_2^2\oplus\chi_1)\otimes \operatorname{Reg}(G_1)\Big)^\vee\\
&\quad= \operatorname{Reg}(G_1\times G_2) = \operatorname{Reg}(G).
\end{aligned}$$

The regular representation contains every $G \simeq \mathbb{Z}_3\times\mathbb{Z}_3$ character, and so, in particular, there is a one-dimensional invariant subspace

$$\operatorname{Ext}^1\Big(\chi_2\mathcal{O}_{B_2}(2f)\otimes I_9, \mathcal{O}_{B_2}(-2f)\Big)^G = 1. \qquad (6.82)$$

Using this extension, we can conclude that $\mathcal{W}$ is a vector bundle.

The second place where we assumed the existence of a nontrivial extension was in the definition of $\mathcal{V}$ itself in equation (6.3). The vector bundle $\mathcal{V}$ can only be stable if the extension is nontrivial. Hence, we must ensure that there are, indeed, nonzero elements of

$$\begin{aligned}
\operatorname{Ext}^1\Big(\mathcal{V}_2, \mathcal{V}_1\Big) &= H^1\Big(\widetilde{X}, \mathcal{V}_1\otimes\mathcal{V}_2^\vee\Big)\\
&= H^1\Big(\widetilde{X}, 2\chi_2\mathcal{O}_{\widetilde{X}}(-2\tau_1+2\tau_2)\otimes\pi_2^*(\mathcal{W})\Big)\\
&= H^0\Big(B_2, 2\chi_2 R^1\pi_{2*}(\mathcal{O}_{\widetilde{X}}(-2\tau_1))\otimes\mathcal{O}_{B_2}(2t)\otimes\mathcal{W}\Big)\\
&= H^0\Big(B_2, 12\chi_2\otimes\mathcal{O}_{B_2}(2t-f)\otimes\mathcal{W}\Big). \qquad (6.83)
\end{aligned}$$

We want to compute this cohomology group by pushing down to the base $\mathbb{P}^1$, for which we need the direct image of $\mathcal{O}_{B_2}(2t-f)\otimes\mathcal{W}$. By twisting the short exact sequence, equation (6.1), we obtain

$$0 \longrightarrow \mathcal{O}_{B_2}(2t-3f) \longrightarrow \mathcal{O}_{B_2}(2t-f)\otimes \mathcal{W} \longrightarrow \chi_2\mathcal{O}_{B_2}(2t+f)\otimes I_9 \longrightarrow 0. \tag{6.84}$$

Because $R^1\beta_{2*}\mathcal{O}_{B_2}(2t-3f)=0$ (for degree reasons), the long exact sequence of push-downs truncates and we obtain another short exact sequence

$$\begin{aligned} 0 \longrightarrow 6\mathcal{O}_{\mathbb{P}^1}(-3) &\longrightarrow \beta_{2*}\Big(\mathcal{O}_{B_2}(2t-f)\otimes \mathcal{W}\Big) \\ &\longrightarrow \chi_2\mathcal{O}_{\mathbb{P}^1}(f)\otimes \beta_{2*}\Big(\mathcal{O}_{B_2}(2t)\otimes I_9\Big) \longrightarrow 0. \end{aligned} \tag{6.85}$$

To make use of this short exact sequence, we first have to compute the push-down in the rightmost term. Using the definition of the ideal sheaf, equation (6.32), we obtain

$$\begin{array}{c} 0 \longrightarrow \mathcal{O}_{B_2}(2t)\otimes I_9 \longrightarrow \mathcal{O}_{B_2}(2t) \longrightarrow \bigoplus_{i=1}^{9}\mathcal{O}_{p_i} \longrightarrow 0 \\ \Downarrow \beta_{2*} \\ 0 \longrightarrow \beta_{2*}\Big(\mathcal{O}_{B_2}(2t)\otimes I_9\Big) \longrightarrow \underbrace{\beta_{2*}\Big(\mathcal{O}_{B_2}(2t)\Big)}_{=6\mathcal{O}_{\mathbb{P}^1}} \longrightarrow \bigoplus_{k=1}^{3} 3\mathcal{O}_{\beta_2(p_k)} \longrightarrow 0. \end{array} \tag{6.86}$$

Keeping in mind that really the Heisenberg group has to act on each term, this leaves us with two possibilities:

$$\beta_{2*}\Big(\mathcal{O}_{B_2}(2t)\otimes I_9\Big) = \begin{cases} 3\mathcal{O}_{\mathbb{P}^1}(-3)\oplus 3\mathcal{O}_{\mathbb{P}^1} \\ \qquad \text{or} \\ 3\mathcal{O}_{\mathbb{P}^1}(-2)\oplus 3\mathcal{O}_{\mathbb{P}^1}(-1). \end{cases} \tag{6.87}$$

Figuring out which possibility is realized turns out to be difficult and lengthy. Since extensions in equation (6.3) exist in any case, we just remark that

$$\beta_{2*}\Big(\mathcal{O}_{B_2}(2t)\otimes I_9\Big) = 3\mathcal{O}_{\mathbb{P}^1}(-2)\oplus 3\mathcal{O}_{\mathbb{P}^1}(-1). \tag{6.88}$$

Second, we have to resolve any extension ambiguities in the above short exact sequence, equation (6.85). We show that it splits in Appendix B and, hence, obtain that

$$\beta_{2*}\Big(\mathcal{O}_{B_2}(2t-f)\otimes \mathcal{W}\Big) = 6\mathcal{O}_{\mathbb{P}^1}(-3)\oplus 3\mathcal{O}_{\mathbb{P}^1}(-1)\oplus 3\mathcal{O}_{\mathbb{P}^1} \tag{6.89}$$

and

$$\dim_{\mathbb{C}} H^0\Big(B_2, 12\mathcal{O}_{B_2}(2t-f)\otimes \mathcal{W}\Big) = 36. \tag{6.90}$$

Now we were not quite careful with the $G=\mathbb{Z}_3\times\mathbb{Z}_3$ group action. One has to keep in mind that sections of $\mathcal{O}_{B_2}(t)$ form representations of the

Heisenberg group. One can then easily show that the 36-dimensional vector space, equation (6.83), decomposes as

$$\operatorname{Ext}^1\left(\mathcal{V}_2, \mathcal{V}_1\right) = 4\ \operatorname{Reg}\left(\mathbb{Z}_3 \times \mathbb{Z}_3\right) \Longrightarrow \operatorname{Ext}^1\left(\mathcal{V}_2, \mathcal{V}_1\right)^G = 4. \tag{6.91}$$

We conclude that there are indeed equivariant extensions in all the places where we assumed their existence.

6.6 Vanishing of the first Chern class

In our construction, we picked an $SU(4)$ subgroup of the E_8 gauge bundle. This is not strictly necessary and one can also work with $U(1)$ subgroups, as in [80]. However, we will only consider simple subgroups in this paper. Using the chosen embedding, an $SU(4)$ principal bundle then gives rise to the desired E_8 bundle. Of course, we are really working with a holomorphic rank 4 vector bundle and its $SL(4, \mathbb{C})$ structure group, and then use the deformation retract $SU(4) \to SL(4, \mathbb{C})$.

But, here it is important that the holomorphic vector bundle does have an $SL(4, \mathbb{C})$ structure group instead of the most general $GL(4, \mathbb{C})$ structure group. Topologically, this manifests itself in the vanishing of the first Chern class. Now, in constructing our vector bundle $\mathcal{V}$, we worked on the universal covering space $\widetilde{X}$, whereas we should have worked on the Calabi–Yau threefold $X = \widetilde{X}/G$. Of course, the trace of the curvature of $\mathcal{V}$ is the same as on $\mathcal{V}/G$, so the de Rham representative $c_1(\mathcal{V}/G) \in H^2(X, \mathbb{R})$ stays zero. But the first Chern class really lives in $H^2(X, \mathbb{Z})$ and quotienting by $G \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$ can generate a torsion part. So, in general, only

$$c_1(\mathcal{V}) = 0 \in H^2(\widetilde{X}, \mathbb{Z}) \Longrightarrow c_1(\mathcal{V}/G) \in H^2_{\text{tors}}(X, \mathbb{Z}) = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \tag{6.92}$$

holds and we must check that $c_1(\mathcal{V}/G) = 0$ separately.

The easiest way to ensure the vanishing of the first Chern class of $\mathcal{V}/G$ is to find a trivialization of the determinant line bundle $\wedge^4\mathcal{V}/G$. This we can discuss on the covering space $\widetilde{X}$ where, by construction, $\wedge^4\mathcal{V} = \chi\mathcal{O}_{\widetilde{X}}$ for some character χ. The quotient is then trivial if and only if this character is the identity representation,

$$\frac{\left(\chi\mathcal{O}_{\widetilde{X}}\right)}{G} = \mathcal{O}_X \Longleftrightarrow \chi = 1. \tag{6.93}$$

The determinant line bundle for our rank 4 bundle $\mathcal{V}$, equation (6.3), is

$$\begin{aligned}\wedge^4\mathcal{V} = \Big(\wedge^2\mathcal{V}_1\Big)\otimes\Big(\wedge^2\mathcal{V}_2\Big) &= \Big(\chi_2\mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2)\Big)^{\otimes 2}\\ &\otimes\Bigg[\Big(\mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2)\otimes\pi_2^*\big(\mathcal{O}_{B_2}(-2f)\big)\Big)\\ &\otimes\Big(\mathcal{O}_{\widetilde{X}}(-\tau_1+\tau_2)\otimes\pi_2^*\big(\chi_2\mathcal{O}_{B_2}(2f)\big)\Big)\Bigg] = \chi_2^3\mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}\end{aligned} \tag{6.94}$$

and, hence,

$$c_1\left(\frac{\mathcal{V}}{G}\right) = 0 \in H^2(X,\mathbb{Z}) \tag{6.95}$$

as it should.

We remark that there is at least one other equivariant action on the rank 4 vector bundle $\mathcal{V}$ which also leads to the vanishing first Chern class. This nicely illustrates the importance of the equivariant actions, and we discuss it in more detail in Appendix C.

7 The low energy spectrum

7.1 Spin(10) gauge theory

First, let us only consider the effect of the $SU(4)$ instanton in the visible E_8 gauge group. Then, the E_8 gauge bosons acquire masses except for the components which commute with the $SU(4)$. In other words, the gauge group is broken to the commutant (or centralizer) of $SU(4) \in E_8$. We pick a regular $SU(4)$ subgroup of E_8. Then, the commutant can simply be read off from the extended Dynkin diagram, see figure 2. The appearance of a Spin(10) gauge group is very desirable, since one full generation of Standard Model matter quarks and leptons (including a right-handed neutrino) fill out one **<u>16</u>** representation of Spin(10).

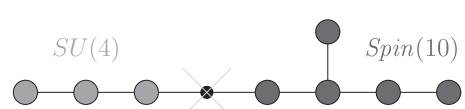

Figure 2: Regular $SU(4)\times\mathrm{Spin}(10)$ subgroup of E_8.

The branching rule for the adjoint representation of E_8 is

$$R[E_8] \ni \underline{\mathbf{248}} = (\underline{\mathbf{1}}, \underline{\mathbf{45}}) \oplus (\underline{\mathbf{15}}, \underline{\mathbf{1}}) \oplus (\underline{\mathbf{6}}, \underline{\mathbf{10}}) \oplus (\underline{\mathbf{4}}, \underline{\mathbf{16}}) \\ \oplus (\overline{\underline{\mathbf{4}}}, \overline{\underline{\mathbf{16}}}) \in R\big[SU(4) \times \mathrm{Spin}(10)\big]. \tag{7.1}$$

Correspondingly, the fermions in the adjoint of E_8 split into fields charged only under $SU(4)$, only under Spin(10), or under both groups. We identify the corresponding zero modes as

$(\underline{\mathbf{4}}, \underline{\mathbf{16}})$: The matter fields transforming in the $\underline{\mathbf{16}}$ of Spin(10). The number of such chiral multiplets is

$$H^1\left(X, \frac{\mathcal{V}}{G}\right) = H^1\big(\widetilde{X}, \mathcal{V}\big)^G. \tag{7.2}$$

$(\overline{\underline{\mathbf{4}}}, \overline{\underline{\mathbf{16}}})$: Likewise, the number of $\overline{\underline{\mathbf{16}}}$ matter fields. Similarly, their number is

$$H^1\left(X, \frac{\mathcal{V}^\vee}{G}\right) = H^1(\widetilde{X}, \mathcal{V}^\vee)^G. \tag{7.3}$$

$(\underline{\mathbf{6}}, \underline{\mathbf{10}})$: The matter fields transforming in the $\underline{\mathbf{10}}$ of Spin(10). Notice that it is a real representation, in particular,

$$\wedge^2 \underline{\mathbf{4}} = \underline{\mathbf{6}} = \wedge^2 \overline{\underline{\mathbf{4}}} \in R\big[SU(4)\big]. \tag{7.4}$$

The number of chiral multiplets equals

$$H^1\big(\widetilde{X}, \wedge^2 \mathcal{V}\big)^G = H^1\big(\widetilde{X}, \wedge^2 \mathcal{V}^\vee\big)^G. \tag{7.5}$$

$(\underline{\mathbf{1}}, \underline{\mathbf{45}})$: The gauginos of the Spin(10) gauge group.

$(\underline{\mathbf{15}}, \underline{\mathbf{1}})$: The superpartners of the moduli fields (they are neutral under the Spin(10) gauge group). As

$$\underline{\mathbf{15}} = \mathrm{ad}_{SU(4)} = \underline{\mathbf{4}} \otimes \overline{\underline{\mathbf{4}}} - \underline{\mathbf{1}} \in R\big[SU(4)\big], \tag{7.6}$$

their number equals

$$\Big[H^1\big(\widetilde{X}, \mathcal{V} \otimes \mathcal{V}^\vee\big) - \overbrace{H^1\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\big)}^{=0}\Big]^G = H^1\big(\widetilde{X}, \mathcal{V} \otimes \mathcal{V}^\vee\big)^G. \tag{7.7}$$

7.2 Wilson lines

Of course, Spin(10) is not the right gauge group for phenomenological purposes. It must be broken down to the standard model $SU(3)_C \times SU(2)_L \times U(1)_Y$. Actually, to incorporate the long lifetime of the nucleons, we postulate an extra $U(1)_{B-L}$ which naturally suppresses nucleon decay. Hence, we want to break

$$\mathrm{Spin}(10) \longrightarrow SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L}. \tag{7.8}$$

The obvious mechanism to do this is to make use of the $\pi_1(X) = \mathbb{Z}_3 \times \mathbb{Z}_3$ fundamental group and add suitable Wilson lines. We showed in [61] that there is a $\mathbb{Z}_3 \times \mathbb{Z}_3 \subset \mathrm{Spin}(10)$ subgroup whose commutant is precisely $SU(3) \times SU(2) \times U(1)^2$. Moreover, we computed the decomposition of the $\underline{\mathbf{10}}$ and $\underline{\mathbf{16}}$ representations of Spin(10) under this $\mathbb{Z}_3 \times \mathbb{Z}_3 \times SU(3) \times SU(2) \times U(1)^2$ subgroup and found[10]

$$\begin{aligned}
\underline{\mathbf{16}} &= \chi_1^2\chi_2(\underline{\mathbf{3}}, \underline{\mathbf{2}}, 1, 1) \oplus \chi_1^2(\underline{\mathbf{1}}, \underline{\mathbf{1}}, 6, 3) \oplus \chi_1^2\chi_2^2(\underline{\overline{\mathbf{3}}}, \underline{\mathbf{1}}, -4, -1) \\
&\quad \oplus \chi_2^2(\underline{\overline{\mathbf{3}}}, \underline{\mathbf{1}}, 2, -1) \oplus (\underline{\mathbf{1}}, \underline{\overline{\mathbf{2}}}, -3, -3) \oplus \chi_1(\underline{\mathbf{1}}, \underline{\mathbf{1}}, 0, 3) \\
\underline{\mathbf{10}} &= \chi_1(\underline{\mathbf{1}}, \underline{\mathbf{2}}, 3, 0) \oplus \chi_1\chi_2(\underline{\mathbf{3}}, \underline{\mathbf{1}}, -2, -2) \oplus \chi_1^2(\underline{\mathbf{1}}, \underline{\overline{\mathbf{2}}}, -3, 0) \\
&\quad \oplus \chi_1^2\chi_2^2(\underline{\overline{\mathbf{3}}}, \underline{\mathbf{1}}, 2, 2).
\end{aligned} \tag{7.9}$$

This means the following in terms of vector bundles. On $\widetilde{X}$, we are concerned with the rank 248 vector bundle $\mathcal{E}_8^V$ which is associated to the visible E_8 principal bundle via the adjoint representation $\underline{\mathbf{248}} \in R[E_8]$. Thanks to the decomposition, equation (7.1), this vector bundle can be written in terms of the $SU(4)$ vector bundle $\mathcal{V}$ as

$$\begin{aligned}
\mathcal{E}_8^V &= \Big(\mathcal{O}_{\widetilde{X}} \otimes \theta(\underline{\mathbf{45}})\Big) \oplus \Big(\mathrm{ad}(\mathcal{V}) \otimes \theta(\underline{\mathbf{1}})\Big) \oplus \Big(\wedge^2 \mathcal{V} \otimes \theta(\underline{\mathbf{10}})\Big) \\
&\quad \oplus \Big(\mathcal{V} \otimes \theta(\underline{\mathbf{16}})\Big) \oplus \Big(\mathcal{V}^\vee \otimes \theta(\underline{\overline{\mathbf{16}}})\Big).
\end{aligned} \tag{7.10}$$

Here, the vector bundles $\theta(R)$ associated with a Spin(10) representation R are just trivial rank $\dim(R)$ bundles on $\widetilde{X}$. However, they inherit certain $\mathbb{Z}_3 \times \mathbb{Z}_3$ actions from the representations detailed in equation (7.9) and, therefore, their $\mathbb{Z}_3 \times \mathbb{Z}_3$ quotient can be nontrivial.

7.3 Matter fields

The matter fields in the low energy effective action are massless relative to the compactification scale and are computed as the index of the Dirac operator coupled to the $\mathcal{E}_8^V$, see equation (7.10). This index can be computed by the cohomology groups of $\mathcal{E}_8^V$, which is what we are now going to do.

[10]Here, we are picking generators for the $U(1)^2$ which coincide with the conventional hypercharge and $B - L$. Our normalization is the one that gets rid of any fractions.

For example, let us focus on the $\wedge^2\mathcal{V} \otimes \theta(\underline{\mathbf{10}})$ summand. The vector bundle $\theta(\underline{\mathbf{10}})$ decomposes as

$$\begin{aligned}\theta(\underline{\mathbf{10}}) &= \chi_1\theta(\underline{\mathbf{1}}, \underline{\mathbf{2}}, 3, 0) \oplus \chi_1\chi_2\theta(\underline{\mathbf{3}}, \underline{\mathbf{1}}, -2, -2)\\ &\quad \oplus \chi_1^2\theta(\underline{\mathbf{1}}, \underline{\overline{\mathbf{2}}}, -3, 0) \oplus \chi_1^2\chi_2^2\theta(\underline{\overline{\mathbf{3}}}, \underline{\mathbf{1}}, 2, 2)\\ &\simeq 2\chi_1\mathcal{O}_{\widetilde{X}} \oplus 3\chi_1\chi_2\mathcal{O}_{\widetilde{X}} \oplus 2\chi_1^2\mathcal{O}_{\widetilde{X}} \oplus 3\chi_1^2\chi_2^2\mathcal{O}_{\widetilde{X}}.\end{aligned} \tag{7.11}$$

Therefore, the corresponding summand in equation (7.10) decomposes into

$$\begin{aligned}\wedge^2\mathcal{V} \otimes \theta(\underline{\mathbf{10}}) &= \Big(\wedge^2 \mathcal{V} \otimes \chi_1\theta(\underline{\mathbf{1}}, \underline{\mathbf{2}}, 3, 0)\Big) \oplus \Big(\wedge^2 \mathcal{V} \otimes \chi_1\chi_2\theta(\underline{\mathbf{3}}, \underline{\mathbf{1}}, -2, -2)\Big)\\ &\quad \oplus \Big(\wedge^2 \mathcal{V} \otimes \chi_1^2\theta(\underline{\mathbf{1}}, \underline{\overline{\mathbf{2}}}, -3, 0)\Big) \oplus \Big(\wedge^2 \mathcal{V} \otimes \chi_1^2\chi_2^2\theta(\underline{\overline{\mathbf{3}}}, \underline{\mathbf{1}}, 2, 2)\Big).\end{aligned} \tag{7.12}$$

The $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ invariant part of the first cohomology group are the zero modes. In our case, we obtain

$$\begin{aligned}H^1\Big(\widetilde{X}, \wedge^2\mathcal{V} \otimes \theta(\underline{\mathbf{10}})\Big)^G &= \Big[\chi_1 \otimes H^1\Big(\widetilde{X}, \wedge^2\mathcal{V}\Big)\Big]^G \otimes (\underline{\mathbf{1}}, \underline{\mathbf{2}}, 3, 0)\\ &\quad \oplus \Big[\chi_1\chi_2 \otimes H^1\Big(\widetilde{X}, \wedge^2\mathcal{V}\Big)\Big]^G \otimes (\underline{\mathbf{3}}, \underline{\mathbf{1}}, -2, -2)\\ &\quad \oplus \Big[\chi_1^2 \otimes H^1\Big(\widetilde{X}, \wedge^2\mathcal{V}\Big)\Big]^G \otimes (\underline{\mathbf{1}}, \underline{\overline{\mathbf{2}}}, -3, 0)\\ &\quad \oplus \Big[\chi_1^2\chi_2^2 \otimes H^1\Big(\widetilde{X}, \wedge^2\mathcal{V}\Big)\Big]^G \otimes (\underline{\overline{\mathbf{3}}}, \underline{\mathbf{1}}, 2, 2).\end{aligned} \tag{7.13}$$

Hence, $\big[\chi_1 \otimes H^1(\widetilde{X}, \wedge^2\mathcal{V})\big]^G$ is the number of fields transforming in the representation $(\underline{\mathbf{1}}, \underline{\mathbf{2}}, 3, 0)$, and so on. The same can be done for the other matter fields coming from the $\underline{\mathbf{16}}$ and $\underline{\overline{\mathbf{16}}}$ of Spin(10).

The resulting spectrum can then be read off from the cohomology groups which we computed in equations (6.47) and (6.80). We give the spectrum[11] in table 1, which is precisely three families of quarks and leptons, together with two pairs of Higgs doublets.

[11] To be precise, we only list the left chiral half, that is the number of $N = 1$ left chiral multiplets. It is always understood that the particles are accompanied by their CPT conjugates.

Table 1: Low energy spectrum.

Multiplicity	Representation	Name
$3 = \left[\chi_1^2\chi_2 \otimes H^1\left(\widetilde{X}, \mathcal{V}\right)\right]^G$	$(\underline{\mathbf{3}}, \underline{\mathbf{2}}, 1, 1)$	Left-handed quark
$3 = \left[\chi_1^2 \otimes H^1\left(\widetilde{X}, \mathcal{V}\right)\right]^G$	$(\underline{\mathbf{1}}, \underline{\mathbf{1}}, 6, 3)$	Left-handed charged anti-lepton
$3 = \left[\chi_1^2\chi_2^2 \otimes H^1\left(\widetilde{X}, \mathcal{V}\right)\right]^G$	$(\underline{\overline{\mathbf{3}}}, \underline{\mathbf{1}}, -4, -1)$	Left-handed anti-up
$3 = \left[\chi_2^2 \otimes H^1\left(\widetilde{X}, \mathcal{V}\right)\right]^G$	$(\underline{\overline{\mathbf{3}}}, \underline{\mathbf{1}}, 2, -1)$	Left-handed anti-down
$3 = \left[H^1\left(\widetilde{X}, \mathcal{V}\right)\right]^G$	$(\underline{\mathbf{1}}, \underline{\overline{\mathbf{2}}}, -3, -3)$	Left-handed lepton
$3 = \left[\chi_1 \otimes H^1\left(\widetilde{X}, \mathcal{V}\right)\right]^G$	$(\underline{\mathbf{1}}, \underline{\mathbf{1}}, 0, 3)$	Left-handed anti-neutrino
$0 = \left[\chi_1\chi_2^2 \otimes H^1\left(\widetilde{X}, \mathcal{V}^\vee\right)\right]^G$	$(\underline{\overline{\mathbf{3}}}, \underline{\overline{\mathbf{2}}}, -1, -1)$	Exotic
$0 = \left[\chi_1 \otimes H^1\left(\widetilde{X}, \mathcal{V}^\vee\right)\right]^G$	$(\underline{\mathbf{1}}, \underline{\mathbf{1}}, -6, -3)$	Exotic
$0 = \left[\chi_1\chi_2 \otimes H^1\left(\widetilde{X}, \mathcal{V}^\vee\right)\right]^G$	$(\underline{\mathbf{3}}, \underline{\mathbf{1}}, 4, 1)$	Exotic
$0 = \left[\chi_2 \otimes H^1\left(\widetilde{X}, \mathcal{V}^\vee\right)\right]^G$	$(\underline{\mathbf{3}}, \underline{\mathbf{1}}, -2, 1)$	Exotic
$0 = \left[H^1\left(\widetilde{X}, \mathcal{V}^\vee\right)\right]^G$	$(\underline{\mathbf{1}}, \underline{\mathbf{2}}, 3, 3)$	Exotic
$0 = \left[\chi_1^2 \otimes H^1\left(\widetilde{X}, \mathcal{V}^\vee\right)\right]^G$	$(\underline{\mathbf{1}}, \underline{\mathbf{1}}, 0, -3)$	Exotic
$2 = \left[\chi_1 \otimes H^1\left(\widetilde{X}, \wedge^2\mathcal{V}\right)\right]^G$	$(\underline{\mathbf{1}}, \underline{\mathbf{2}}, 3, 0)$	Up Higgs
$0 = \left[\chi_1\chi_2 \otimes H^1\left(\widetilde{X}, \wedge^2\mathcal{V}\right)\right]^G$	$(\underline{\mathbf{3}}, \underline{\mathbf{1}}, -2, -2)$	Exotic
$2 = \left[\chi_1^2 \otimes H^1\left(\widetilde{X}, \wedge^2\mathcal{V}\right)\right]^G$	$(\underline{\mathbf{1}}, \underline{\overline{\mathbf{2}}}, -3, 0)$	Down Higgs
$0 = \left[\chi_1^2\chi_2^2 \otimes H^1\left(\widetilde{X}, \wedge^2\mathcal{V}\right)\right]^G$	$(\underline{\overline{\mathbf{3}}}, \underline{\mathbf{1}}, 2, 2)$	Exotic

Note that all exotic representations of the gauge group come with multiplicity zero.

8 An alternative hidden sector

8.1 Anomaly cancellation without five-branes

Instead of using five-branes, we can also cancel the anomaly by adding another $SU(2)$ instanton to the hidden E_8. In total, we then have an $SU(2) \times SU(2)$ gauge bundle embedded into the hidden E_8. We denote the additional hidden $SU(2)$ bundle by $\mathcal{S}$ and define it as a pullback from B_1, that is,

$$\mathcal{S} \stackrel{\text{def}}{=} \pi_1^*(\mathcal{S}_B). \tag{8.1}$$

The rank 2 bundle $\mathcal{S}_B$ on B_1 is defined as the generic extension in the short exact sequence

$$0 \longrightarrow \mathcal{O}(-2f) \longrightarrow \mathcal{S}_B \longrightarrow \mathcal{O}(2f) \otimes I_6^f \longrightarrow 0. \tag{8.2}$$

The definition of the ideal sheaf I_6^f is delicate, and we postpone it for the moment. Its detailed definition is important for the cohomology groups of $\mathcal{S}$, but the intricacies are not detected by Chern classes.

The second Chern class of the ideal sheaf I_6^f on B_1 is simply 6, the number of points. Therefore,

$$\operatorname{ch}(\mathcal{S}) = e^{-2f} + e^{2f}\Big(1 - 6\tau_1^2\Big) = 2 - 6\tau_1^2. \tag{8.3}$$

We can immediately read off that

$$c_2(\mathcal{S}) = 6\tau_1^2, \tag{8.4}$$

which is precisely the Poincaré dual of the curve on which we had to wrap the five-brane in the previous section. This is exactly what we want, since for two $SU(2)$ bundles, the second Chern classes just adds,

$$c_2\Big(\mathcal{H} \oplus \mathcal{S}\Big) = c_2(\mathcal{H}) + c_2(\mathcal{S}). \tag{8.5}$$

Hence, we can use $\mathcal{H} \oplus \mathcal{S}$ as a hidden gauge instanton and get an anomaly free model without any five-branes.

$$c_2(T\widetilde{X}) - c_2(\mathcal{V}) - c_2(\mathcal{H} \oplus \mathcal{S}) = 0. \tag{8.6}$$

8.2 The ideal sheaf

As mentioned above, the ideal sheaf I_6^f requires additional discussion. So far, all ideal sheaves used a generic G orbit, which has $|G| = 9$ points. But, on the dP_9 surface B_1, there are two shorter orbits, both of length 3. One comprises the three g_1 fixed points and the other the three g_2 fixed points. For the ideal sheaf I_6^f, we take the three g_2 fixed points p_1, p_2, and p_3 with multiplicity 2.

The three fixed points p_i are in three different I_1 fibers[12] of the elliptic fibration. But the ideal sheaf is not uniquely determined by knowing that there are three points of multiplicity two. Recall that the ideal sheaf of

[12]Since g_2 is translation by a section, its fixed points necessarily lie in singular fibers.

points with multiplicity 1 is just the sheaf of analytic functions vanishing at these points. Now, the multiplicity 2 means that the function and a first derivative has to vanish at the three points p_1, p_2, and p_3. But, at any point, there are two independent first derivatives, since the dP_9 surface is two dimensional. We are going to demand that the first derivative in the fiber direction vanishes and express that by the superscript "f" in I_6^f. The extension equation (8.2) still satisfies the Cayley–Bacharach condition, since a global section of $\mathcal{O}_{B_1}(4f) \otimes K_{B_1}$ vanishing at a point in a fiber vanishes identically on that fiber. Therefore, a generic extension in the short exact sequence, equation (8.2), is a rank 2 vector bundle.

In order to compute the cohomology groups of $\mathcal{S}$, we need to know the pushdown of I_6^f to the base $\mathbb{P}^1$. The ideal sheaf fits into a short exact sequence

$$0 \longrightarrow I_6^f \longrightarrow \mathcal{O}_{B_1} \longrightarrow \bigoplus_{i=1}^{3} \mathcal{O}_{2p_i} \longrightarrow 0 \tag{8.7}$$

leading to a long exact sequence for the pushdown

$$\begin{array}{l} 0 \longrightarrow \beta_{1*}(I_6^f) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \xrightarrow{\ r\ } \bigoplus_{i=1}^{3} \mathcal{O}_{2\beta_1(p_i)} \longrightarrow \\ \longrightarrow R^1\beta_{1*}(I_6^f) \longrightarrow \chi_1 \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow 0 \longrightarrow 0. \end{array} \tag{8.8}$$

Now the restriction map r works as follows. Let f be a local section of $\mathcal{O}_{\mathbb{P}^1}$, that is, a holomorphic function over a small open set. Then, this function is pulled back to B_1, that is, it is taken to be constant along the fiber of the elliptic fibration $\beta_1 : B_1 \to \mathbb{P}^1$. Now restrict to the skyscraper sheaf $\oplus \mathcal{O}_{2p_i}$. If the open set does not contain $\beta_1(p_i)$, then the restriction is simply zero. And if the open set does contain $\beta_1(p_i)$, then the pullback function $f \circ \beta_1$ is restricted to its value at p_i and its first derivative in the fiber direction at p_i. By construction, this first derivative is always zero for a function pulled back from the base $\mathbb{P}^1$.

To summarize, the fiber of the skyscraper sheaf $\oplus_{i=1}^{3} \mathcal{O}_{2\beta_1(p_i)}$ over p_i is $\mathbb{C}^2$, and the image of the restriction map is $\mathbb{C} \oplus \{0\}$. The generator g_1 acts in the regular representation on the skyscraper sheaf, permuting the three points $\beta_1(p_1)$, $\beta_1(p_2)$, and $\beta_1(p_3)$, while the second generator g_2 acts as[13]

[13]The first derivative in the fiber direction picks up this subtle phase. However, we can afford to gloss over it because the torsion part of $R^1\pi_{1*}\big(I_6^f\big)$ is absorbed in a nontrivial extension, see equation (8.14).

$1 \oplus \chi_2$. Hence, we can split the long exact sequence into

$$\begin{gathered} 0 \longrightarrow \beta_{1*}(I_6^f) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \bigoplus_{i=1}^{3} \mathcal{O}_{\beta_1(p_i)} \longrightarrow 0, \\ 0 \longrightarrow \bigoplus_{i=1}^{3} \chi_2 \mathcal{O}_{\beta_1(p_i)} \longrightarrow R^1\beta_{1*}(I_6^f) \longrightarrow \chi_1 \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow 0. \end{gathered} \tag{8.9}$$

These short exact sequences uniquely determine the push-down of the ideal sheaf to be

$$\beta_{1*}(I_6^f) = \mathcal{O}_{\mathbb{P}^1}(-3), \qquad R^1\beta_{1*}(I_6^f) = \bigoplus_{i=1}^{3} \chi_2 \mathcal{O}_{\beta_1(p_i)} \oplus \chi_1 \mathcal{O}_{\mathbb{P}^1}(-1). \tag{8.10}$$

8.3 Cohomology of $\mathcal{S}$

To be able to apply the Leray spectral sequence, we need to know the push down of $\mathcal{S}_B$ from B_1 to $\mathbb{P}^1$. By definition, $\mathcal{S}_B$ fits into the short exact sequence, equation (8.2). Hence, we obtain the following long exact sequence for the pushdown

$$\begin{gathered} 0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow \beta_{1*}\mathcal{S}_B \longrightarrow \beta_{1*}(I_6^f) \otimes \mathcal{O}_{\mathbb{P}^1}(2) \xrightarrow{\delta} \\ \chi_1 \mathcal{O}_{\mathbb{P}^1}(-3) \longrightarrow R^1\beta_{1*}\mathcal{S}_B \longrightarrow \Big(R^1\beta_{1*}(I_6^f)\Big) \otimes \mathcal{O}_{\mathbb{P}^1}(2) \longrightarrow 0. \end{gathered} \tag{8.11}$$

The coboundary map

$$\delta : \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-3) \tag{8.12}$$

is zero for degree reasons and, therefore, the long exact sequence splits into two short exact sequences

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow \beta_{1*}(\mathcal{S}_B) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow 0\ , \tag{8.13}$$

$$0 \longrightarrow \chi_1 \mathcal{O}_{\mathbb{P}^1}(-3) \longrightarrow R^1\beta_{1*}(\mathcal{S}_B) \longrightarrow \bigoplus_{i=1}^{3} \chi_2 \mathcal{O}_{\beta_1(p_i)} \oplus \chi_1 \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0\,. \tag{8.14}$$

The first short exact sequence uniquely determines

$$\beta_{1*}(\mathcal{S}_B) = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1), \tag{8.15}$$

but the extension in the second short exact sequence is not unique. We fix the ambiguity by combining the fact that $\underline{\mathbf{2}} = \overline{\underline{\mathbf{2}}}$ in $SU(2)$, hence the bundle

$\mathcal{S}_B$ is isomorphic to its dual $\mathcal{S}_B^\vee$ and relative duality to get

$$\begin{aligned}\beta_{1*}(\mathcal{S}_B) = \Big(R^1\beta_{1*}(\mathcal{S}_B \otimes K_{B_1|\mathbb{P}^1})\Big)^\vee = \Big(R^1\beta_{1*}(\mathcal{S}_B)\Big)^\vee \otimes \chi_1\mathcal{O}_{\mathbb{P}^1}(-1)\\ \Longrightarrow R^1\beta_{1*}(\mathcal{S}_B) = \Big(\beta_{1*}(\mathcal{S}_B)\Big)^\vee \otimes \chi_1\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \text{torsion}\\ = \chi_1\mathcal{O}_{\mathbb{P}^1}(1) \oplus \chi_1\mathcal{O}_{\mathbb{P}^1} \oplus \text{torsion}.\end{aligned} \tag{8.16}$$

The only extension in the short exact sequence, equation (8.14), that also satisfies equation (8.16) is

$$R^1\beta_{1*}(\mathcal{S}_B) = \chi_1\mathcal{O}_{\mathbb{P}^1}(1) \oplus \chi_1\mathcal{O}_{\mathbb{P}^1}. \tag{8.17}$$

Now it is a simple application of the Leray spectral sequence to compute the cohomology of $\mathcal{S}$. In the first step, we push $\mathcal{S}$ back down to B_1 and obtain

$$E_2^{p,q}(\widetilde{X},\mathcal{S}) = H^p\Big(B_1, R^q\pi_{1*}(\pi_1^*\mathcal{S}_B)\Big) = \begin{cases} H^p\Big(B_1, R^1\pi_{1*}(\pi_1^*\mathcal{S}_B)\Big), & q=1\\ H^p\Big(B_1,\mathcal{S}_B\Big), & q=0.\end{cases} \tag{8.18}$$

The cohomology groups of the two sheaves on B_1 can, in turn, again be computed using another Leray spectral sequence. Starting with $H^p(B_1,\mathcal{S}_B)$, one finds

$$E_2^{p,q}\Big(B_1,\mathcal{S}_B\Big) = H^p\Big(\mathbb{P}^1, R^q\beta_{1*}\mathcal{S}_B\Big) = \begin{array}{r|cc} {\scriptstyle q=1} & 2\chi_1+\chi_1^2 & 0\\ {\scriptstyle q=0} & 0 & \chi_1^2\\ \hline & {\scriptstyle p=0} & {\scriptstyle p=1}\end{array} = E_\infty^{p,q}\Big(B_1,\mathcal{S}_B\Big) \tag{8.19}$$

Each entry in the E_2 tableau above can easily be computed using equation (5.16). For example, $E_2^{0,1}(B_1,\mathcal{S}_B)$ is

$$\begin{aligned}H^1\Big(\mathbb{P}^1,\beta_{1*}\mathcal{S}_B\Big) &= H^1\Big(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(-2)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)\Big)\\ &= H^0\Big(\mathbb{P}^1,(\mathcal{O}_{\mathbb{P}^1}(-2)\oplus\mathcal{O}_{\mathbb{P}^1}(-1))^\vee\otimes K_{\mathbb{P}^1}\Big)^\vee\\ &= H^0\Big(\mathbb{P}^1,\chi_1\mathcal{O}_{\mathbb{P}^1}\oplus\chi_1\mathcal{O}_{\mathbb{P}^1}(-1)\Big)^\vee = (\chi_1)^\vee = \chi_1^2.\end{aligned} \tag{8.20}$$

Next, we need the cohomology groups of $R^1\pi_{1*}(\pi_1^*\mathcal{S}_B)$. Using the projection formula and relative duality, we identify

$$R^1\pi_{1*}(\pi_1^*\mathcal{S}_B) = \mathcal{S}_B\otimes R^1\pi_{1*}(\mathcal{O}_{\widetilde{X}}) = \mathcal{S}_B\otimes(\pi_{1*}K_{\widetilde{X}|B_1})^\vee = \mathcal{S}_B\otimes\mathcal{O}_{B_1}(-f). \tag{8.21}$$

Therefore, the Leray spectral sequence for pushing it down to the base $\mathbb{P}^1$ is

$$E_2^{p,q}\Big(B_1, R^1\pi_{1*}(\pi_1^*\mathcal{S}_B)\Big) = H^p\Big(\mathbb{P}^1, R^q\beta_{1*}(\mathcal{S}_B)\otimes\mathcal{O}_{\mathbb{P}^1}(-1)\Big)$$

$$= \begin{array}{c|cc} q{=}1 & \chi_1 & 0 \\ q{=}0 & 0 & 2\chi_1^2+\chi_1 \\ \hline & p{=}0 & p{=}1 \end{array} = E_\infty^{p,q}\Big(B_1, R^1\pi_{1*}(\pi_1^*\mathcal{S}_B)\Big). \tag{8.22}$$

We determined all entries of the $E_2^{p,q}(\widetilde{X},\mathcal{S})$ tableau to be

$$E_2^{p,q}\Big(\widetilde{X},\mathcal{S}\Big) = \begin{array}{c|ccc} q{=}1 & 0 & 2\chi_1+2\chi_1^2 & 0 \\ q{=}0 & 0 & 2\chi_1+2\chi_1^2 & 0 \\ \hline & p{=}0 & p{=}1 & p{=}2 \end{array}. \tag{8.23}$$

The potential d_3 differential vanishes by dimension and the spectral sequence collapses. Summing up the diagonals, we determine the cohomology of $\mathcal{S}$ to be

$$H^p\Big(\widetilde{X},\mathcal{S}\Big) = \begin{cases} 0, & p=3 \\ 2\chi_1+2\chi_1^2, & p=2 \\ 2\chi_1+2\chi_1^2, & p=1 \\ 0, & p=0. \end{cases} \tag{8.24}$$

Note that there is no invariant part,

$$H^*\Big(\widetilde{X},\mathcal{S}\Big)^G = 0. \tag{8.25}$$

8.4 Cohomology of $\mathcal{H}\otimes\mathcal{S}$

To compute the cohomology groups of $\mathcal{H}\otimes\mathcal{S}$, we again utilize the Leray spectral sequence. First, we push down to B_1. The short exact sequence, equation (5.1), tensored with $\mathcal{S}$ yields

$$0 \longrightarrow \underbrace{\mathcal{O}_{\widetilde{X}}(2\tau_1+\tau_2-\phi)\otimes\mathcal{S}}_{=\pi_1^*\left(\mathcal{O}_{B_1}(2t)\otimes\mathcal{S}_B\right)\otimes\pi_2^*\mathcal{O}_{B_2}(t-f)} \longrightarrow \mathcal{H}\otimes\mathcal{S} \longrightarrow \underbrace{\mathcal{O}_{\widetilde{X}}(-2\tau_1-\tau_2+\phi)\otimes\mathcal{S}}_{=\pi_1^*\left(\mathcal{O}_{B_1}(-2t)\otimes\mathcal{S}_B\right)\otimes\pi_2^*\mathcal{O}_{B_2}(-t+f)} \longrightarrow 0. \tag{8.26}$$

Now we push down to B_1, and from the associated long exact sequence, we can immediately read off

$$\begin{aligned} R_1\pi_{1*}(\mathcal{H}\otimes\mathcal{S}) &= \mathcal{O}_{B_1}(-2t)\otimes\mathcal{S}_B\otimes\Big(3\mathcal{O}_{B_1}\Big) \\ \pi_{1*}(\mathcal{H}\otimes\mathcal{S}) &= \mathcal{O}_{B_1}(2t)\otimes\mathcal{S}_B\otimes\Big(3\mathcal{O}_{B_1}(-f)\Big). \end{aligned} \tag{8.27}$$

To compute the subsequent pushdown to the base $\mathbb{P}^1$, we first have to push down $I_6^f\otimes\mathcal{O}_{B_1}(2t)$. This can be obtained from the defining short exact sequence

$$\begin{gathered} 0 \longrightarrow \mathcal{O}_{B_1}(2t)\otimes I_6^f \longrightarrow \mathcal{O}_{B_1}(2t) \longrightarrow \bigoplus_{i=1}^{3}\mathcal{O}_{2p_i} \longrightarrow 0 \\ \Downarrow \beta_{1*} \\ 0 \longrightarrow \beta_{1*}\Big(\mathcal{O}_{B_1}(2t)\otimes I_6^f\Big) \longrightarrow \underbrace{\beta_{1*}\Big(\mathcal{O}_{B_2}(2t)\Big)}_{=6\mathcal{O}_{\mathbb{P}^1}} \longrightarrow \bigoplus_{i=1}^{3}\mathcal{O}_{2\beta_1(p_i)} \longrightarrow 0. \end{gathered} \tag{8.28}$$

Considering that a section of $\mathcal{O}_{B_1}(2t)$ and its derivative do not vanish simultaneously, we conclude that

$$\beta_{1*}\Big(\mathcal{O}_{B_1}(2t)\otimes I_6^f\Big) = 6\mathcal{O}_{\mathbb{P}^1}(-1), \quad R^1\beta_{1*}\Big(\mathcal{O}_{B_1}(2t)\otimes I_6^f\Big) = 0. \tag{8.29}$$

Tensoring the short exact sequence defining S_B, equation (8.2), with $\mathcal{O}_{B_1}(2t)$ and pushing down one obtains

$$0 \longrightarrow 6\mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow \beta_{1*}\Big(\mathcal{O}_{B_1}(2t)\otimes\mathcal{S}_B\Big) \longrightarrow 6\mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0. \tag{8.30}$$

At this point, we assume that the extension is generic, that is,

$$\beta_{1*}\Big(\mathcal{O}_{B_1}(2t)\otimes\mathcal{S}_B\Big) = 6\mathcal{O}_{\mathbb{P}^1}(-1)\oplus 6\mathcal{O}_{\mathbb{P}^1}. \tag{8.31}$$

By relative duality and the fact that $\mathcal{S}_B = \mathcal{S}_B^\vee$, this implies

$$\beta_{1*}\Big(\mathcal{O}_{B_1}(-2t)\otimes\mathcal{S}_B\Big) = 6\mathcal{O}_{\mathbb{P}^1}(-1)\oplus 6\mathcal{O}_{\mathbb{P}^1}. \tag{8.32}$$

Putting everything together, we can easily compute every entry in the Leray spectral sequence. The group action is necessarily the regular representation coming from the tensor product of Heisenberg group representations. Therefore, one obtains

$$H^p\Big(\widetilde{X},\mathcal{H}\otimes\mathcal{S}\Big) = \begin{cases} 0, & p=3 \\ 2\ \mathrm{Reg}(\mathbb{Z}_3\times\mathbb{Z}_3), & p=2 \\ 2\ \mathrm{Reg}(\mathbb{Z}_3\times\mathbb{Z}_3), & p=1 \\ 0, & p=0. \end{cases} \tag{8.33}$$

8.5 The hidden spectrum

Having constructed the two $SU(2)$ bundles $\mathcal{H}$ and $\mathcal{S}$, we must specify an embedding $SU(2) \times SU(2) \subset E_8$ to construct the hidden E_8 gauge bundle. We will stick to the simplest possibility and choose the maximal regular subgroup

$$SU(2) \times SU(2) \times \mathrm{Spin}(12) \subset E_8. \tag{8.34}$$

Hence, the hidden E_8 gauge groups is broken down to Spin(12). The corresponding branching rule for the adjoint of E_8 is

$$\underline{\mathbf{248}} = (\underline{\mathbf{1}}, \underline{\mathbf{1}}, \underline{\mathbf{66}}) \oplus (\underline{\mathbf{3}}, \underline{\mathbf{1}}, \underline{\mathbf{1}}) \oplus (\underline{\mathbf{1}}, \underline{\mathbf{3}}, \underline{\mathbf{1}}) \oplus (\underline{\mathbf{1}}, \underline{\mathbf{2}}, \underline{\mathbf{32}}) \oplus (\underline{\mathbf{2}}, \underline{\mathbf{1}}, \underline{\mathbf{32}}) \oplus (\underline{\mathbf{2}}, \underline{\mathbf{2}}, \underline{\mathbf{12}}). \tag{8.35}$$

The first three summands correspond to Spin(12) gauginos and moduli for $\mathcal{H}$ and $\mathcal{S}$. Then, there are potentially matter fields corresponding to $H^1(X, \mathcal{H}/G)$ and $H^1(X, \mathcal{S}/G)$. As we saw previously, there are no such light matter fields by equations (5.21) and (8.25).

However, there is a final contribution to the spectrum consisting of matter fields transforming in the $\underline{\mathbf{12}}$ of Spin(12). Their multiplicity is

$$H^1\Big(X, (\mathcal{H} \otimes \mathcal{S})/G\Big) = H^1\Big(\widetilde{X}, \mathcal{H} \otimes \mathcal{S}\Big)^G = 2, \tag{8.36}$$

where we have used the cohomology groups computed in equation (8.33). Note that because of the occurrence of the regular representation of $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ in equation (8.33), we cannot project out these matter fields using Wilson lines. This is why we choose not to turn on any Wilson lines in the hidden sector.

To summarize, this alternative hidden sector can be used if one wants to work in the weakly or strongly coupled heterotic string. It does not rely on five-branes to cancel the anomaly. The hidden E_8 gauge group is broken to Spin(12), and there are two matter fields transforming in the $\underline{\mathbf{12}}$ of Spin(12). These are, of course, uncharged under the visible $SU(3) \times SU(2) \times U(1)^2$ gauge group.

Appendix A Pushdown formulae for invariant line bundles on a dP$_9$ surface

In the course of the computation of the cohomology groups, we need to know the pushdown of invariant line bundles on a dP_9 surface B with elliptic fibration $\beta : B \to \mathbb{P}^1$ (that is, either B_1 or B_2). The invariant degree-2 cohomology is two-dimensional and generated by divisor classes f and t,

$$H^2\Big(B, \mathbb{Q}\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} = \mathbb{Z}f \oplus \mathbb{Z}t. \tag{A1}$$

Therefore, all line bundles on B are of the form $\mathcal{O}_B(nf + mt)$, $n, m \in \mathbb{Z}$. But the fiber class f is, by definition, just the preimage of a point on the base $\mathbb{P}^1$. By the projection formula, we immediately conclude that

$$R^k\beta_*\mathcal{O}_B(nf + mt) = \Big(R^k\beta_*\mathcal{O}_B(mt)\Big) \otimes \mathcal{O}_{\mathbb{P}^1}(n), \quad n, m \in \mathbb{Z}. \tag{A2}$$

Combined with relative duality, equation (4.24), we find

$$\beta_*\mathcal{O}_B(nf) = \mathcal{O}_{\mathbb{P}^1}(n), \qquad R^1\beta_*\mathcal{O}_B(nf) = K^\vee_{B|\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(n). \tag{A3}$$

Pushing down line bundles that are not vertical is more complicated. For that, recall that t is the sum of three distinct sections of the elliptic fibration,

$$t = \xi + \alpha\xi + (\eta \boxplus \xi). \tag{A4}$$

For convenience, we list the intersection numbers of these sections in table 2. Now each section s of the elliptic fibration is, by definition, also a divisor. Hence, for each line bundle $\mathcal{O}_B(D) \in \operatorname{Pic}(B)$, there is a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_B(D - s) \longrightarrow \mathcal{O}_B(D) \longrightarrow \mathcal{O}_s(D \cdot s) \longrightarrow 0 \tag{A5}$$

coming from the restriction to the (complex) codimension-one variety $s \subset B$. For example, choosing $s = D = \xi$, we obtain

$$0 \longrightarrow \mathcal{O}_B \longrightarrow \mathcal{O}_B(\xi) \longrightarrow \mathcal{O}_\xi(-1) \longrightarrow 0. \tag{A6}$$

The associated long exact sequence for the pushdown is then

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \beta_*\Big(\mathcal{O}_B(\xi)\Big) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\delta} \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow 0. \tag{A7}$$

Table 2: Intersection table on the dP_9.

$\bullet$	ξ	$\alpha\xi$	$\eta \boxplus \xi$
ξ	-1	1	0
$\alpha\xi$	1	-1	1
$\eta \boxplus \xi$	0	1	-1

The coboundary map δ has to be an isomorphism for degree reasons and, therefore,

$$\beta_*\Big(\mathcal{O}_B(\xi)\Big) = \mathcal{O}_{\mathbb{P}^1}. \tag{A8}$$

We can use this method to inductively find the pushdown as we add one section at a time. For example, for $\mathcal{O}_B(\xi \oplus \alpha\xi)$, we find a short exact sequence

$$0 \longrightarrow \mathcal{O}_B(\xi) \longrightarrow \mathcal{O}_B(\xi + \alpha\xi) \longrightarrow \mathcal{O}_{\alpha\xi} \longrightarrow 0, \tag{A9}$$

which pushes down to the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \beta_*\Big(\mathcal{O}_B(\xi + \alpha\xi)\Big) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0 \tag{A10}$$

which determines unambiguously the pushdown to be

$$\beta_*\Big(\mathcal{O}_B(\xi + \alpha\xi)\Big) = 2\mathcal{O}_{\mathbb{P}^1}. \tag{A11}$$

Continuing this way, we can, in principle, determine the pushdown for all $\mathcal{O}_B(nt)$. However, eventually one encounters extension ambiguities that make it more difficult to find a unique answer. Happily, one can avoid the extension ambiguities up to $\mathcal{O}_B(5t)$ if one just adds sections in the correct order. Reading from left to right, the preferred order to add the sections is

$$\begin{aligned}
t &= \xi + \alpha\xi + (\eta \boxplus \xi) \\
2t &= \xi + \alpha\xi + (\eta \boxplus \xi) + \alpha\xi + \xi + (\eta \boxplus \xi) \\
3t &= \xi + \alpha\xi + (\eta \boxplus \xi) + \alpha\xi + \xi + \alpha\xi + \xi + (\eta \boxplus \xi) + (\eta \boxplus \xi) \\
4t &= \xi + \alpha\xi + (\eta \boxplus \xi) + \alpha\xi + \xi + \alpha\xi \\
&\quad + \xi + (\eta \boxplus \xi) + \alpha\xi + \xi + (\eta \boxplus \xi) + (\eta \boxplus \xi) \\
5t &= \xi + \alpha\xi + (\eta \boxplus \xi) + \alpha\xi + \xi + \alpha\xi \\
&\quad + \xi + (\eta \boxplus \xi) + \alpha\xi + (\eta \boxplus \xi) + \alpha\xi + \xi + \xi + (\eta \boxplus \xi) + (\eta \boxplus \xi).
\end{aligned} \tag{A12}$$

The corresponding pushdown is then easily determined to be

$$\begin{aligned}
\beta_*\Big(\mathcal{O}_B(t)\Big) &= 3\mathcal{O}_{\mathbb{P}^1} \\
\beta_*\Big(\mathcal{O}_B(2t)\Big) &= 6\mathcal{O}_{\mathbb{P}^1} \\
\beta_*\Big(\mathcal{O}_B(3t)\Big) &= 8\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \\
\beta_*\Big(\mathcal{O}_B(4t)\Big) &= 9\mathcal{O}_{\mathbb{P}^1} \oplus 3\mathcal{O}_{\mathbb{P}^1}(1) \\
\beta_*\Big(\mathcal{O}_B(5t)\Big) &= 9\mathcal{O}_{\mathbb{P}^1} \oplus 6\mathcal{O}_{\mathbb{P}^1}(1).
\end{aligned} \tag{A13}$$

One can also deal with the extension ambiguities for $\beta_* \mathcal{O}(nt)$, $n > 5$, and arrive at a unique answer. However, this is not necessary for our purposes.

Appendix B Direct image of an extension

The purpose of this Appendix is to show that the pushdown map

$$\begin{aligned} \beta_{2*}: \ & \operatorname{Ext}^1\Big(\chi_2 \mathcal{O}_{B_2}(2t+f) \otimes I_9,\ \mathcal{O}_{B_2}(2t-3f)\Big)^G \\ \longrightarrow & \operatorname{Ext}^1\Bigg(\underbrace{\beta_{2*}\Big(\chi_2 \mathcal{O}_{B_2}(2t+f) \otimes I_9\Big)}_{=3\mathcal{O}_{\mathbb{P}^1}(-1) \oplus 3\mathcal{O}_{\mathbb{P}^1}},\ \underbrace{\beta_{2*}\Big(\mathcal{O}_{B_2}(2t-3f)\Big)}_{=6\mathcal{O}_{\mathbb{P}^1}(-3)}\Bigg)^G \end{aligned} \tag{B1}$$

of the $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ invariant part of the Ext groups is the zero map. To conclude this, we will look at the local to global spectral sequence for the Ext groups. For the rest of this Appendix, let $\underline{\mathrm{Hom}}$ and $\underline{\mathrm{Ext}}$ be the local Hom and Ext, that is, the sheaf of homomorphisms and the corresponding derived functor. Note that a quick application of the long exact sequence associated to the short exact sequence of sheaves, equation (6.32), yields

$$\underline{\operatorname{Ext}}^i\Big(I_9, \mathcal{O}_{B_2}\Big) = \begin{cases} \mathcal{O}_{B_2}, & i=0 \\ \bigoplus_{i=1}^{9} \mathcal{O}_{p_i}, & i=1 \\ 0, & \text{otherwise.} \end{cases} \tag{B2}$$

Using this and elementary properties of the local extensions, see [81], we find

$$\underline{\operatorname{Ext}}^i\Big(\chi_2 \mathcal{O}_{B_2}(2t+f) \otimes I_9,\ \mathcal{O}_{B_2}(2t-3f)\Big) = \begin{cases} \chi_2^2 \mathcal{O}_{B_2}(-4f), & i=0 \\ \bigoplus_{i=1}^{9} \mathcal{O}_{p_i}, & i=1 \\ 0, & \text{otherwise.} \end{cases} \tag{B3}$$

Of course, we are only interested in the global Ext which classifies extensions of sheaves. The global Ext groups are determined from the local Ext sheaves by means of a spectral sequence. In general, for arbitrary sheaves $\mathcal{E}$ and $\mathcal{F}$, this spectral sequence starts with

$$E_2^{p,q} = H^p\Big(\underline{\operatorname{Ext}}^q(\mathcal{E}, \mathcal{F})\Big) \implies \operatorname{Ext}^{p+q}(\mathcal{E}, \mathcal{F}). \tag{B4}$$

In our case, the local to global spectral sequence

$$\begin{aligned} E_2^{p,q}(B_2) = H^p\Big(\underline{\mathrm{Ext}}^q\left(\chi_2\mathcal{O}_{B_2}(2t+f)\otimes I_9,\ \mathcal{O}_{B_2}(2t-3f)\right)\Big) \\ \Longrightarrow \mathrm{Ext}^{p+q}\left(\chi_2\mathcal{O}_{B_2}(2t+f)\otimes I_9,\ \mathcal{O}_{B_2}(2t-3f)\right) \end{aligned} \tag{B5}$$

for the sheaves on B_2 starts with

$$E_2^{p,q}(B_2) = \begin{array}{r|ccc} q{=}1 & \mathrm{Reg}(G) & 0 & 0 \\ q{=}0 & 0 & \chi_2^2+\chi_1\chi_2^2+\chi_1^2\chi_2^2 & \chi_2^2+\chi_1\chi_2^2+2\chi_1^2\chi_2^2 \\ \hline & p{=}0 & p{=}1 & p{=}2 \end{array} \quad \text{with } d_3: \mathrm{Reg}(G)\to \chi_2^2+\chi_1\chi_2^2+2\chi_1^2\chi_2^2. \tag{B6}$$

The d_3 differential is nontrivial. One way to determine it is to compare with the result for the global Ext^1 group, see equation (6.81). From that, we can conclude that the image of d_3 has to be $\chi_2^2+\chi_1\chi_2^2+\chi_1^2\chi_2^2$ and, hence, the third (and final) tableau must be

$$E_\infty^{p,q}(B_2) = \begin{array}{r|ccc} q{=}1 & (1+\chi_2)(1+\chi_2+\chi_2^2) & 0 & 0 \\ q{=}0 & 0 & \chi_2^2(1+\chi_1+\chi_1^2) & \chi_1^2\chi_2^2 \\ \hline & p{=}0 & p{=}1 & p{=}2 \end{array} . \tag{B7}$$

On the other hand side, the local Ext of the direct image sheaves is

$$\underline{\mathrm{Ext}}^i\left(3\mathcal{O}_{\mathbb{P}^1}(-1)\oplus 3\mathcal{O}_{\mathbb{P}^1},\ 6\mathcal{O}_{\mathbb{P}^1}(-3)\right) \tag{B8}$$

$$= \begin{cases} 18\mathcal{O}_{\mathbb{P}^1}(-2)\oplus 18\mathcal{O}_{\mathbb{P}^1}(-3), & i=0 \\ 0, & \text{otherwise.} \end{cases} \tag{B9}$$

The corresponding local to global spectral sequence then has only one nonvanishing entry and looks schematically like

$$E_2^{p,q}(\mathbb{P}^1) = E_\infty^{p,q}(\mathbb{P}^1) = \begin{array}{r|ccc} q{=}1 & 0 & 0 & 0 \\ q{=}0 & 0 & \cdots & 0 \\ \hline & p{=}0 & p{=}1 & p{=}2 \end{array} . \tag{B10}$$

But the pushdown image of the G-invariant part of the global Ext groups, equation (B1), has to be induced from a map of spectral sequences. But because it has to respect the (p, q) degree, there is no nonzero map

$$E_2^{p,q}(B_2)^G \xrightarrow{0} E_2^{p,q}(\mathbb{P}^1)^G. \tag{B11}$$

Hence, the map, equation (B1), is the zero map, as claimed.

Appendix C Other equivariant actions

There is a different way to distribute the characters occurring in our standard model vector bundle. For that, we change the rank 2 bundle $\mathcal{V}_1$ to $\mathcal{V}_1'$, defined as

$$\mathcal{V}_1' \stackrel{\text{def}}{=} \mathcal{O}_{\widetilde{X}}(-\tau_1 + \tau_2) \oplus \chi_2^2 \mathcal{O}_{\widetilde{X}}(-\tau_1 + \tau_2). \tag{C1}$$

The rank 4 vector bundle $\mathcal{V}'$ is again defined as a generic extension of the form

$$0 \longrightarrow \mathcal{V}_1' \longrightarrow \mathcal{V}' \longrightarrow \mathcal{V}_2 \longrightarrow 0. \tag{C2}$$

If one were only to look at the underlying holomorphic vector bundles, then $\mathcal{V}$ and $\mathcal{V}'$ are the same bundles[14]. But as equivariant bundles they differ, because the equivariant $\mathbb{Z}_3 \times \mathbb{Z}_3$ action is different. Hence, the quotient $\mathcal{V}/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ and $\mathcal{V}'/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ are different vector bundles on $X = \widetilde{X}/(\mathbb{Z}_3 \times \mathbb{Z}_3)$.

This is illustrated by the cohomology groups. For example, consider the cohomology of $\wedge^2\mathcal{V}$ versus $\wedge^2\mathcal{V}'$. Of course, the dimension is the same since they are equal as ordinary vector bundles,

$$\dim_C H^1\Big(\widetilde{X},\ \wedge^2\mathcal{V}\Big) = 14 = \dim_C H^1\Big(\widetilde{X},\ \wedge^2\mathcal{V}'\Big). \tag{C3}$$

But the 14-dimensional $\mathbb{Z}_3 \times \mathbb{Z}_3$ representation differs. One can easily show that

$$H^1\Big(\widetilde{X}, \wedge^2\mathcal{V}\Big) = 2 \oplus 2\chi_1 \oplus 2\chi_2 \oplus 2\chi_1^2 \oplus 2\chi_2^2 \oplus 2\chi_1\chi_2^2 \oplus 2\chi_1^2\chi_2, \tag{C4}$$

whereas

$$H^1\Big(\widetilde{X}, \wedge^2\mathcal{V}'\Big) = 2 \oplus \chi_1 \oplus \chi_1^2 \oplus 2\chi_2 \oplus 2\chi_1\chi_2 \oplus \chi_1^2\chi_2 \oplus 2\chi_2^2 \oplus \chi_1\chi_2^2 \oplus 2\chi_1^2\chi_2^2. \tag{C5}$$

[14]This is not quite correct: $\mathcal{V}_1'$ and $\mathcal{V}_1$ are the same underlying bundle, but the extensions $\mathcal{V}'$ and $\mathcal{V}$ are different, because one must choose different extensions if one wants the extension to be equivariant. Indeed, stable holomorphic bundles have uniquely determined equivariant structures, up to multiplication by an overall phase.

We can now read off the spectrum if one were to use the different bundle $\mathcal{V}'$ for compactification and obtain one pair of Higgs, two $\underline{\mathbf{3}}$, and two $\underline{\overline{\mathbf{3}}}$ multiplets. Again, doublets and triplets are split but not in the desired way.

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